# Fourier expansion of generalized prolate spheroidal wave function concerning 

## generalized polynomials ,Aleph-function and multivariable Aleph-function

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## ABSTRACT

The object of the paper is to establish an integral formula involving the generalized prolate spheroidal wave function, generalized hypergeometric function, generalized polynomials, Aleph-function of one variable and multivariable Aleph-function. This integral formula has been employed to obtain an expansion formula for multivariable Aleph-function, generalized hypergeometric function, a class of polynomial and Aleph-function in terms of generalized prolate spheroidal wave function. This expansion formula being of very general nature can be transformed to provide many new results involving various commonly used special functions occuring in applied mathematics, mathematics physics and mchanics. During the course of finding, we establish several particular cases.

Keywords :generalized multivariable Aleph-function, Aleph-function, class of multivariable polynomials, generalized hypergeometric function, finite integral, generalized prolate spheroidal wave function, expansion formula

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## 1. Introduction and preliminaries.

The Aleph- function, introduced by Südland [10] et al, however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

for all $z$ different to 0 and

$$
\begin{equation*}
\Omega_{P_{i}, Q_{i}, c_{i} ; r}^{M, N}(s)=\frac{\prod_{j=1}^{M} \Gamma\left(b_{j}+B_{j} s\right) \prod_{j=1}^{N} \Gamma\left(1-a_{j}-A_{j} s\right)}{\sum_{i=1}^{r} c_{i} \prod_{j=N+1}^{P_{i}} \Gamma\left(a_{j i}+A_{j i} s\right) \prod_{j=M+1}^{Q_{i}} \Gamma\left(1-b_{j i}-B_{j i} s\right)} \tag{1.2}
\end{equation*}
$$

With :
$|\arg z|<\frac{1}{2} \pi \Omega \quad$ Where $\Omega=\sum_{j=1}^{M} \beta_{j}+\sum_{j=1}^{N} \alpha_{j}-c_{i}\left(\sum_{j=M+1}^{Q_{i}} \beta_{j i}+\sum_{j=N+1}^{P_{i}} \alpha_{j i}\right)>0$ with $i=1, \cdots, r$
For convergence conditions and other details of Aleph-function, see Südland et al [10].
Serie representation of Aleph-function is given by Chaurasia et al [2].
$\aleph_{P_{i}, Q_{i}, c_{i} ; r}^{M, N}(z)=\sum_{G=1}^{M} \sum_{g=0}^{\infty} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r}^{M, N}(s)}{B_{G} g!} z^{-s}$
With $s=\eta_{G, g}=\frac{b_{G}+g}{B_{G}}, P_{i}<Q_{i},|z|<1$ and $\Omega_{P_{i}, Q_{i}, c_{i} ; r}^{M, N}(s)$ is given in (1.2)
The generalized polynomials of multivariables defined by Srivastava [9], is given in the following manner :
$S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left[y_{1}, \cdots, y_{s}\right]=\sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} \frac{\left(-N_{1}\right)_{M_{1} K_{1}}}{K_{1}!} \cdots \frac{\left(-N_{s}\right)_{M_{s} K_{s}}}{K_{s}!}$
$A\left[N_{1}, K_{1} ; \cdots ; N_{s}, K_{s}\right] y_{1}^{K_{1}} \cdots y_{s}^{K_{s}}$
where $M_{1}, \cdots, M_{s}$ are arbitrary positive integers and the coefficients are $A\left[N_{1}, K_{1} ; \cdots ; N_{s}, K_{s}\right]$ arbitrary constants, real or complex.
The generalized hypergeometric function serie is defined as follows :
${ }_{p^{\prime}} F_{q^{\prime}}(y)=\sum_{s^{\prime}=0}^{\infty} \frac{\left[\left(a_{p^{\prime}}\right)\right]_{s^{\prime}}}{\left[\left(b_{q^{\prime}}\right)\right]_{s^{\prime}}} y^{s^{\prime}}$
Here $\left[\left(a_{p^{\prime}}\right)\right]_{s^{\prime}}=\left(a_{1}\right)_{s^{\prime}} \cdots\left(a_{p^{\prime}}\right)_{s^{\prime}} ;\left[\left(b_{q^{\prime}}\right)\right]_{s^{\prime}}=\left(b_{1}\right)_{s^{\prime}} \cdots\left(b_{q^{\prime}}\right)_{s^{\prime}}$.
The serie (1.7) converge if $p^{\prime} \leqslant q^{\prime}$ and $|y|<1$.
In the document, we note : $a_{1}=\frac{\left(-N_{1}\right)_{M_{1} K_{1}}}{K_{1}!} \cdots \frac{\left(-N_{s}\right)_{M_{s} K_{s}}}{K_{s}!} A\left[N_{1}, K_{1} ; \cdots ; N_{s}, K_{s}\right]$
The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [3], itself is an a generalisation of G and H -functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have $: \aleph\left(z_{1}, \cdots, z_{r}\right)=\aleph_{p_{i}, q_{i}, \tau_{i} ; R: p_{i(1)}, q_{i(1)}, \tau_{i(1)} ; R^{(1)} ; \cdots ; p_{i(r)}, q_{i(r)} ; \tau_{i(r)} ; R^{(r)}}\left(\left.\begin{array}{c}\mathrm{z}_{1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_{r}\end{array} \right\rvert\,\right.$

$$
\begin{array}{cl}
{\left[\left(\mathrm{a}_{j} ; \alpha_{j}^{(1)}, \cdots, \alpha_{j}^{(r)}\right)_{1, \mathfrak{n}}\right]} & ,\left[\tau_{i}\left(a_{j i} ; \alpha_{j i}^{(1)}, \cdots, \alpha_{j i}^{(r)}\right)_{\mathfrak{n}+1, p_{i}}\right]: \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & ,\left[\tau_{i}\left(b_{j i} ; \beta_{j i}^{(1)}, \cdots, \beta_{j i}^{(r)}\right)_{m+1, q_{i}}\right]:
\end{array}
$$

$$
\left.\left.\left[\left(\mathrm{c}_{j}^{(1)}\right), \gamma_{j}^{(1)}\right)_{1, n_{1}}\right],\left[\tau_{i(1)}\left(c_{j i^{(1)}}^{(1)}, \gamma_{j i^{(1)}}^{(1)}\right)_{n_{1}+1, p_{i}^{(1)}}\right] ; \cdots ; ;\left[\left(\mathrm{c}_{j}^{(r)}\right), \gamma_{j}^{(r)}\right)_{1, n_{r}}\right],\left[\tau_{i^{(r)}}\left(c_{j i^{(r)}}^{(r)}, \gamma_{j i(r)}^{(r)}\right)_{n_{r}+1, p_{i}^{(r)}}\right]
$$

$$
\left.\left.\left[\left(\mathrm{d}_{j}^{(1)}\right), \delta_{j}^{(1)}\right)_{1, m_{1}}\right],\left[\tau_{i(1)}\left(d_{j i(1)}^{(1)}, \delta_{j i^{(1)}}^{(1)}\right)_{m_{1}+1, q_{i}^{(1)}}\right] ; \cdots ; ;\left[\left(\mathrm{d}_{j}^{(r)}\right), \delta_{j}^{(r)}\right)_{1, m_{r}}\right],\left[\tau_{i(r)}\left(d_{j i(r)}^{(r)}, \delta_{j i(r)}^{(r)}\right)_{m_{r}+1, q_{i}^{(r)}}\right]
$$

$$
\begin{equation*}
=\frac{1}{(2 \pi \omega)^{r}} \int_{L_{1}} \cdots \int_{L_{r}} \psi\left(s_{1}, \cdots, s_{r}\right) \prod_{k=1}^{r} \theta_{k}\left(s_{k}\right) z_{k}^{s_{k}} \mathrm{~d} s_{1} \cdots \mathrm{~d} s_{r} \tag{1.8}
\end{equation*}
$$

with $\omega=\sqrt{-1}$
For more details, see Ayant [1]. The reals numbers $\tau_{i}$ are positives for $i=1, \cdots, R, \tau_{i(k)}$ are positives for $i^{(k)}=1, \cdots, R^{(k)}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H -function given by as :
$\left|\arg z_{k}\right|<\frac{1}{2} A_{i}^{(k)} \pi$, where
$A_{i}^{(k)}=\sum_{j=1}^{\mathfrak{n}} \alpha_{j}^{(k)}-\tau_{i} \sum_{j=\mathfrak{n}+1}^{p_{i}} \alpha_{j i}^{(k)}-\tau_{i} \sum_{j=1}^{q_{i}} \beta_{j i}^{(k)}+\sum_{j=1}^{n_{k}} \gamma_{j}^{(k)}-\tau_{i(k)} \sum_{j=n_{k}+1}^{p_{i(k)}} \gamma_{j i(k)}^{(k)}$
$+\sum_{j=1}^{m_{k}} \delta_{j}^{(k)}-\tau_{i^{(k)}} \sum_{j=m_{k}+1}^{q_{i}(k)} \delta_{j i(k)}^{(k)}>0$, with $k=1 \cdots, r, i=1, \cdots, R, i^{(k)}=1, \cdots, R^{(k)}$
The complex numbers $z_{i}$ are not zero.Throughout this document, we assume the existence and absolute convergence
conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :
$\aleph\left(z_{1}, \cdots, z_{r}\right)=0\left(\left|z_{1}\right|^{\alpha_{1}} \ldots\left|z_{r}\right|^{\alpha_{r}}\right), \max \left(\left|z_{1}\right| \ldots\left|z_{r}\right|\right) \rightarrow 0$
$\aleph\left(z_{1}, \cdots, z_{r}\right)=0\left(\left|z_{1}\right|^{\beta_{1}} \ldots\left|z_{r}\right|^{\beta_{r}}\right), \min \left(\left|z_{1}\right| \ldots\left|z_{r}\right|\right) \rightarrow \infty$
where, with $k=1, \cdots, r: \alpha_{k}=\min \left[\operatorname{Re}\left(d_{j}^{(k)} / \delta_{j}^{(k)}\right)\right], j=1, \cdots, m_{k}$ and

$$
\beta_{k}=\max \left[\operatorname{Re}\left(\left(c_{j}^{(k)}-1\right) / \gamma_{j}^{(k)}\right)\right], j=1, \cdots, n_{k}
$$

We will use these following notations in this paper
$U=p_{i}, q_{i}, \tau_{i} ; R ; V=m_{1}, n_{1} ; \cdots ; m_{r}, n_{r}$
$\mathrm{W}=p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}} ; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}} ; R^{(r)}$
$A=\left\{\left(a_{j} ; \alpha_{j}^{(1)}, \cdots, \alpha_{j}^{(r)}\right)_{1, n}\right\},\left\{\tau_{i}\left(a_{j i} ; \alpha_{j i}^{(1)}, \cdots, \alpha_{j i}^{(r)}\right)_{n+1, p_{i}}\right\}$
$B=\left\{\tau_{i}\left(b_{j i} ; \beta_{j i}^{(1)}, \cdots, \beta_{j i}^{(r)}\right)_{m+1, q_{i}}\right\}$
$\left.\left.C=\left\{\left(c_{j}^{(1)} ; \gamma_{j}^{(1)}\right)_{1, n_{1}}\right\}, \tau_{i^{(1)}}\left(c_{j i^{(1)}}^{(1)} ; \gamma_{j i^{(1)}}^{(1)}\right)_{n_{1}+1, p_{i(1)}}\right\}, \cdots,\left\{\left(c_{j}^{(r)} ; \gamma_{j}^{(r)}\right)_{1, n_{r}}\right\}, \tau_{i^{(r)}}\left(c_{j i(r)}^{(r)} ; \gamma_{j i(r)}^{(r)}\right)_{n_{r}+1, p_{i}(r)}\right\}$
$\left.\left.D=\left\{\left(d_{j}^{(1)} ; \delta_{j}^{(1)}\right)_{1, m_{1}}\right\}, \tau_{i(1)}\left(d_{j i(1)}^{(1)} ; \delta_{j i^{(1)}}^{(1)}\right)_{m_{1}+1, q_{i}(1)}\right\}, \cdots,\left\{\left(d_{j}^{(r)} ; \delta_{j}^{(r)}\right)_{1, m_{r}}\right\}, \tau_{i(r)}\left(d_{j i}^{(r)} ; \delta_{j i(r)}^{(r)}\right)_{m_{r}+1, q_{i}(r)}\right\}$
The multivariable Aleph-function write :
$\aleph\left(z_{1}, \cdots, z_{r}\right)=\aleph_{U: W}^{0, \mathfrak{n}: V}\left(\begin{array}{c|c}\mathrm{z}_{1} & \mathrm{~A}: \mathrm{C} \\ \cdot & : \\ \cdot & \cdots \cdot \cdot \\ \cdot \cdot & \mathrm{B}: \mathrm{D}\end{array}\right)$

## 2. Generalized prolate spheroidal wave function

The generalized prolate spheroidal wave functions has been recently defined by Gupta [4] as the solution of the differential equation
$\left(1-x^{2}\right) y^{\prime \prime}+[\beta-\alpha-(\alpha+\beta+2) x] y^{\prime}+\left[\zeta(c)-c^{2} x^{2}\right] y=0$
in the form of an infinite sum :
$\phi_{n}^{(\alpha, \beta)}(c, x)=\sum_{j=0}^{\infty} d_{j, n}^{(\alpha, \beta)}(c) P_{n+j}^{(\alpha, \beta)}(x)$
where $\zeta(c)$ being separation constants for every value of constant parameter $c$ and the coefficients $d_{j, n}^{(\alpha, \beta)}(c)$ can be determined by a five term recursion formula in a manner quite parallel to prolate spheroidal wave functions. More recently, Sharma [8] has developed multiple generalized prolate spheroidal wave transforms by using the orthogonality property given by Gupta [4].

$$
\begin{equation*}
\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} \phi_{n}^{(\alpha, \beta)}(c, x) \phi_{m}^{(\alpha, \beta)}(c, x) \mathrm{d} x=N_{n, m}^{(\alpha, \beta)} \delta_{m, n} \quad \text {,where } \tag{2.3}
\end{equation*}
$$

$N_{n, n}^{(\alpha, \beta)}=2^{\alpha+\beta+1} \sum_{j=0}^{\infty}\left[d_{j, n}^{(\alpha, \beta)}\right]^{2} \frac{\Gamma(\alpha+j+1) \Gamma(\beta+j+1)}{(1+\alpha+\beta+2 n+2 j) \Gamma(n+j+1) \Gamma((1+\alpha+\beta+n+j)}$
$\delta_{m, n}$ is the Kronecker symbol.

## 3.Required integral

We have the following integral ( Erdelyi et al [5], page 284, eq.3)
$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma} P_{n}^{(\alpha, \beta)}(x) \mathrm{d} x=\frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1)}{n \Gamma(\rho+\sigma+2)}$
${ }_{3} F_{2}[-n, \alpha+\beta+n+1, \rho+1 ; \alpha+1, \rho+\sigma+2 ; 1], \operatorname{Re}(\rho)>0, \operatorname{Re}(\sigma)>0$

## 4. Main integral

The integral formula to be established here is

$$
\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma} \phi_{n}^{(\alpha, \beta)}\{c, x\} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right)
$$

$$
{ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}
\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\
\dot{\cdot} \cdot \\
\mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}
\end{array}\right)
$$

$$
\aleph_{U: W}^{0, \mathfrak{n}: V}\binom{\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}}}{\mathrm{z}_{r}(1-x)^{h_{r}}(1+x)^{k_{r}}} \mathrm{~d} x=\sum_{p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M, N}\left(\eta_{G, g}\right)}{B_{G} g!}
$$

$$
\frac{\prod_{i=1}^{M}\left(a_{i}\right)_{q}}{\prod_{i=1}^{N}\left(b_{i}\right)_{q} q!} t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} 2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right)} d_{p, n}^{(\alpha, \beta)}(c)
$$

$$
\sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} \aleph_{U_{21}: W}^{0, \mathfrak{n}+2: V}\left(\begin{array}{c}
2^{h_{1}+k_{1}} z_{1} \\
\cdots \\
\underset{2^{h_{r}+k_{r}} z_{r}}{\cdots}
\end{array}\right)
$$

$$
\left.\begin{array}{c}
\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, \cdots, h_{r}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, \cdots, k_{r}\right), A: C  \tag{4.1}\\
\left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\dot{\sum_{i=1}^{s}} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, \cdots, h_{r}+k_{r}\right), B: D
\end{array}\right)
$$

where $U_{21}=p_{i}+2, q_{i}+1, \tau_{i} ; R$

## Provided

а ) $\min \left\{e, f, e^{\prime}, f^{\prime}, g_{i}, w_{i}, h_{j}, k_{j}\right\}>0, i=1, \cdots, s ; j=1, \cdots, r$
b) $R e\left[\rho+e \min _{1 \leqslant j \leqslant M} \frac{b_{j}}{B_{j}}+\sum_{i=1}^{r} h_{i} \min _{1 \leqslant j \leqslant m_{i}} \frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right]>-1$
c) $R e\left[\sigma+f \min _{1 \leqslant j \leqslant M} \frac{b_{j}}{B_{j}}+\sum_{i=1}^{r} k_{i} \min _{1 \leqslant j \leqslant m_{i}} \frac{d_{j}^{(i)}}{\delta_{j}^{(i)}}\right]>-1$
d) $\left|\arg z_{k}\right|<\frac{1}{2} A_{i}^{(k)} \pi$, where $A_{i}^{(k)}$ is given in (1.10)
e) $|\arg t|<\frac{1}{2} \pi \Omega \quad$ where $\Omega=\sum_{j=1}^{M} \beta_{j}+\sum_{j=1}^{N} \alpha_{j}-c_{i}\left(\sum_{j=M+1}^{Q_{i}} \beta_{j i}+\sum_{j=N+1}^{P_{i}} \alpha_{j i}\right)>0$
f) $\alpha>-1, \beta>-1$ and $M \leqslant N(M=N+1$ and $|y|<1)$

Proof Let $M=\frac{1}{(2 \pi \omega)^{r}} \int_{L_{1}} \cdots \int_{L_{r}} \psi\left(s_{1}, \cdots, s_{r}\right) \prod_{k=1}^{r} \theta_{k}\left(s_{k}\right) z_{k}^{s_{k}}$
To establish (4.1), express the generalized prolate spheroidal wave function as given in (2.2), the generalized hypergeometric function in serie with the help of (1.6) ,the Aleph-function of one variable in serie with the help of (1.3), the general polynomials with the help of (1.5) an and the multivariable Aleph-function in terms of Mellin-Barnes type contour integral with the help of (1.8), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process), we obtain
$\sum_{p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M,\left(\eta_{G, g}\right) \prod_{i=1}^{M}\left(a_{i}\right)_{q}}}{B_{G} g!} \prod_{i=1}^{N}\left(b_{i}\right)_{q} q!\quad t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} d_{p, n}^{(\alpha, \beta)}(c)$
$M\left[\int_{-1}^{1}(1-x)^{\rho+e \eta_{G, g}+e^{\prime} q+\sum_{i=1}^{s} K_{i} g_{i}+\sum_{i=1}^{r} h_{i} s_{i}}(1+x)^{\rho+f \eta_{G, g}+f^{\prime} q+\sum_{i=1}^{s} K_{i} w_{i}+\sum_{i=1}^{r} k_{i} s_{i}}\right.$
$\left.P_{n+p}^{(\alpha, \beta)}(x) \mathrm{d} x\right] \mathrm{d} s_{1} \cdots \mathrm{~d} s_{r}$
Now evaluating the inner $x$-integral with the help of (3.1). Writing series representation for ${ }_{3} F_{2}$, changing the order of integration and summation involved therein and expressing the multiple contour integral as the multivariable Alephfunction, we obtain the right hand side of (4.1).

## 5. Expansion formula

We have the general formula expansion
$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right){ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right)$
$S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\ \dot{\cdot} \cdot \\ \mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}\end{array}\right) \aleph_{U: W}^{0, \mathfrak{n}: V}\left(\begin{array}{c}\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\ \dot{\cdot} \cdot \dot{子} \\ \mathrm{z}_{r}(1-x)^{h_{r}}(1+x)^{k_{r}}\end{array}\right)=\sum_{l, p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty}$
$\sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M, N}\left(\eta_{G, g}\right) \prod_{i=1}^{M}\left(a_{i}\right)_{q}}{B_{G} g!} \prod_{i=1}^{N}\left(b_{i}\right)_{q} q!\quad t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} \frac{D_{p, n}^{(\alpha, \beta)}(c)}{N_{l, l}^{(\alpha, \beta)}}$
$2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right)} \sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} \aleph_{U_{21}: W}^{0, n+2: V}\left(\begin{array}{c}2^{h_{1}+k_{1}} z_{1} \\ \cdots \\ \underset{\sim}{h_{r}+k_{r}} z_{r}\end{array}\right)$
$\left.\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, \cdots, h_{r}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, \cdots, k_{r}\right), A: C\right)$ $\left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\sum_{i=1}^{s} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, \cdots, h_{r}+k_{r}\right), B: D$
$\phi_{l}^{(\alpha, \beta)}(c, x)$

## Proof

To establish (5.1), let $f(x)=\aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right)(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}$
${ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\ \dot{\cdots} \cdot \\ \mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}\end{array}\right)$
$\aleph_{U: W}^{0, \mathfrak{n}: V}\binom{\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}}}{\dot{\mathrm{k}^{\prime}} \cdot \dot{)_{r}}(1-x)^{h_{r}}(1+x)^{k_{r}}}=\sum_{l=0}^{\infty} A_{l} \phi_{l}^{(\alpha, \beta)}(c, x)$

The equation is valid if $f(x)$ is continuous and bounded variation in the domain $(-1,1)$. Multiplying both sides of (5.2) by $(1-x)^{\alpha}(1+x)^{\beta} \phi_{n}^{(\alpha, \beta)}(c, x), \alpha>-1, \beta>-1$ and integrating with respect to $x$ from -1 to 1 , change the order and summation ( which is permissible) on the right, we obtain :
$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma} \phi_{n}^{(\alpha, \beta)}\{c, x\} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right)$
${ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\ \dot{\sim} \cdot \\ \mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}\end{array}\right)$
$\aleph_{U: W}^{0, \mathfrak{n}: V}\left(\begin{array}{c}\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\ \dot{\cdot} \cdot \\ \mathrm{z}_{r}(1-x)^{h_{r}}(1+x)^{k_{r}}\end{array}\right) \mathrm{d} x=\sum_{l=0}^{\infty} A_{l} \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta} \phi_{n}^{(\alpha, \beta)}\{c, x\} \phi_{l}^{(\alpha, \beta)}(c, x) \mathrm{d} x$ (5.3)

Using the orthogonality property of the generalized prolate spheroidal wave function (2.3) on the right hand side and the result (4.1) on the left hand side of (5.3), we obtain
on subtituting the value of $A_{l}$ from (5.4) in (5.2) and using the lemma([4], page 57, eq.2), we obtain the desired result.

## 6. Multivariable I-function

If $\tau_{i}, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. in this section we have the following expansion formulae with the multivariable I-function defined by Sharma et al [5].

$$
\begin{aligned}
& (1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right){ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) \\
& S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}
\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\
\cdot \cdot \\
\mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}
\end{array}\right) I_{U: W}^{0, \mathfrak{n}: V}\left(\begin{array}{c}
\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\
\cdot \cdot \cdot \\
\mathrm{z}_{r}(1-x)^{h_{r}}(1+x)^{k_{r}}
\end{array}\right)=\sum_{l, p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty}
\end{aligned}
$$

$$
\sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M, N}\left(\eta_{G, g}\right)}{B_{G} g!} \frac{\prod_{i=1}^{M}\left(a_{i}\right)_{q}}{\prod_{i=1}^{N}\left(b_{i}\right)_{q} q!} t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} \frac{D_{p, n}^{(\alpha, \beta)}(c)}{N_{l, l}^{(\alpha, \beta)}}
$$

$$
2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right)} \sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} I_{U_{21}: W}^{0, \mathfrak{n}+2: V}\left(\begin{array}{c}
2^{h_{1}+k_{1}} z_{1} \\
\cdots \\
\cdots \\
2^{h_{r}+k_{r}} z_{r}
\end{array}\right)
$$

$$
\left.\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, \cdots, h_{r}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, \cdots, k_{r}\right), A: C\right)
$$

$$
\left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\stackrel{\sum_{i=1}^{s}}{\cdot} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, \cdots, h_{r}+k_{r}\right), B: D
$$

$$
\begin{equation*}
\phi_{l}^{(\alpha, \beta)}(c, x) \tag{6.1}
\end{equation*}
$$

$$
\begin{align*}
& A_{l}=\sum_{p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M,\left(\eta_{G, g}\right)} \prod_{i=1}^{M}\left(a_{i}\right)_{q}}{B_{G} g!} \prod_{i=1}^{N}\left(b_{i}\right)_{q} q!\quad t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} \frac{D_{p, n}^{(\alpha, \beta)}(c)}{N_{l, l}^{(\alpha, \beta)}} \\
& 2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right.} \sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} \aleph_{U_{21}: W}^{0, \mathfrak{n}+2: V}\left(\begin{array}{c}
2^{h_{1}+k_{1}} z_{1} \\
\cdots \\
\dot{.} \\
2^{h_{r}+k_{r}} z_{r}
\end{array}\right) \\
& \left.\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, \cdots, h_{r}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, \cdots, k_{r}\right), A: C\right)  \tag{5.4}\\
& \left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\dot{\sum_{i=1}^{s}} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, \cdots, h_{r}+k_{r}\right), B: D
\end{align*}
$$

with the same notations and conditions that (4.1) with $\tau_{i}, \tau_{i^{(1)}}, \cdots, \tau_{i(r)} \rightarrow 1$

## 7. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [7], and we have the two following results.

$$
\begin{aligned}
& (1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right){ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) \\
& S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}
\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\
\cdot \cdot \cdot \\
\mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}
\end{array}\right) \aleph_{U: W}^{0, \mathfrak{n}: V}\left(\begin{array}{c}
\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\
\cdot \cdot \cdot \\
\mathrm{z}_{2}(1-x)^{h_{2}}(1+x)^{k_{2}}
\end{array}\right)=\sum_{l, p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty}
\end{aligned}
$$

$$
\sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{\left.M, \eta_{G, g}\right)}}{B_{G} g!} \frac{\prod_{i=1}^{M}\left(a_{i}\right)_{q}}{\prod_{i=1}^{N}\left(b_{i}\right)_{q} q!} t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} \frac{D_{p, n}^{(\alpha, \beta)}(c)}{N_{l, l}^{(\alpha, \beta)}}
$$

$$
2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right)} \sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} \aleph_{U_{21}: W}^{0, \mathfrak{n}+2: V}\left(\begin{array}{c}
2^{h_{1}+k_{1}} z_{1} \\
\cdots \\
\cdots \\
2^{h_{2}+k_{2}} z_{r}
\end{array}\right)
$$

$$
\left.\begin{array}{c}
\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, h_{2}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, k_{2}\right), A: C \\
\left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\sum_{i=1}^{s} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, h_{2}+k_{2}\right), B: D
\end{array}\right)
$$

$$
\begin{equation*}
\phi_{l}^{(\alpha, \beta)}(c, x) \tag{7.1}
\end{equation*}
$$

with the same notations and validity conditions of (4.1) and $r=2$

## 8. I-function of two variables

If $\tau_{i}, \tau_{i}^{\prime}, \tau_{i}^{\prime \prime} \rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [6] and we obtain :

$$
\begin{aligned}
& (1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta} \aleph_{P_{i}, Q_{i}, c_{i} ; r^{\prime}}^{M, N}\left(t(1-x)^{e}(1+x)^{f}\right){ }_{M} F_{N}\left(\left(a_{M}\right) ;\left(b_{N}\right) ; y(1-x)^{e^{\prime}}(1+x)^{f^{\prime}}\right) \\
& S_{N_{1}, \cdots, N_{s}}^{M_{1}, \cdots, M_{s}}\left(\begin{array}{c}
\mathrm{t}_{1}(1-x)^{g_{1}}(1+x)^{w_{1}} \\
\cdot \cdot \\
\mathrm{t}_{s}(1-x)^{g_{s}}(1+x)^{w_{s}}
\end{array}\right) \underset{U: W}{0, \mathfrak{n}: V}\left(\begin{array}{c}
\mathrm{z}_{1}(1-x)^{h_{1}}(1+x)^{k_{1}} \\
\cdot \cdot \cdot \\
\mathrm{z}_{2}(1-x)^{h_{2}}(1+x)^{k_{2}}
\end{array}\right)=\sum_{l, p, q=0}^{\infty} \sum_{G=1}^{M} \sum_{g=0}^{\infty}
\end{aligned}
$$

$$
\left.\begin{array}{l}
\sum_{K_{1}=0}^{\left[N_{1} / M_{1}\right]} \cdots \sum_{K_{s}=0}^{\left[N_{s} / M_{s}\right]} a_{1} \frac{(-)^{g} \Omega_{P_{i}, Q_{i}, c_{i}, r^{\prime}}^{M,}\left(\eta_{G, g}\right)}{B_{G} g!} \frac{\prod_{i=1}^{M}\left(a_{i}\right)_{q}}{\prod_{i=1}^{N}\left(b_{i}\right)_{q} q!} t^{\eta_{G, g}} x_{1}^{K_{1}} \cdots t_{s}^{K_{s}} y^{q} \frac{D_{p, n}^{(\alpha, \beta)}(c)}{N_{l, l}^{(\alpha, \beta)}} \\
2^{\left(\eta_{G, g}(e+f)+q\left(e^{\prime}+f^{\prime}\right)+\sum_{i=0}^{s} K_{i}\left(g_{i}+w_{i}\right)\right)} \sum_{m=0}^{n+p} \frac{(-n-p)_{m}(\alpha+\beta+n+p+1)_{m}}{m!(\alpha+1)_{m}} I_{U_{21}: W}^{0, n+2: V}\left(\begin{array}{c}
2^{h_{1}+k_{1}} z_{1} \\
\cdots \\
\cdots \\
2^{h_{2}+k_{2}} z_{r}
\end{array}\right) \\
\left(-\mathrm{m}-\rho-e \eta_{G, g}-e^{\prime} q-\sum_{i=1}^{s} K_{i} g_{i} ; h_{1}, h_{2}\right),\left(-\sigma-f \eta_{G, g}-f^{\prime} q-\sum_{i=1}^{s} K_{i} w_{i} ; k_{1}, k_{2}\right), A: C \\
\cdots  \tag{8.1}\\
\left(-1-\mathrm{m}-\rho-\sigma-(e+f) \eta_{G, g}-\left(e^{\prime}+f^{\prime}\right) q-\sum_{i=1}^{s} K\left(g_{i}+w_{i}\right) ; h_{1}+k_{1}, h_{2}+k_{2}\right), B: D
\end{array}\right)
$$

with the same notations and validity conditions of (4.1) and $r=2$ and $\tau_{i}, \tau_{i}^{\prime}, \tau_{i}^{\prime \prime} \rightarrow 1$

## 9. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore, on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable Ifunction ,multivariable Fox's H-function, Fox's H-function, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function, binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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