Integral involving the multivariable Aleph-function, a general class of polynomial, Aleph-function and integral function of two complex variables

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ABSTRACT
The object of this paper is to evaluate an integral involving the Laguerre polynomial and an integral function of two complex variables of order ρ, based on the properties of the Aleph-function, general class of polynomials and multivariable Aleph-function. Further we establish some special cases.

KEYWORDS : Aleph-function of several variables, general class of polynomials, Aleph-function of one variable, Aleph-function of two variables, I-function of several variables, I-function of two variables, integral function of two complex variables of order ρ,.

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1.Introduction and preliminaries.

The Aleph- function , introduced by Südland [10] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z\ \middle|\ \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r'}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r'}\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\ \Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)z^{-s}\mathrm{d}s \quad (1.1)$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r'}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \quad (1.2)$$

With $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0, i=1,\cdots,r'$

For convergence conditions and other details of Aleph-function , see Südland et al [10]. The serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_Gg!}z^{-s} \quad (1.3)$$

With $s=\eta_{G,g}=\frac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [9], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s]=\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \quad (1.5)$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1=\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.6}$$

The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [6] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{l}[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]\\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.7}$$

with $\omega=\sqrt{-1}$

For more details, see Ayant [1]. The reals numbers τ_i are positives for $i=1,\cdots,R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)}=1,\cdots,R^{(k)}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$A_i^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with }k=1\cdots,r,i=1,\cdots,R\ ,i^{(k)}=1,\cdots,R^{(k)} \tag{1.8}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}\,),max(\,|z_1|\ldots|z_r|\,)\rightarrow 0$

$\aleph(z_1,\cdots,z_r)=0(\,|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}\,),min(\,|z_1|\ldots|z_r|\,)\rightarrow\infty$

where, with $k=1,\cdots,r:\alpha_k=min[Re(d_j^{(k)}/\delta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k=max[Re((c_j^{(k)}-1)/\gamma_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \; ; \; V = m_1, n_1; \cdots ; m_r, n_r \tag{1.9}$$

$$\text{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.10}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.11}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.12}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots , \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.13}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.14}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{U:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \begin{array}{c} \text{A : C} \\ \cdot\;\cdot\;\cdot \\ \text{B : D} \end{array} \right) \tag{1.15}$$

$$\text{Let } F(Z_1, Z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{A_{n_1,n_2}}{n_1! n_2!} Z_1^{n_1} Z_2^{n_2} \tag{1.16}$$

be an integral function of two complex variables Z_1 and Z_2. Denote by

$M_F(r) = max\big(|Z_1| + |Z_2|\big) = r|F(Z_1, Z_2)|$, the maximum modulus of $F(Z_1, Z_2)$.

Following Dzrbasjan [4], the integral function $F(Z_1, Z_2)$ is said to be of finite order ρ if

$$\lim_{n\rightarrow\infty} sup \left\{ \frac{loglogM_F(r)}{logr} \right\} = \rho \quad (0 \leqslant \rho < \infty) \tag{1.17}$$

2. Lemme

We have the following integral

$$\int_0^{\infty} x^{\gamma} e^{-x} L_k^{(\sigma)}(x) \, \aleph_{P_i,Q_i,c_i;r'}^{M,N}(tx^a) \, S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s} \left(\begin{array}{c} \text{t}_1 x^{b_1} \\ \cdot\;\cdot\;\cdot \\ \text{t}_s x^{b_s} \end{array} \right) \aleph_{U:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c} \text{z}_1 x^{\eta_1} \\ \cdot\;\cdot\;\cdot \\ \text{z}_r x^{\eta_r} \end{array} \right) \text{d}x$$

$$= \frac{(-)^k}{k!} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} t^{\eta_{G,g}} t_1^{K_1} \cdots t_s^{K_s} \aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V} \left(\begin{array}{c} \text{z}_1 \\ \cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot \\ \text{z}_r \end{array} \right|$$

$$\left. \begin{array}{c} (\text{-}\gamma - a\eta_{G,g} - \sum_{i=1}^s K_i b_i; \eta_1, \cdots , \eta_r), (\sigma - \gamma - a\eta_{G,g} - \sum_{i=1}^s K_i b_i; \eta_1, \cdots , \eta_r), A : C \\ \cdot\;\cdot\;\cdot \\ (\sigma - \gamma + k - a\eta_{G,g} - \sum_{i=1}^s K_i b_i; \eta_1, \cdots , \eta_r), B : D \end{array} \right) \tag{2.1}$$

with $U_{21} = p_i + 2; q_i + 1; \tau_i; R$

Provided

a) $min\{a,b_i,\eta_j\}>0, i=1,\cdots,s; j=1,\cdots,r$

b) $Re[\gamma+1+a\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}\eta_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

d) $|argt|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

Proof

Expressing the Aleph-function of one variable in series with the help of equation (1.3), the general class of polynomials of several variables in series with the help of equation (1.5), and the Aleph-function of r variables in Mellin-Barnes contour integral with the help of equation (1.7), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the inner integral by using a result given by Gradshteyn L.S and Ryzhik I ([5], eq.11,page 1244). Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

3. Main integral

Let $|\zeta_l|\neq 0, |arg\zeta_l|<\frac{\pi}{2\rho},(l=1,2)$; $F(Z_1,Z_2)=\sum_{n_1,n_2=0}^{\infty}\frac{A_{n_1,n_2}}{n_1!n_2!}Z_1^{n_1}Z_2^{n_2}$ be an integral function of two complex variables Z_1 and Z_2 of order ρ $(0\leqslant\rho<\infty)$. Then for $arg(\zeta_1)=arg(\zeta_2)$,

$$A_{n_1,n_2}(\zeta_1,\zeta_2)=\int_0^\infty\int_0^\infty(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^\rho\}L_k^\sigma(\{(t_1\zeta_1+t_2\zeta_2)^\rho)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1}\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_r(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_r}\end{array}\right)$$

$$F(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$=\frac{(-)^k}{k!}\sum_{n_1,n_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}\frac{A_{n_1,n_2}}{(n_1+n_2+1)!\zeta_1^{n_1+1}\zeta_2^{n_2+1}}x^{\eta_{G,g}}$$

$$x_1^{K_1}\cdots x_s^{K_s}\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{z}_r\end{array}\middle|\begin{array}{c}(\text{-}\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\cdots,\eta_r),\\ \cdot\ \cdot\ \cdot\\ (\sigma-\gamma+k-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\cdots,\eta_r),\end{array}\right)$$

$$(\sigma-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\textstyle\sum_{i=1}^{s}K_ib_i;\eta_1,\cdots,\eta_r),A:C \atop \cdots \atop \mathrm{B}:\mathrm{D}\Bigg) \tag{3.1}$$

with $U_{21}=p_i+2;q_i+1;\tau_i;R$

Provided

a) $min\{a,b_i,\eta_j\}>0, i=1,\cdots,s;j=1,\cdots,r$

b) $Re[\gamma+\dfrac{n_1+n_2+1}{\rho}+a\min\limits_{1\leqslant j\leqslant M}\dfrac{b_j}{B_j}+\sum\limits_{i=1}^{r}\eta_i\min\limits_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c) $|argz_k|<\dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

d) $|argx|<\dfrac{1}{2}\pi\Omega$ where $\Omega=\sum\limits_{j=1}^{M}\beta_j+\sum\limits_{j=1}^{N}\alpha_j-c_i(\sum\limits_{j=M+1}^{Q_i}\beta_{ji}+\sum\limits_{j=N+1}^{P_i}\alpha_{ji})>0$

Proof of (3.1)

Consider the following integral

$$I_\rho=\int_0^\infty\int_0^\infty(x_1+x_2)^{\gamma\rho-1}exp\{-(x_1+x_2)^\rho\}L_k^\sigma(\{(x_1+x_2)^\rho)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(x_1+x_2)^{a\rho})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{y}_1(x_1+x_2)^{\rho b_1}\\ \cdots\\ \mathrm{y}_s(x_1+x_2)^{\rho b_s}\end{pmatrix}\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1(x_1+x_2)^{\rho\eta_1}\\ \cdots\\ \mathrm{z}_r(x_1+x_2)^{\rho\eta_r}\end{pmatrix}\mathrm{d}x_1\mathrm{d}x_2 \tag{3.2}$$

where η_i are positive numbers and $0\leqslant n_i, i=1,2$ are integers. Changing the variables

$x_1=t(1-u),x_2=tu,0\leqslant u\leqslant 1;0\leqslant t<+\infty$, we have

$$I_\rho=\int_0^\infty\int_0^\infty t^{n_1+n_2+\gamma\rho}e^{-t^\rho}L_k^{(\sigma)}(t^\rho)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{y}_1(t^\rho)^{b_1}\\ \cdots\\ \mathrm{y}_s(t^\rho)^{b_s}\end{pmatrix}\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1(t^\rho)^{\eta_1}\\ \cdots\\ \mathrm{z}_r(t^\rho)^{\eta_r}\end{pmatrix}$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(ta^\rho)\,(1-u)^{n_1}u^{n_2}\frac{\partial(x_1,x_2)}{\partial(t,u)}\mathrm{d}t\mathrm{d}u \tag{3.3}$$

$$=\int_0^\infty\int_0^1 t^{n_1+n_2+\gamma\rho}e^{-t^\rho}L_k^{(\sigma)}(t^\rho)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{y}_1(t^\rho)^{b_1}\\ \cdots\\ \mathrm{y}_s(t^\rho)^{b_s}\end{pmatrix}\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix}\mathrm{z}_1(t^\rho)^{\eta_1}\\ \cdots\\ \mathrm{z}_r(t^\rho)^{\eta_r}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}(xt^{a\rho})$$

$$(1-u)^{n_1}u^{n_2}\,\mathrm{d}t\mathrm{d}u \tag{3.4}$$

Evaluating u-integral with the help of the Eulerian integral of the first kind ([3], Copson, 1961 page 212), we obtain

$$I_\rho = \frac{n_1!n_2!}{(n_1+n_2+1)!}\int_0^\infty t^{n_1+n_2+\gamma\rho}e^{-t^\rho}L_k^{(\sigma)}(t^\rho)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(t^\rho)^{b_1} \\ \cdots \\ \mathrm{y}_s(t^\rho)^{b_s} \end{array}\right)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(xt^{a\rho})$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1(t^\rho)^{\eta_1} \\ \cdots \\ \mathrm{z}_r(t^\rho)^{\eta_r} \end{array}\right)\mathrm{d}t = \frac{n_1!n_2!}{(n_1+n_2+1)!}\int_0^\infty x^{\gamma+\frac{n_1+n_2+1}{\rho}-1}\, e^{-x}L_k^{(\sigma)}(x)\,\aleph_{P_i,Q_i,c_i;r'}^{M,N}(tx^a)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1x^{b_1} \\ \cdots \\ \mathrm{y}_sx^{b_s} \end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1x^{\eta_1} \\ \cdots \\ \mathrm{z}_rx^{\eta_r} \end{array}\right)\mathrm{d}x \tag{3.5}$$

Now evaluating the x-integral with the help of Lemme (2.1), we have

$$I_\rho = \frac{(-)^k}{k!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}\frac{n_1!n_2!}{(n_1+n_2+1)!}x^{\eta G,g}$$

$$y_1^{K_1}\cdots y_s^{K_s}\ \aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdots \\ \cdots \\ \mathrm{z}_r \end{array}\left|\begin{array}{c} (\text{-}\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_ib_i;\eta_1,\cdots,\eta_r), \\ \cdots \\ (\sigma-\gamma+k-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_ib_i;\eta_1,\cdots,\eta_r), \end{array}\right.\right.$$

$$\left.\begin{array}{c} (\sigma-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_ib_i;\eta_1,\cdots,\eta_r),A:C \\ \cdots \\ \mathrm{B}:\mathrm{D} \end{array}\right) \tag{3.6}$$

with $U_{21}=p_i+2;q_i+1;\tau_i;R$

Provided

a) $min\{a,b_i,\eta_j\}>0, i=1,\cdots,s; j=1,\cdots,r$

b) $Re[\gamma+\frac{n_1+n_2+1}{\rho}+a\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^r\eta_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

c)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

d) $|argx|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^M\beta_j+\sum_{j=1}^N\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

We have $A_{n_1,n_2}(\zeta_1,\zeta_2)=\int_0^\infty\int_0^\infty(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^\rho\}L_k^\sigma(\{(t_1\zeta_1+t_2\zeta_2)^\rho)$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1} \\ \cdots \\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s} \end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} \mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1} \\ \cdots \\ \mathrm{z}_r(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_r} \end{array}\right)$$

$$F(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 \tag{3.7}$$

Now changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) , we have

$$A_{n_1,n_2}(\zeta_1,\zeta_2) = \sum_{n_1,n_2=0}^{\infty} \frac{A_{n_1,n_2}}{n_1!n_2!} \int_0^{\infty}\int_0^{\infty} t_1^{n_1}t_2^{n_2}(t_1\zeta_1+t_2\zeta_2)^{\gamma\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^{\rho}\}$$

$$L_k^{\sigma}(\{(t_1\zeta_1+t_2\zeta_2)^{\rho})\ \aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1}\\ \cdots\\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s}\end{pmatrix}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1}\\ \cdots\\ \mathrm{z}_r(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_r}\end{pmatrix}\mathrm{d}t_1\mathrm{d}t_2 = \sum_{n_1,n_2=0}^{\infty}\frac{A_{n_1,n_2}}{\zeta_1^{n_1}\zeta_2^{n_2}n_1!n_2!}\int_0^{\infty}\int_0^{\infty}\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(x_1+x_2)^{a\rho})$$

$$(x_1+x_2)^{\gamma\rho-1}exp\{-(x_1+x_2)^{\rho}\}L_k^{\sigma}(\{(x_1+x_2)^{\rho})\ S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(x_1+x_2)^{\rho b_1}\\ \cdots\\ \mathrm{y}_s(x_1+x_2)^{\rho b_s}\end{pmatrix}$$

$$\aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1(x_1+x_2)^{\rho\eta_1}\\ \cdots\\ \mathrm{z}_r(x_1+x_2)^{\rho\eta_r}\end{pmatrix}\mathrm{d}x_1\mathrm{d}x_2 = \sum_{n_1,n_2=0}^{\infty}\frac{A_{n_1,n_2}}{\zeta_1^{n_1}\zeta_2^{n_2}n_1!n_2!}I_{\rho} \tag{3.8}$$

Now we use the equation (3.6) , we obtain the desired result.

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. The formula have been derived in this section for multivariable I-functions defined by Sharma et al [6].

We have the following result

corollary 1

Let $|\zeta_l|\neq 0, |arg\zeta_l| < \frac{\pi}{2\rho}, (l=1,2)$; $F(Z_1,Z_2) = \sum_{n_1,n_2=0}^{\infty}\frac{A_{n_1,n_2}}{n_1!n_2!}Z_1^{n_1}Z_2^{n_2}$ be an integral function of two complex variables Z_1 and Z_2 of order ρ $(0\leqslant\rho<\infty)$. Then for $arg(\zeta_1)=arg(\zeta_2)$,

$$A_{n_1,n_2}(\zeta_1,\zeta_2) = \int_0^{\infty}\int_0^{\infty}(t_1\zeta_1+t_2\zeta_2)^{\gamma_\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^{\rho}\}L_k^{\sigma}(\{(t_1\zeta_1+t_2\zeta_2)^{\rho})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1}\\ \cdots\\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s}\end{pmatrix}I_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1}\\ \cdots\\ \mathrm{z}_r(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_r}\end{pmatrix}$$

$$F(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$= \frac{(-)^k}{k!}\sum_{n_1,n_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\frac{A_{n_1,n_2}}{(n_1+n_2+1)!\zeta_1^{n_1+1}\zeta_2^{n_2+1}} x^{\eta G,g}$$

$$y_1^{K_1}\cdots y_s^{K_s}\ I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r \end{array}\middle|\begin{array}{c} (\text{-}\gamma-(n_1+n_2+1)/\rho - a\eta_{G,g}-\sum_{i=1}^s K_i b_i;\eta_1,\cdots,\eta_r), \\ \cdot\cdot\cdot \\ (\sigma-\gamma+k-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_i b_i;\eta_1,\cdots,\eta_r), \end{array}\right.$$

$$\left.\begin{array}{c} (\sigma-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_i b_i;\eta_1,\cdots,\eta_r), A : C \\ \cdot\cdot\cdot \\ \mathrm{B}:\mathrm{D} \end{array}\right) \tag{4.1}$$

with the same notation and conditions that (3.1)

5. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [8], and we have the two following result

corollary 2

Let $|\zeta_l| \neq 0, |arg\zeta_l| < \frac{\pi}{2\rho}, (l=1,2)$; $F(Z_1,Z_2)=\sum_{n_1,n_2=0}^{\infty}\frac{A_{n_1,n_2}}{n_1!n_2!}Z_1^{n_1}Z_2^{n_2}$ be an integral function of two complex variables Z_1 and Z_2 of order ρ $(0 \leqslant \rho < \infty)$. Then for $arg(\zeta_1)=arg(\zeta_2)$,

$$A_{n_1,n_2}(\zeta_1,\zeta_2)=\int_0^\infty\int_0^\infty (t_1\zeta_1+t_2\zeta_2)^{\gamma_\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^\rho\}L_k^\sigma(\{(t_1\zeta_1+t_2\zeta_2)^\rho)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1}\\ \cdot\cdot\cdot \\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1}\\ \cdot\cdot\cdot \\ \mathrm{z}_2(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_2}\end{array}\right)$$

$$F(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$= \frac{(-)^k}{k!}\sum_{n_1,n_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\frac{A_{n_1,n_2}}{(n_1+n_2+1)!\zeta_1^{n_1+1}\zeta_2^{n_2+1}} x^{\eta G,g}$$

$$y_1^{K_1}\cdots y_s^{K_s}\ \aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \mathrm{z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_2 \end{array}\middle|\begin{array}{c} (\text{-}\gamma-(n_1+n_2+1)/\rho - a\eta_{G,g}-\sum_{i=1}^s K_i b_i;\eta_1,\eta_2), \\ \cdot\cdot\cdot \\ (\sigma-\gamma+k-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^s K_i b_i;\eta_1,\eta_2), \end{array}\right.$$

$$\left.\begin{matrix}(\sigma-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\eta_2),A:C\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right) \tag{5.1}$$

with the same notation and conditions that (3.1) and $r=2$

6. I-function of two variables

If $\tau_i,\tau_i',\tau_i''\rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [7] and we obtain the same formulas with the I-function of two variables.

corollary 2

Let $|\zeta_l|\neq 0,|arg\zeta_l|<\dfrac{\pi}{2\rho},(l=1,2)$; $F(Z_1,Z_2)=\sum_{n_1,n_2=0}^{\infty}\dfrac{A_{n_1,n_2}}{n_1!n_2!}Z_1^{n_1}Z_2^{n_2}$ be an integral function of two complex variables Z_1 and Z_2 of order ρ $(0\leqslant\rho<\infty)$. Then for $arg(\zeta_1)=arg(\zeta_2)$,

$$A_{n_1,n_2}(\zeta_1,\zeta_2)=\int_0^{\infty}\int_0^{\infty}(t_1\zeta_1+t_2\zeta_2)^{\gamma_\rho-1}exp\{-(t_1\zeta_1+t_2\zeta_2)^{\rho}\}L_k^{\sigma}(\{(t_1\zeta_1+t_2\zeta_2)^{\rho})$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(x(t_1\zeta_1+t_2\zeta_2)^{a\rho})S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{matrix}\mathrm{y}_1(t_1\zeta_1+t_2\zeta_2)^{\rho b_1}\\ \cdots\\ \mathrm{y}_s(t_1\zeta_1+t_2\zeta_2)^{\rho b_s}\end{matrix}\right)I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_1}\\ \cdots\\ \mathrm{z}_2(t_1\zeta_1+t_2\zeta_2)^{\rho\eta_2}\end{matrix}\right)$$

$$F(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2$$

$$=\frac{(-)^k}{k!}\sum_{n_1,n_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}\frac{A_{n_1,n_2}}{(n_1+n_2+1)!\zeta_1^{n_1+1}\zeta_2^{n_2+1}}x^{\eta G,g}$$

$$y_1^{K_1}\cdots y_s^{K_s}\ I_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{matrix}\mathrm{z}_1\\ \cdots\\ \cdots\\ \mathrm{z}_2\end{matrix}\ \middle|\ \begin{matrix}(-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\eta_2),\\ \cdots\\ (\sigma-\gamma+k-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\eta_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma-\gamma-(n_1+n_2+1)/\rho-a\eta_{G,g}-\sum_{i=1}^{s}K_ib_i;\eta_1,\eta_2),A:C\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right) \tag{6.1}$$

with the same notation and conditions that (3.1), $r=2$

7. Conclusion

In this paper we have evaluated a simple integral involving the multivariable Aleph-function, a class of polynomials of several variables, a Aleph-function of one variable and a integral function of two complex variables. The integral established in this paper is of very general nature as it contains Multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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