On certain expansions

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Abstract
The aim of the present paper is to evaluate firstly the six finite integrals involving Jacobi polynomials, the Aleph-function of one variable, a class of polynomials of several variables and the generalized multivariable Aleph-function. Then we derive six expansions formulae for Jacobi polynomials, the Aleph-function of one variable, a class of polynomials of several variables and the generalized multivariable Aleph-function.

Keywords : Generalized Aleph-function of several variable, aleph-function of one variable, Jacobi polynomials, Class of polynomials of several variables, expansion formulae.

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1.Introduction and preliminaries.

In this document, we will evaluate firstly the six finite integrals involving Jacobi polynomials, the Aleph-function of one variable, a class of polynomials of several variables and the generalized multivariable Aleph-function. Then we will derive six expansions formulae for Jacobi polynomials, the Aleph-function of one variable, a class of polynomials of several variables and the generalized multivariable Aleph-function. The generalized Aleph-function of several variables generalize the multivariable I-function defined by H.M. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.The generalized Aleph-function of several variables is defined as following.

We have : $\aleph(z_1,\cdots,z_r)=\aleph^{m,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right|$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]\ ,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$[(\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})_{1,m}]\ ,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$[(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$\left.[(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]\right)$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\sum_{k=1}^{r}\beta_j^{(k)}s_j)\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

$$\text{and}\ \theta_k(s_k)=\frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi$,

where $A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \sum_{j=1}^{m} \beta_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=m+1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1\cdots,r, i=1,\cdots,R \ , i^{(k)} = 1,\cdots,R^{(k)} \tag{1.4}$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.5}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.6}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.7}$$

$$B = \{(b_j; \beta_j, \cdots, \beta_j)_{1,m}\} ; \ \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.8}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.9}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.10}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{m,\mathfrak{n}:V}\left(\begin{array}{c|c} z_1 & \text{A : C} \\ \cdot & \\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \\ z_r & \text{B : D} \end{array}\right) \tag{1.11}$$

The Aleph- function , introduced by Südland [7] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z \ \middle|\ \begin{array}{c} (\text{ a'}_j, A'_j)_{1,N}, [c'_i(a'_{ji}, A'_{ji})]_{N+1,P_i;r'} \\ (\text{ b'}_j, B'_j)_{1,M}, [c_i(b'_{ji}, B'_{ji})]_{M+1,Q_i;r'} \end{array}\right)$$

$$= \frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r'}^{M,N}(s) z^{-s} \mathrm{d}s \tag{1.12}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s) = \frac{\prod_{j=1}^{M}\Gamma(b'_j + B'_j s)\prod_{j=1}^{N}\Gamma(1 - a'_j - A'_j s)}{\sum_{i=1}^{r'} c_i \prod_{j=N+1}^{P_i}\Gamma(a'_{ji} + A'_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b'_{ji} - B'_{ji}s)} \tag{1.13}$$

With :

$|argz| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M} B'_j + \sum_{j=1}^{N} A'_j - c_i(\sum_{j=M+1}^{Q_i} B'_{ji} + \sum_{j=N+1}^{P_i} A'_{ji}) > 0$ with $i = 1, \cdots, r'$

For convergence conditions and other details of Aleph-function , see Südland et al [7]. The serie representation of Aleph-function is given by Chaurasia et al [1].

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^{-s} \tag{1.14}$$

With $s=\eta_{G,g}=\dfrac{b'_G+g}{B'_G}, P_i<Q_i, |z|<1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r'}(s)$ is given in (1.2) (1.15)

The generalized polynomials defined by Srivastava [8], is given in the following manner :

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.16}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notations.

$A'=\dfrac{(-N_1)_{M_1K_1}}{K_1!}\cdots\dfrac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s]$ and $U_{hk}=p_i+h,q_i+k,\tau_i;R_i$ (1.17)

h,k are integers

2.Required formulas

a) $$\int_{-1}^{1}(1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(\sigma+n+1)\Gamma(-n-\sigma)}{\Gamma(\rho+\sigma+n+2)\Gamma(\rho+n+1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+n+1)\Gamma(\beta+k+n+1)\Gamma(\alpha+\beta+k+n-\rho-\sigma)}{\Gamma(\alpha+\beta+k-\sigma+n+1)\Gamma(\alpha+k-\sigma-\rho)n!} \tag{2.1}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(-n-\sigma)>0, Re(1+\alpha)>0$ and

$Re(\alpha+\beta+n+k-\rho-\sigma)>0$

b) $$\int_{-1}^{1}(1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{(-)^n2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\rho+n+1)\Gamma(-n-\rho)}{\Gamma(\rho+\sigma+n+2)\Gamma(\sigma+n+1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k+n+1)\Gamma(\sigma+k+n+1)\Gamma(\alpha+\beta+k+n-\rho-\sigma)}{\Gamma(\alpha+\beta+n+k-\rho)\Gamma(\beta+k-\sigma-\rho)n!} \tag{2.2}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(-n-\sigma)>0, Re(-\rho)>0, Re(1+\beta)>0$ and

$Re(\alpha+\beta+n+k-\rho-\sigma)>0$

c) $$\int_{-1}^{1}(1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\rho+1)}{\Gamma(\rho+n+1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+n+1)\Gamma(\rho+k+n+1)\Gamma(1-\beta+k+\sigma)}{\Gamma(\sigma+2+k+\rho)\Gamma(\alpha+k+n+2+\sigma)n!} \tag{2.3}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\alpha)>0, Re(1-\beta+k+\sigma)>0$

d) $$\int_{-1}^{1}(1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(\rho+1)}{\Gamma(\sigma+n+1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k+n+1)\Gamma(\sigma+k+n+1)\Gamma(1-\alpha+k+\rho)}{\Gamma(\sigma+2+k+\rho)\Gamma(\beta+k+n+2+\rho)n!} \tag{2.4}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\beta)>0, Re(1-\alpha+k+\sigma)>0$

e) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{2^{\rho+\sigma+1}\Gamma(\rho+n+1)\Gamma(\sigma+1)\Gamma(-n-\rho)}{\Gamma(\rho+\sigma+n+2)\Gamma(-\rho-\sigma-1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-\rho-\sigma-1-k)\Gamma(1-\beta+k+\sigma)}{\Gamma(-\alpha-\beta-n+k+\sigma)\Gamma(-\beta+k-n-\rho)n!} \tag{2.5}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\alpha+\beta+2n)>0, Re(-\alpha-\beta-2n)>0$ and

$Re(-\alpha-\beta-n+k)>0, Re(-\rho-n)>0, Re(-1-\rho-\sigma+k)>0, Re(1-\beta+\sigma+k)>0$

f) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)\mathrm{d}x=\frac{(-)^n2^{\rho+\sigma+1}\Gamma(\sigma+n+1)\Gamma(\rho+1)\Gamma(-n-\sigma)}{\Gamma(\rho+\sigma+n+2)\Gamma(-\rho-\sigma-1)n!}\times$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha-\beta-n+k)\Gamma(-\rho-\sigma-1-k)\Gamma(1-\beta+k+\rho)}{\Gamma(-\beta-n+k-\sigma)\Gamma(-\alpha-\beta+k-n+\rho)n!} \tag{2.6}$$

with $Re(\rho+1)>0, Re(\sigma+1)>0, Re(1+\alpha+\beta+2n)>0, Re(-\alpha-\beta-2n)>0$ and

$Re(-\alpha-\beta-n+k)>0, Re(-\sigma-n)>0, Re(-1-\rho-\sigma+k)>0, Re(1-\beta+\rho+k)>0$

See respectively ([2] ,p.254(eq.2),p.255(eq.8), p.254(eq.1),p.254(eq.3), p.255(eq.7), p.255(eq.9)).

Throughout this paper, we will use the notations.

$$P_1=\min_{1\leqslant j\leqslant m,1\leqslant \mathrm{j}\leqslant m_1}\left(\sum_{j=1}^{m}\frac{b_j}{\beta_j^{(1)}}+\frac{d_{\mathrm{j}}^{(1)}}{\delta_{\mathrm{j}}^{(1)}}\right),\cdots,P_r=\min_{1\leqslant j\leqslant m,1\leqslant \mathrm{j}\leqslant m_r}\left(\sum_{j=1}^{m}\frac{b_j}{\beta_j^{(r)}}+\frac{d_{\mathrm{j}}^{(r)}}{\delta_{\mathrm{j}}^{(r)}}\right) \text{ and } p=\min_{1\leqslant j\leqslant M}\frac{b'_j}{B'_j}$$

3. Integral formulas

In this section, we will evaluate six finite integrals involving Jacobi polynomials, the Aleph-function of one variable, a class of polynomials of several variables and the generalized multivariable Aleph-function. See the end of this paper (*) concerning the additionnel validity conditions. For the validity conditions, $j=1,\cdots,r$

a) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^c(1+x)^d\right)$$

$$\aleph\begin{pmatrix}\text{(1-x)}^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ \text{(1-x)}^{h_r}(1+x)^{k_r}z_r\end{pmatrix}\mathrm{d}x\ =\frac{2^{\rho+\sigma+1}}{n!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A'$$

$$\frac{(-)^g \Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)\Gamma(1+\alpha+n+k)}{B_G g! \qquad k!} x_1^{K_1}\cdots x_s^{K_s} z^{\eta_{G,g}} \aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c} 2^{h_1+k_1} z_1 \\ \cdot \\ \cdot \\ \cdot \\ 2^{h_r+k_r} z_r \end{array}\right|$$

$$\begin{array}{c}(\text{-}\rho-\textstyle\sum_{i=1}^s c_iK_i - c\eta_{G,g};h_1,\cdots,h_r), \quad (\text{-}\sigma-\sum_{i=1}^s d_iK_i - d\eta_{G,g};k_1,\cdots,k_r), \\ \cdots \qquad\qquad \cdots \\ (\text{-}\rho-n-\sum_{i=1}^s c_iK_i - c\eta_{G,g};h_1,\cdots,h_r),(\text{-}\sigma-n-\sum_{i=1}^s d_iK_i - d\eta_{G,g};k_1,\cdots,k_r),\end{array}$$

$$\begin{array}{c}(\text{-}\rho-k-n-\textstyle\sum_{i=1}^s c_iK_i - c\eta_{G,g};h_1,\cdots,h_r),(1+\alpha+\beta+n+k-\sigma-\sum_{i=1}^s d_iK_i - d\eta_{G,g};k_1,\cdots,k_r) \\ \cdots \\ (\text{-}1\text{-}\rho-\sigma-n-\sum_{i=1}^s (c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),\end{array}$$

$$\left.\begin{array}{c}(\alpha+k-\rho-\sigma-\textstyle\sum_{i=1}^s (c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C \\ \cdots \\ (\alpha+\beta+k+n-\rho-\sigma-\sum_{i=1}^s (c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right) \tag{3.1}$$

Provided that :

$$Re(1+\rho+\sum_{i=1}^r h_iP_j+cp+1)>0, Re(\sigma+1+n+\sum_{i=1}^r k_iP_j+dp)>0, Re(1+\alpha)>0$$

$$Re(\alpha+\beta+n+k-\rho-\sigma-\sum_{i=1}^r(h_i+k_i)P_j)-(c+d)p>0, Re(1+n+k+\rho+\sum_{i=1}^r h_iP_j+cp)>0$$

$Re(-n-\sigma-\sum_{i=1}^r k_iP_j-dp)>0, |argz_k|< \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and

$|argz|<\frac{1}{2}\pi\Omega$

b) $$\int_{-1}^1 (1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x) S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\begin{pmatrix} \mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1} \\ \cdots \\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix} \aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(z(1-x)^c(1+x)^d\right)$$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1 \\ \cdots \\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}\mathrm{d}x = \frac{2^{\rho+\sigma+1}}{n!}\sum_{G=1}^M\sum_{g=0}^\infty\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^\infty 2^{\sum_{i=1}^s K_i(c_i+d_i)+(c+d)\eta_{G,g}} A'$$

$$\frac{(-)^g \Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)\Gamma(1+\beta+n+k)}{B_G g! \qquad k!} x_1^{K_1}\cdots x_s^{K_s} z^{\eta_{G,g}} \aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c} 2^{h_1+k_1} z_1 \\ \cdot \\ \cdot \\ \cdot \\ 2^{h_r+k_r} z_r \end{array}\right|$$

$(\text{-}\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \quad (\text{-n-k-}\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$

$(\text{-}\rho-n-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\text{-}\sigma-n-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$

$(\text{-}\rho-k-n-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(1+\alpha+\beta+n+k-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)$

$(\text{-1-}\rho-\sigma-n-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),$

$$\left.\begin{matrix}(\beta+k-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\alpha+\beta+k+n-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \quad (3.2)$$

Provided that :

$Re(1+\sigma+\sum_{i=1}^{r}k_iP_j+dp+1)>0, Re(\rho+n+1+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(1+\beta)>0$

$Re(\alpha+\beta+n+k-\rho-\sigma-\sum_{i=1}^{r}(h_i+k_i)P_j-(c+d)p)>0, Re(1+n+k+\sigma+\sum_{i=1}^{r}k_iP_j+dp)>0$

$Re(-n-\rho-\sum_{i=1}^{r}h_iP_j-cp)>0$, $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz| < \frac{1}{2}\pi\Omega$

c) $\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^{c}(1+x)^{d}\right)$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}\mathrm{d}x = \frac{2^{\rho+\sigma+1}}{n!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)\Gamma(1+\beta+n+k)}{B_G g!\qquad k!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{43}:W}^{m,\mathfrak{n}+4:V}\left(\begin{matrix}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{matrix}\ \middle|\right.$$

$(\text{-}\rho-k-n-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\beta-k-\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r)$

$(\text{-1-k-}\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),$

$$\left.\begin{matrix}(\text{-}\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \quad (\text{-}\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),A:C\\ \cdots\\ (\text{n-}\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\text{-1-n-k-}\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),B:D\end{matrix}\right) \quad (3.3)$$

Provided that : $Re(1+\sigma+\sum_{i=1}^{r}k_iP_j+dp)>0$ and $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4)

$Re(1-\beta+k+\sigma+\sum_{i=1}^{r}k_iP_j+1+dp)>0, Re(\rho+1+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(1+\alpha)>0$ and $|argz|<\frac{1}{2}\pi\Omega$

d) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^c(1+x)^d\right)$$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}\mathrm{d}x=\frac{2^{\rho+\sigma+1}}{n!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)\Gamma(1+\beta+n+k)}{B_Gg!\qquad k!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{43}:W}^{m,\mathfrak{n}+4:V}\left(\begin{matrix}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{matrix}\ \middle|\right.$$

$$\begin{matrix}(\text{-}\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), & (\text{-}\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\\ \cdots & \cdots\\ (\text{-n-}\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), & (\text{-1-}\beta-n-k-\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\end{matrix}$$

$$\left.\begin{matrix}(\text{-}\sigma-k-n-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(\alpha-k-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),A:C\\ \cdots\\ (\text{-1-k-}\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \tag{3.4}$$

Provided that : $Re(1+\rho+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(1+\beta)>0, Re(1+\sigma+\sum_{i=1}^{r}k_iP_j+dp)>0$

$Re(1-\alpha+k+\rho+\sum_{i=1}^{r}c_iP_j+cp)>0$

$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

e) $$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^c(1+x)^d\right)$$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}\mathrm{d}x=\frac{2^{\rho+\sigma+1}}{n!}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A'$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)\Gamma(-\alpha-\beta-n+k)}{B_G g! \qquad\qquad k!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{54}:W}^{m+2,\mathfrak{n}+3:V}\left(\begin{array}{c}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$\begin{array}{c}(\text{-n-}\rho-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \qquad (\text{-}\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\\ \cdots \qquad\qquad \cdots\\ (\text{-}\rho-n-1-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),\text{-}\beta+n-k-\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\end{array}$$

$$\begin{array}{c}(\beta-k-\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\beta-\rho-n+k-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)\\ \cdots\\ (\text{-1-}\rho-\sigma+k-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r)\end{array}$$

$$\left.\begin{array}{c}(\text{-1+k-}\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\text{-1-n-}\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right) \tag{3.5}$$

Provided that : $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and $|argz| < \frac{1}{2}\pi\Omega$

$$Re(1+n+\rho+\sum_{i=1}^r h_iP_j+1+cp)>0, Re(\sigma+1+\sum_{i=1}^r k_iP_j+dp)>0, Re(1+\alpha+\beta)>0$$

$$Re(-1+k-\rho-\sigma-\sum_{i=1}^r(h_i+k_i)P_j-(c+d)p)>0, Re(-\alpha-\beta-2n)>0$$

$$Re(1-\beta+k+\sigma-\sum_{i=1}^r k_iP_j-dp)>0, Re(-\rho-n-\rho-\sum_{i=1}^r h_iP_j-cp)>0$$

f) $$\int_{-1}^1(1-x)^\rho(1+x)^\sigma P_n^{(\alpha,\beta)}(x)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\text{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \text{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{array}\right)\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^c(1+x)^d\right)$$

$$\aleph\left(\begin{array}{c}(\text{1-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (\text{1-x})^{h_r}(1+x)^{k_r}z_r\end{array}\right)\text{dx} = \frac{(-)^n2^{\rho+\sigma+1}}{n!}\sum_{G=1}^M\sum_{g=0}^\infty\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^\infty 2^{\sum_{i=1}^s K_i(c_i+d_i)+(c+d)\eta_{G,g}}$$

$$\frac{\Gamma(-\alpha-\beta-n+k)}{k!}A_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\ \aleph_{U_{54}:W}^{m+2,\mathfrak{n}+3:V}\left(\begin{array}{c}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$\begin{array}{c}(\text{-}\rho-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \qquad (\text{-}\sigma-n-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\\ \cdots \qquad\qquad \cdots\\ (\text{-}\rho-n-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\text{1+}\alpha+\beta+n-k-\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\end{array}$$

$$(\beta-k-\sigma-\sum_{i=1}^{s} d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\beta-\rho-n+k-\sum_{i=1}^{s} c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)$$
$$\cdots$$
$$(\text{-}1+\rho+\sigma-n+\sum_{i=1}^{s}(c_i+d_i)K_i+(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r)$$

$$\left.\begin{matrix}(\text{-}1\text{-}\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots \\ (\text{-}1\text{-n-}\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \tag{3.6}$$

Provided that $|argz_k|< \frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$:

$$Re(1+n+\sigma+\sum_{i=1}^{r}k_iP_j+1+dp)>0, Re(\rho+1+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(1+\alpha+\beta)>0|$$

$$Re(-1+k-\rho-\sigma-\sum_{i=1}^{r}(h_i+k_i)P_j-(c+d)p)>0, Re(-\alpha-\beta-2n)>0, Re(-\alpha-\beta-n+k)>0$$

$$Re(1-\beta+k+\rho-\sum_{i=1}^{r}h_iP_j-cp)>0, Re(-\sigma-n-\sum_{i=1}^{r}k_iP_j-dp)>0$$

Proof of (3.1)

To establish the finite integral (3.1), express the generalized class of polynomials occuring on the L.H.S in the series form given by (1.16), the Aleph-function of one variable in serie form given by (1.14) and the generalized multivariable Aleph-function involving there in terms of Mellin-Barnes contour integral by (1.1). We interchange the order of summation and integration (which is permissible under the conditions stated). Now evaluating the x-integral by using the formula (2.1), after simplifications and on reinterpreting the Mellin-Barnes contour integral, we get the desired result. To prove the integrals (3.2) to (3.6), we use the similar method with the help of results (2.2) to (2.6) respectively.

4. Expansion formula

Let $A_1=\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+k+n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)k!}A'$

$$A_2=\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(-\alpha-\beta+k-n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)k!}A' \text{ and}$$

$$A_3=\frac{2^{\rho+\sigma}(1+\alpha+\beta+2n)\Gamma(1+\alpha+\beta+n)\Gamma(1+\alpha+\beta+k-n)}{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)k!}A'$$

For the validity conditions, $j=1,\cdots,r$, we have the following expansions.

a) $(1-x)^{\rho}(1+x)^{\sigma}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\big(z(1-x)^c(1+x)^d\big)$

$$\aleph\begin{pmatrix}(\text{1-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (\text{1-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_1P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c} 2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$(\text{-}\alpha-\rho-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\text{-}\sigma-\beta-n-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$$
$$\cdots \qquad \cdots$$
$$(\text{-}\sigma-n-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(\text{-}\alpha-\rho-n-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),$$

$$(\text{-n-k-}\rho-\alpha-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(1+\alpha-\sigma+n+k-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r)$$
$$\cdots$$
$$(\text{-}\rho-\sigma-1-\alpha-\beta-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r)$$

$$\left.\begin{array}{l}(\text{k-}\beta-\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\text{k+n-}\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right) \qquad (4.1)$$

Provided that : $Re(1+\beta)>0$, $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz| < \frac{1}{2}\pi\Omega$

$$Re(1+\rho+\alpha+\sum_{i=1}^r h_iP_j+cp)>0, Re(\sigma+1+\beta+n+\sum_{i=1}^r k_iP_j+dp)>0, Re(1+\alpha)>0$$

$$Re(n+k-\rho-\sigma-\sum_{i=1}^r(h_i+k_i)P_j)-(c+d)p>0, Re(-n-\sigma-\sum_{i=1}^r k_iP_j-dp)>0, Re(\beta+1)>0$$

b) $(1-x)^\rho(1+x)^\sigma S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\text{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \text{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{array}\right)\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(z(1-x)^c(1+x)^d\right)$

$$\aleph\left(\begin{array}{c}\text{(1-x)}^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ \text{(1-x)}^{h_r}(1+x)^{k_r}z_r\end{array}\right)=\sum_{G=1}^M\sum_{g=0}^\infty\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^\infty 2^{\sum_{i=1}^s K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_1P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c} 2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$(\text{-}\beta-\rho-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \qquad (\text{-}\rho-\alpha-\sum_{i=1}^s c_iK_i-c\eta_{G,g}h_1,\cdots,h_r),$$
$$\cdots \qquad \cdots$$
$$(\text{-}\beta-n-\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(\text{-}\alpha-\rho-n-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),$$

$$(\text{-n-k-}\sigma-\alpha-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(1+\beta-\rho+n+k-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)$$
$$\cdots$$
$$(\text{-}\rho-\sigma-1-\alpha-\beta-n-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r)$$

$$\left.\begin{matrix}(\text{k-}\alpha-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\text{k+n-}\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \quad (4.2)$$

Provided that : $Re(1+\beta)>0\,,|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

$$Re(1+\rho+n+\alpha+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(\sigma+1+\beta+\sum_{i=1}^{r}k_iP_j+dp)>0, Re(1+\alpha)>0$$

$$Re(n+k-\rho-\sigma-\sum_{i=1}^{r}(h_i+k_i)P_j)-(c+d)p>0, Re(-n-\alpha-\rho-\sum_{i=1}^{r}k_iP_j-cp)>0$$

$$Re(1+n+k+\alpha+\sigma+\sum_{i=1}^{r}k_iP_j+dp)>0$$

$Re(1+\beta)>0\,,|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

c) $(1-x)^{\rho}(1+x)^{\sigma}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\text{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \text{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\big(z(1-x)^c(1+x)^d\big)$

$$\aleph\begin{pmatrix}(\text{1-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (\text{1-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_1P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_Gg!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{43}:W}^{m,\mathfrak{n}+4:V}\left(\begin{matrix}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{matrix}\ \middle|\right.$$

$$(\text{-}\sigma-\beta-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\rho-\alpha-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)$$
$$\cdots$$
$$(\text{-1-}\alpha-\beta-\sigma-n-k-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$$

$$(\text{-n-k-}\alpha-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(-k-\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r)$$
$$\cdots$$
$$(\text{-}\rho-\alpha-n-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),$$

$$\left.\begin{matrix}\cdots,\text{A : C}\\ \cdots\\ (\text{-1-k-}\alpha-\beta-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \quad (4.3)$$

Provided that : $Re(1+\beta)>0\,,|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

$$Re(1+\rho+\alpha+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(\sigma+1+\beta+\sum_{i=1}^{r}k_iP_j+dp)>0, Re(1+\alpha)>0)$$

$$Re(1+k+\sigma+\sum_{i=1}^{r}k_iP_j+dp)>0, Re(\beta+1)>0$$

d) $$(1-x)^{\rho}(1+x)^{\sigma}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^{c}(1+x)^{d}\right)$$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_1P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_Gg!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{43}:W}^{m,\mathfrak{n}+4:V}\left(\begin{array}{c|}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right.$$

$$\begin{array}{c}(-\sigma-\beta-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\rho-\alpha-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)\\ \cdots\\ (-1-\alpha-\beta-\rho-\sigma-k-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),\end{array}$$

$$\begin{array}{c}(\text{n-k-}\beta-\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-k-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)\\ \cdots\\ (\text{-n-}\sigma-\beta-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\end{array}$$

$$\left.\begin{array}{c}\cdots,\mathrm{A}:\mathrm{C}\\ \cdots\\ (\text{-1-k-}\alpha-\beta-n-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),B:D\end{array}\right)\tag{4.4}$$

Provided that : $Re(1+\beta)>0$, $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

$$Re(1+\rho+\alpha+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(\sigma+1+\beta+\sum_{i=1}^{r}k_iP_j+dp)>0, Re(1+\alpha)>0$$

$$Re(\beta+1)>0$$

e) $$(1-x)^{\rho}(1+x)^{\sigma}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^{c}(1+x)^{d}\right)$$

$$\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_2P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$\begin{array}{c}(\text{-n-}\alpha-\rho-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),\quad(\text{-}\sigma-\beta-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),\\ \cdots\qquad\qquad\cdots\\ (1+\alpha+n-k-\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\alpha-\rho-n-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),\end{array}$$

$$\begin{array}{c}(\text{-k-}\sigma-\sum_{i=1}^s d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(-\alpha-\beta-\rho-n+k-\sum_{i=1}^s c_iK_i-c\eta_{G,g};h_1,\cdots,h_r)\\ \cdots\\ (\text{-1+k-}\alpha-\beta-\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r)\end{array}$$

$$\left.\begin{array}{c}(\text{-1-}\alpha-\beta-\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\text{-1-n-}\rho-\sigma-\sum_{i=1}^s(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right)\quad(4.5)$$

Provided that : $Re(-1+\sigma+k+\sum_{i=1}^r d_iP_j-dp)>0, Re(1+\beta+\sigma+\sum_{i=1}^r k_iP_j+dp)>0$

$Re(1+n+\rho+\alpha+\sum_{i=1}^r h_iP_j+cp)>0, Re(-\rho-\alpha-n-\sum_{i=1}^r h_iP_j-cp)>0$

$Re(-1+k-\alpha-\sigma-\rho-\beta-\sum_{i=1}^r(h_i+k_i)P_j)-(c+d)p>0, Re(-\alpha-\beta-n+k)>0$

$Re(1+\alpha+\beta+n)>0$, $|argz_k| < \frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz| < \frac{1}{2}\pi\Omega$

f) $(1-x)^\rho(1+x)^\sigma S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{array}\right)\aleph^{M,N}_{P_i,Q_i,c_i;r'}(z(1-x)^c(1+x)^d)$

$$\aleph\left(\begin{array}{c}(\text{1-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (\text{1-x})^{h_r}(1+x)^{k_r}z_r\end{array}\right)=\sum_{G=1}^M\sum_{g=0}^\infty\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k,n=0}^\infty 2^{\sum_{i=1}^s K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_3P_n^{(\alpha,\beta)}(x)$$

$$\frac{(-)^{g+n}\Omega^{M,N}_{P_i,Q_i,c_i,r'}(s)}{B_G g!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph^{m+2,\mathfrak{n}+3:V}_{U_{54}:W}\left(\begin{array}{c}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{array}\right|$$

$$(\text{-}\alpha-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r), \qquad (\text{-}\sigma-\beta-n-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$$

$$(\text{-}\alpha-n-\sigma-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(1+\beta+n-\rho-k-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),$$

$$(\beta-\alpha-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(-2-\beta-\sigma-n+k-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$$

$$(\text{-}1\text{-}\alpha-k-\beta-n-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),$$

$$\left.\begin{matrix}(\text{-}1\text{-}\alpha-\beta+k-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),A:C\\ \cdots\\ (\text{-}1\text{-}\alpha-\beta-\rho-\sigma-\sum_{i=1}^{s}(c_i+d_i)K_i-(c+d)\eta_{G,g};h_1+k_1,\cdots,h_r+k_r),B:D\end{matrix}\right) \tag{4.6}$$

Provided that : $Re(-\sigma-\alpha-n-\sum_{i=1}^{r}k_iP_j-dp)>0, Re(1-\beta+\alpha+k+\rho+\sum_{i=1}^{r}h_iP_j+cd)>0$

$Re(1+\alpha+\rho+\sum_{i=1}^{r}h_iP_j+cp)>0, Re(\sigma+1+n+\beta+\sum_{i=1}^{r}k_iP_j+dp)>0$

$Re(-1-k+\alpha-\beta-\rho-\sigma-\sum_{i=1}^{r}(h_i+k_i)P_j-(c+d)p)>0, Re(-\alpha-\beta-n+k)>0$

$Re(1+\alpha+\beta+n)>0$, $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.4) and $|argz|<\frac{1}{2}\pi\Omega$

Proof of (4.1)

To establish (4.1), let $f(x)=(1-x)^{\rho}(1+x)^{\sigma}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix}\mathrm{x}_1(1-x)^{c_1}(1+x)^{d_1}\\ \cdots\\ \mathrm{x}_s(1-x)^{c_s}(1+x)^{d_s}\end{pmatrix}\times$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(z(1-x)^c(1+x)^d\right)\aleph\begin{pmatrix}(1\text{-x})^{h_1}(1+x)^{k_1}z_1\\ \cdots\\ (1\text{-x})^{h_r}(1+x)^{k_r}z_r\end{pmatrix}=\sum_{R=0}^{\infty}c_RP_R^{(\alpha,\beta)}(x) \tag{4.7}$$

The equation (4.7) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(-1,1)$, multiplying both the sides of (4.7) by $(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$ and integrate with respect to x between the limits -1 to 1, and use the orthogonal property of Jacobi polynomial and the integral (3.1), with substitution we get :

$$C_n=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}2^{\sum_{i=1}^{s}K_i(c_i+d_i)+(c+d)\eta_{G,g}}A_1\times$$

$$\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_Gg!}x_1^{K_1}\cdots x_s^{K_s}z^{\eta_{G,g}}\aleph_{U_{54}:W}^{m+2,\mathfrak{n}+3:V}\left(\begin{matrix}2^{h_1+k_1}z_1\\ \cdot\\ \cdot\\ \cdot\\ 2^{h_r+k_r}z_r\end{matrix}\ \middle|\right.$$

$$(\text{-}\alpha-\rho-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),(\text{-}\sigma-\beta-n-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),$$

$$(\text{-}\sigma-n-\sum_{i=1}^{s}d_iK_i-d\eta_{G,g};k_1,\cdots,k_r),(\text{-}\alpha-\rho-n-\sum_{i=1}^{s}c_iK_i-c\eta_{G,g};h_1,\cdots,h_r),$$

$$(\text{-n-k-}\rho - \alpha - \sum_{i=1}^{s} c_i K_i - c\eta_{G,g}; h_1, \cdots, h_r), (1 + \alpha - \sigma + n + k - \sum_{i=1}^{s} d_i K_i - d\eta_{G,g}; k_1, \cdots, k_r)$$
$$\cdots$$
$$(\text{-}\rho - \sigma - 1 - \alpha - \beta - \sum_{i=1}^{s} (c_i + d_i) K_i - (c+d)\eta_{G,g}; h_1 + k_1, \cdots, h_r + k_r)$$

$$\left. \begin{matrix} (\text{k-}\beta - \rho - \sigma - \sum_{i=1}^{s} (c_i + d_i) K_i - (c+d)\eta_{G,g}; h_1 + k_1, \cdots, h_r + k_r), A : C \\ \cdots \\ (\text{k+n-}\rho - \sigma - \sum_{i=1}^{s} (c_i + d_i) K_i - (c+d)\eta_{G,g}; h_1 + k_1, \cdots, h_r + k_r), B : D \end{matrix} \right) \quad (4.8)$$

Substituting the value C_n in (4.7), we get the desired result (4.1).

Remarks: We have the same expansion series with the generalized multivariable I-function, the generalized Aleph-function of two variables, the multivariable I-function defined by Sharma et al [3], the Aleph-function defined by Sharma [5] and the I-function of two variables defined by Sharma et al [4].

***Throughout this document, we suppose that :** $min(c, d, c_i, d_i, h_j, k_j) > 0, i = 1, \cdots, s; j = 1, \cdots, r$

5. Conclusion

In this paper, we have established six general Fourier-Jacobi expansions formulas involving the generalized multivariable Aleph-function, the Aleph-function of one variable and a class of multivariable polynomials. Due to general nature of the generalized multivariable aleph-function involving here, our formulas are capable to be reduced into many known and news expansion formulas involving the special functions of one and several variables.

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