

Degree of Approximation of functions by their Fourier Series in the Besov space by Generalized Matrix Mean

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Abstract: The paper studies the degree of approximation of functions by their Fourier series in the Besov space by matrix means and this generalizing many known results.

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1 Introduction

Let f be a 2π periodic function and let $f \in L_p[0, 2\pi]$ for $p \geq 1$. Let the Fourier series of f at x is given by

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (1.1)$$

Let $s_n(x)$ denote the nth partial sums of (1.1). We know ([10], p.50) that

$$s_n(x) - f(x) = \frac{1}{\pi} \int_0^\pi \phi_x(u) D_n(u) du \quad (1.2)$$

where

$$\phi_x(u) = f(x+u) + f(x-u) - 2f(x) \quad (1.3)$$

$$D_n(u) = \frac{1}{2} + \sum_{k=0}^n \cos ku = \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}} \quad (1.4)$$

$$K_n^i(u) = \sum_{k=0}^{\infty} a_{nk}(i) D_k(u) \quad (1.5)$$

Let \mathbf{A} denote the sequence of matrices $A^i = (a_{nk}(i))$ of complex numbers \mathbb{C} and for a sequence $x = (x_k)$ we write

$$A_n^i(x) = \sum_{k=0}^{\infty} a_{nk}(i) x_k$$

if it exists for each n and $i \geq 0$. We write \mathbf{Ax} for $(A_n^i(x))_{i,n=0}^{\infty}$. A_n^i is called the generalized matrix mean.

The sequence x is said to be summable to the value s by the generalized matrix method (\mathbf{A}) if

$$\lim_n A_n^i(x) = s, \text{ uniformly in } i$$

and in the case we write

$$x \rightarrow s(\mathbf{A}).$$

In the case

$$a_{nk}(i) = \begin{cases} \frac{1}{n+1} & (i \leq k \leq i+n) \\ 0 & \text{otherwise} \end{cases}$$

(\mathbf{A}) reduces to the method f . If $\mathbf{A} = A = (a_{n,k})$, then we obtain the usual summability method (A). It is significant to note that there does not exist any regular method (A) equivalent to the method f (see Lorentz [5] Theorems 11 and 12). In this case

$$a_{nk}(i) = \frac{1}{(n+1)} \sum_{r=i}^{n+i} a_{nr}$$

then (\mathbf{A}) reduces to the almost summability method introduced by King [4]. Thus the method (\mathbf{A}) provides a comprehensive generalized summability method.

The method (\mathbf{A}) is called conservative if

$$x \rightarrow s \Rightarrow x \rightarrow s'(A)$$

and is called regular if

$$s = s'.$$

The following characterisation of conservative matrices is due to Stieglitz [7].

The method (\mathbf{A}) is conservative if and only if the following conditions hold:

$$\sum_k |a_{nk}(i)| < \infty \text{ for all } n \text{ and } i$$

(summation without limits is from 0 to ∞) and there exists an integer m such that

$$\sup_{i \geq 0, n \geq m} \sum_k |a_{nk}(i)| < \infty \quad (1.6)$$

there exists $a_k \in \mathbb{C}$ such that

$$\lim_n a_{nk}(i) = a_k \text{ uniformly in } i \quad (1.7)$$

there exists $a \in \mathbb{C}$ such that

$$\lim_n \sum_k a_{nk}(i) = a \text{ uniformly in } i \quad (1.8)$$

The method (\mathbf{A}) is regular, if further, $a_k = 0, a = 1$. We write

$$\delta(A) = a - \sum_{k=0}^{\infty} a_k.$$

The method (\mathbf{A}) is called conull if $\delta(\mathbf{A}) = 0$; otherwise co-regular.

Compare the following two sets of conditions: (Das [1])

There exists a constant $K > 0$ such that

$$\sum |a_{nk}(i)| \leq K \quad (\text{for all } n, \text{ all } i) \quad (1.9)$$

There exists a constant $K > 0$ such that for each i

$$\sum_k |a_{nk}(i)| \leq K \text{ for all } n$$

and for each n .

$$\sum_{k=0}^{\infty} |a_{nk}(i)| < \infty \quad (1.10)$$

uniformly in i .

It may be remarked that since $a_{nk}(i)$ is an arbitrary function of three parameters n, k, i then neither of (1.9) and (1.10) would imply the other. For example if

$$a_{nk}(i) = \begin{cases} 1 & (k = i, \text{ for all } n) \\ 0 & (k \neq i, \text{ for all } n) \end{cases}$$

then (1.9) holds and (1.10) does not hold. But if

$$a_{nk}(i) = \begin{cases} i & (k = 0, \text{ for all } n) \\ 0 & (k > 0, \text{ for all } n) \end{cases}$$

then (1.10) holds but (1.9) fails to hold.

So in what follows we shall write

$$\sup_{n,i} \sum_k |a_{nk}(i)| = \|A\| < \infty \quad (1.11)$$

to mean that (1.9) holds and the series

$$\sum_k |a_{nk}(i)|$$

converges uniformly in i for each n .

2 Definitions and Notations

Modulus of Continuity:

Let $A = R, R + [a, b] \subset R$ or T (which usually taken to be R with identification of points modulo 2π).

The modulus of continuity $w(f, t) = w(t)$ of a function f on A can be defined as

$$w(t) = w(f, t) = \sup_{\substack{|x-y| \leq t, \\ x, y \in A}} |f(x) - f(y)|, t \geq 0.$$

Modulus of Smoothness:

The k^{th} order modulus of smoothness [3] of a function $f : A \rightarrow R$ is defined by

$$w_k(f, t) = \sup_{0 < h \leq t} \{ \sup | \Delta_h^k(f, x) | : x, x + kh \in A \}, t \geq 0 \quad (2.1)$$

where

$$\Delta_h^k(f, x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x + ih), k \in N. \quad (2.2)$$

For $k = 1$, $w_1(f, t)$ is called the modulus of continuity of f . The function w is continuous at $t = 0$ if and only if f is uniformly continuous on A , that is $f \in \tilde{c}(A)$. The k^{th} order modulus of smoothness of $f \in L_p(A), 0 < p < \infty$ or of $f \in \tilde{c}(A)$, if $p = \infty$ is defined by

$$w_k(f, t)_p = \sup_{0 < h \leq t} \| \Delta_h^k(f, \cdot) \|_p, t \geq 0 \quad (2.3)$$

if $p \geq 1, k = 1$, then $w_1(f, t)_p = w(f, t)_p$ is a modulus of continuity (or integral modulus of continuity). If

$p = \infty, k = 1$ and f is continuous then $w_k(f, t)_p$ reduces to modulus of continuity $w_1(f, t)$ or $w(f, t)$.

Lipschitz Space:

If $f \in \tilde{c}(A)$ and

$$w(f, t) = O(t^\alpha), 0 < \alpha \leq 1 \quad (2.4)$$

then we write $f \in Lip\alpha$. If $w(f, t) = O(t)$ as $t \rightarrow 0+$ (in particular (1.9) holds for $\alpha > 1$) then f reduces to a constant.

If $f \in L_p(A), 0 < p < \infty$ and

$$w(f, t)_p = O(t^\alpha), 0 < \alpha \leq 1 \quad (2.5)$$

then we write $f \in Lip(\alpha, p), 0 < p < \infty, 0 < \alpha \leq 1$.

The case $\alpha > 1$ is of no interest as the function reduces to a constant, whenever

$$w(f, t)_p = O(t) \text{ as } t \rightarrow 0+ \quad (2.6)$$

We note that if $p = \infty$ and $f \in c(A)$, then $Lip(\alpha, p)$ class reduces to $Lip\alpha$ class.

Generalized Lipschitz Space:

Let $\alpha > 0$ and suppose that $k = [\alpha] + 1$. For $f \in L_p(A), 0 < p < \infty$, if

$$w_k(f, t) = O(t^\alpha), t > 0 \quad (2.7)$$

then we write

$$f \in Lip^*(\alpha, p), \alpha > 0, 0 < p \leq \infty \quad (2.8)$$

and say that f belongs to generalized Lipschitz space. The seminorm is then

$$|f|_{Lip^*(\alpha, L_p)} = \sup_{t>0} (t^{-\alpha} w_k(f, t)_p).$$

It is known ([3], p-52) that the space $Lip^*(\alpha, L_p)$ contains $Lip(\alpha, L_p)$. For $0 < \alpha < 1$ the spaces coincide, (for $p = \infty$, it is necessary to replace L_∞ by \tilde{c} of uniformly continuous function on A). For $0 < \alpha < 1$ and $p = 1$ the space $Lip^*(\alpha, L_p)$ coincide with $Lip\alpha$.

For $\alpha = 1, p = \infty$, we have

$$Lip(1, \tilde{c}) = Lip 1 \quad (2.9)$$

but

$$Lip^*(1, \tilde{c}) = z \quad (2.10)$$

is the Zygmund space [9] which is characterized by (2.5) with $k = 2$.

Holder (H_α) Space:

For $0 < \alpha \leq 1$, let

$$H_\alpha = \{f \in C_{2\pi} : w(f, t) = O(t^\alpha)\}. \quad (2.11)$$

It is known [6] that H_α is a Banach Space with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_c + \sup_{t>0} t^{-\alpha} w(t), 0 < \alpha \leq 1 \quad (2.12)$$

$$\|f\|_0 = \|f\|_c$$

and

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, 0 < \beta \leq \alpha \leq 1 \quad (2.13)$$

H_(α,p) Space:

For 0 < α ≤ 1, let

$$\begin{aligned} H_{(\alpha,p)} = \{f \in L_p[0,2\pi] : 0 < p \leq \infty, \\ w(f,t)_p = O(t^\alpha)\} \end{aligned} \quad (2.14)$$

and introduce the norm $\|\cdot\|_{(\alpha,p)}$ as follows

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f,t)_p, 0 < \alpha \leq 1. \quad (2.15)$$

$$\|f\|_{(0,p)} = \|f\|_p.$$

It is known [2] that $H_{(\alpha,p)}$ is a Banach space for $p \geq 1$ and a complete p -normed space for $0 < p < 1$.

Also

$$H_{(\alpha,p)} \subseteq H_{(\beta,p)} \subseteq L_p, 0 < \beta \leq \alpha \leq 1. \quad (2.16)$$

Note that $H_{(\alpha,\infty)}$ is the space H_α defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in H_α and $H_{(\alpha,p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on α . We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besov developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter q (in addition to p on α) and applying $\alpha \cdot q$ norms (rather than α, ∞ norms) to the modulus of smoothness $w_k(f,\cdot)_p$ of f .

Besov space:

Let $\alpha > 0$ be given and let $k = [\alpha] + 1$. For $0 < p, q \leq \infty$, the Besov space ([3], p-54) $B_q^\alpha(L_p)$ is defined as follows:

$$B_q^\alpha(L_p) = \{f \in L_p : |f|_{B_q^\alpha(L_p)} = \|w_k(f,\cdot)\|_{(\alpha,q)} \text{ is finite}\}$$

where

$$\|w_k(f,\cdot)\|_{(\alpha,q)} = \begin{cases} \left(\int_0^\infty (t^{-\alpha} w_k(f,t)_p)^q \frac{dt}{t} \right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0} t^{-\alpha} w_k(f,t)_p, & q = \infty. \end{cases} \quad (2.17)$$

It is known ([3], p-55) that $\|w_k(f,\cdot)\|_{(\alpha,q)}$ is a seminorm if $1 \leq p, q \leq \infty$ and a quasi-seminorm in other cases.

The Besov norm for $B_q^\alpha(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} = \|f\|_p + \|w_k(f,\cdot)\|_{(\alpha,q)} \quad (2.18)$$

It is known ([8], p-237) that for 2π -periodic function f , the integral $(\int_0^\infty (t^{-\alpha} w_k(f, t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$ is replaced by $(\int_0^\pi (t^{-\alpha} w_k(f, t)_p)^q \frac{dt}{t})^{\frac{1}{q}}$.

We know ([3], p-56, [8], p-236) the following inclusion relations.

For fixed α and p

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), q < q_1.$$

For fixed p and q

$$B_q^\alpha(L_p) \subset B_q^\beta(L_p), \beta < \alpha.$$

For fixed α and q

$$B_q^\alpha(L_p) \subset B_{q_1}^\alpha(L_p), p_1 < p.$$

Special cases of Besov space:

For $q = \infty, B_\infty^\alpha(L_p), \alpha > 0, p \geq 1$ is same as $Lip^*(\alpha, L_p)$ the generalized Lipschitz space and the corresponding norm $\|\cdot\|_{B_\infty^\alpha(L_p)}$ is given by

$$\|f\|_{B_\infty^\alpha(L_p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w_k(f, t)_p \quad (2.19)$$

for every $\alpha > 0$ with $k = [\alpha] + 1$.

For the special case when $0 < \alpha < 1, B_\infty^\alpha(L_p)$ space reduces to $H_{(\alpha, p)}$ space due to Das et al. [2] and the corresponding norm is given by

$$\|f\|_{B_\infty^\alpha(L_p)} = \|f\|_{(\alpha, p)} = \|f\|_p + \sup_{t>0} t^{-\alpha} w(f, t)_p, 0 < \alpha < 1. \quad (2.20)$$

For $\alpha = 1$, the norm is given by

$$\|f\|_{B_\infty^1(L_p)} = \|f\|_p + \sup_{t>0} t^{-1} w_2(f, t)_p. \quad (2.21)$$

Note that $\|f\|_{B_\infty^1(L_p)}$ is not same as $\|f\|_{(1, p)}$ and the space $B_\infty^1(L_p)$ includes the space $H(1, p), p \geq 1$. If we further specialize by taking $p = \infty, B_\infty^\alpha, 0 < \alpha < 1$, coincides with H_α space due to Prossodorf [6] and the norm is given by

$$\|f\|_{B_\infty^\alpha(L_\infty)} = \|f\|_\alpha = \|f\|_c + \sup_{t>0} t^\alpha w(f, t), 0 < \alpha < 1. \quad (2.22)$$

For $\alpha = 1, p = \infty$, the norm is given by

$$\|f\|_{B_\infty^1(L_\infty)} = \|f\|_c + \sup_{t>0} t^{-1} w_2(f, t), \alpha = 1 \quad (2.23)$$

which is different from $\|f\|_1$ and $B_\infty^1(L_\infty)$ includes the H_1 space.

3 Main Result

We prove the following theorem.

Theorem Let the matrix $A = (a_{n,k}(i))$ satisfy the following conditions

$$\sup_{n,i} \sum_{k=0}^{\infty} |a_{n,k}(i)| < \infty \quad (3.1)$$

$$\sum_{k=0}^{\infty} a_{n,k}(i) = 1, \text{ for all } n \text{ and } i \quad (3.2)$$

and

$$\sum_{k=\mu_n}^{\infty} (k+1) |a_{n,k}(i)| = O(\mu_n), \text{ uniformly in } i. \quad (3.3)$$

where (μ_n) is a positive non-decreasing sequence with $\mu_1 = 1$. Let

$$\psi(n) = \sup_{i \geq 0} \sum_{k=0}^{\infty} |a_{n,k}(i) - a_{n,k+1}(i)| \quad (3.4)$$

Let $0 < \alpha < 2$ and $0 \leq \beta < \alpha$. If $f \in B_q^\alpha(L_p)$, $p \geq 1$ and $1 < q \leq \infty$ and let $t_n^i(x)$ be the A -transform of the Fourier series of f , that is,

$$t_n^i(f) = t_n^i(f; x) = \sum_{k=0}^{\infty} a_{n,k}(i) s_k(x)$$

Then

Case 1 ($1 < q < \infty$)

$$\|T_n^i(\cdot)\|_{B_q^\beta(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{\frac{1-\frac{1}{q}}{q-1}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i.$$

Case 2 ($q = \infty$)

$$\|T_n^i(\cdot)\|_{B_q^\beta(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n)) \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)}{\mu_k^{\alpha-\beta}} \right), \text{ uniformly in } i.$$

4 Additional notations and Lemmas

We need the following additional notations

$$\phi(x, t, u) = \begin{cases} \phi_{x+t}(u) - \phi_x(u), & 0 < \alpha < 1 \\ \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u), & 1 \leq \alpha < 2 \end{cases}$$

For $k = [\alpha] + 1$, we have for $p \geq 1$

$$w_k(f, t)_p = \begin{cases} w_1(f, t)_p, & 0 < \alpha < 1 \\ w_2(f, t)_p, & 1 \leq \alpha < 2 \end{cases}$$

Let

$$T_n^i(x, t) = \begin{cases} T_n^i(x+t) - T_n^i(x), & 0 < \alpha < 1 \\ T_n^i(x+t) + T_n^i(x-t) & 1 \leq \alpha < 2 \\ -2T_n^i(x), & \end{cases}$$

Using above equation and definition of $w_k(f, t)_p$, we have

$$w_k(T_n^i, t)_p = \|T_n^i(\cdot, t)\|_p$$

We require the following lemmas for the proof of the theorem.

Lemma 1 Let $1 \leq p \leq \infty$ and $0 < \alpha < 2$. If $f \in L_p[0, 2\pi]$, then for $0 < t, u \leq \pi$

- (i) $\|\phi(\cdot, t, u)\|_p \leq 4w_k(f, t)_p$
- (ii) $\|\phi(\cdot, t, u)\|_p \leq 4w_k(f, u)_p$
- (iii) $\|\phi(u)\|_p \leq 2w_k(f, u)_p$,

where $k = [\alpha] + 1$.

Proof: Case $0 < \alpha < 1$.

Clearly $k = [\alpha] + 1 = 1$. By virtue of (1.3)

$\phi(x, t, u) = \phi_{x+t}(u) - \phi_x(u)$, can be written as

$$\phi(x, t, u) = \begin{cases} \{f(x+t+u) - f(x+u)\} \\ + \{f(x+t-u) - f(x-u)\} \\ - 2\{f(x+t) - f(x)\} \quad (4.1) \end{cases}$$

$$\phi(x, t, u) = \begin{cases} \{f(x+t+u) - f(x+t)\} \\ + \{f(x-u+t) - f(x+t)\} \\ - \{f(x+u) - f(x)\} \\ - \{f(x-u) - f(x)\} \quad (4.2) \end{cases}$$

Applying Minkowski's inequality to (4.1), we get for $p \geq 1$

$$\begin{aligned} \|\phi(\cdot, t, u)\|_p &\leq \|f(\cdot+t+u) - f(\cdot+u)\|_p + \|f(\cdot+t-u) - f(\cdot-u)\|_p \\ &\quad + 2\|f(\cdot+t) - f(\cdot)\|_p \\ &\leq 4w_1(f, t)_p, \quad 0 < \alpha < 1 \end{aligned}$$

Similarly applying Minkowski's inequality to (4.2), we get for $p \geq 1$

$$\|\phi(\cdot, t, u)\|_p \leq 4w_1(f, u)_p.$$

Case $1 \leq \alpha < 2$.

Clearly $k = [\alpha] + 1 = 2$. By virtue of (1.3)

$\phi(x, t, u) = \phi_{x+t}(u) + \phi_{x-t}(u) - 2\phi_x(u)$, can be written as

$$\phi(x, t, u) = \begin{cases} \{f(x+t+u) + f(x+t-u) - 2f(x+t)\} \\ + \{f(x-t+u) + f(x-t-u) - 2f(x-t)\} \\ - 2\{f(x+u) + f(x-u) - 2f(x)\} & (4.3) \\ \{f(x+t+u) + f(x-t+u) - 2f(x+u)\} \\ + \{f(x+t-u) + f(x-t-u) - 2f(x-u)\} \\ - 2\{f(x+t) + f(x-t) - 2f(x)\} & (4.4) \end{cases}$$

Applying Minkowski's inequality to (4.3), we obtain for $p \geq 1$

$$\begin{aligned} \|\phi(\cdot, t, u)\|_p &\leq \|f(\cdot+t+u) + f(\cdot+t-u) - 2f(\cdot+t)\|_p \\ &\quad + \|f(\cdot-t+u) + f(\cdot-t-u) - 2f(\cdot-t)\|_p \\ &\quad + 2\|f(\cdot+u) + f(\cdot-u) - 2f(\cdot)\|_p \\ &\leq 4w_2(f, u)_p \end{aligned}$$

Using (4.4) and proceeding as above, we get

$$\|\phi(\cdot, t, u)\|_p \leq 4w_2(f, t)_p$$

this completes the proof of part (i) and (ii). We omit the proof of (iii) as it is trivial.

Lemma 2 Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $f \in B_q^\alpha(L_p)$, $p \geq 1$, $1 < q < \infty$, then

$$\begin{aligned} \text{(i)} \int_0^\pi |K_n^i(u)| \left(\int_0^u \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du &= O(1) \left\{ \int_0^\pi (u^{\alpha-\beta} |K_n^i(u)|)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\ \text{(ii)} \int_0^\pi |K_n^i(u)| \left(\int_u^\pi \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} du &= O(1) \left\{ \int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}} |K_n^i(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \end{aligned}$$

where $K_n^i(u)$ is defined as in (1.5).

Proof: Applying Lemma 1(i), we have

$$\begin{aligned} &\int_0^\pi |K_n^i(u)| \left(\int_0^u \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} du \\ &= O(1) \int_0^\pi |K_n^i(u)| \left(\int_0^u \left(\frac{w_k(f, t)_p}{t^\alpha} \right)^q t^{(\alpha-\beta)q} \frac{dt}{t} \right)^{\frac{1}{q}} du \\ &= O(1) \int_0^\pi |K_n^i(u)| u^{(\alpha-\beta)} du \left(\int_0^u \frac{w_k(f, t)_p}{t^\alpha} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= O(1) \int_0^\pi |K_n^i(u)| u^{(\alpha-\beta)} du \end{aligned}$$

by Second Mean Value theorem and by the definition of Besov space.

Applying Holders inequality

$$= O(1) \left\{ \int_0^\pi (|K_n^i(u)| u^{(\alpha-\beta)})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \left(\int_0^\pi 1^q du \right)^{\frac{1}{q}}$$

$$= O(1) \left\{ \int_0^\pi \left(K_n^i(u) |u^{(\alpha-\beta)} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

For the second part, applying Lemma 1(ii), we get

$$\begin{aligned} & \int_0^\pi |K_n^i(u)| du \left(\int_u^\pi \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q+1}} dt \right)^{\frac{1}{q}} \\ &= O(1) \int_0^\pi |K_n^i(u)| w_k(f, u)_p du \left(\int_u^\pi \frac{dt}{t^{\beta q+1}} \right)^{\frac{1}{q}} \\ &= O(1) \int_0^\pi |K_n^i(u)| w_k(f, u)_p u^{-\beta} du \\ &= O(1) \int_0^\pi \left(\frac{w_k(f, u)_p}{u^{\frac{\alpha+1}{q}}} \right)^{\alpha-\beta+\frac{1}{q}} |K_n^i(u)| du \end{aligned}$$

Applying Holder's inequality

$$\begin{aligned} &= O(1) \left\{ \int_0^\pi \left(\frac{w_k(f, u)_p}{u^\alpha} \right)^q \frac{du}{u} \right\}^{\frac{1}{q}} \left\{ \int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}} |K_n^i(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\ &= O(1) \left\{ \int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}} |K_n^i(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \end{aligned}$$

by definition of Besov space.

Lemma 3 Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $f \in B_q^\alpha(L_p)$, $p \geq 1$ and $q = \infty$, then

$$\sup_{0 < t, u \leq \pi} t^{-\beta} \|\phi(\cdot, t, u)\|_p = O(u^{\alpha-\beta})$$

Proof: For $0 < t \leq u \leq \pi$, applying Lemma 1(i), we have

$$\begin{aligned} \sup_{\substack{t, \\ 0 < t \leq u \leq \pi}} t^{-\beta} \|\phi(\cdot, t, u)\|_p &= \sup_{\substack{t, \\ 0 < t \leq u \leq \pi}} t^{\alpha-\beta} (t^{-\alpha} \|\phi(\cdot, t, u)\|_p) \\ &\leq 4u^{\alpha-\beta} \sup_t (t^{-\alpha} w_k(f, t)_p) \\ &= O(u^{\alpha-\beta}), \quad \text{by the hypothesis.} \end{aligned}$$

Next for $0 < u \leq t \leq \pi$, applying Lemma 1(ii), we get

$$\begin{aligned} \sup_{\substack{t, \\ 0 < u \leq t \leq \pi}} t^{-\beta} \|\phi(\cdot, t, u)\|_p &\leq 4w_k(f, u)_p \sup_{\substack{t, \\ 0 < u \leq t \leq \pi}} t^{-\beta} \\ &\leq 4u^{\alpha-\beta} \sup_u (u^{-\alpha} w_k(f, u)_p) \\ &= O(u^{\alpha-\beta}), \quad \text{by the hypothesis} \end{aligned}$$

and this completes the proof.

Lemma (4.1).

1. Let the matrix $A = a_{n,k}(i)$ and the kernel $K_n^i(u)$ be defined as in (1.5). Let there exist a positive non-decreasing sequence (μ_n) with $\mu_1 = 1$, then for $0 < u \leq \pi$

$$K_n^i(u) = O(\mu_n), \text{uniformly in } i.$$

2. Let $\psi(n) = \sup \sum_{k=0}^{\infty} |a_{n,k}(i) - a_{n,k+1}(i)|$. Then for $0 < u \leq \pi$,

$$K_n^i(u) = O(u^{-2}\psi(n)), \text{uniformly in } i.$$

Proof. (a) From (1.4), we have

$$\begin{aligned} |D_k(u)| &= \left| \frac{1}{2} + \sum_{v=0}^k \cos vu \right| \\ &\leq k+1 \end{aligned} \quad (4.1)$$

Then

$$\begin{aligned} |K_n^i(u)| &\leq \sum_{k=0}^{\infty} |a_{n,k}(i)D_k(u)| \\ &\leq \sum_{k=0}^{\mu_n} |a_{n,k}(i)|(k+1) + \sum_{k=\mu_n}^{\infty} |a_{n,k}(i)|(k+1) (\text{by using (4.1)}) \\ &\leq O(\mu_n) \sum_{k=0}^{\mu_n} |a_{n,k}(i)| + \sum_{k=\mu_n}^{\infty} |a_{n,k}(i)|(k+1) (\text{Since } \sum_{k=\mu_n}^{\infty} |a_{n,k}(i)|(k+1) = O(\mu_n), \text{uniformly in } i) \\ &= O(\mu_n) + O(\mu_n) \\ &= O(\mu_n), \text{uniformly in } i. \end{aligned}$$

(b) Applying Abel's transformation, we have

$$\begin{aligned} &\sum_{k=0}^{\infty} a_{n,k}(i) \sin(k + \frac{1}{2})u \\ &= O(u^{-1}) \sum_{k=0}^{\infty} |a_{n,k}(i) - a_{n,k+1}(i)| \\ &= O(u^{-1}\psi(n)), \text{uniformly in } i. \end{aligned}$$

from which it follows that

$$K_n^i(u) = O(u^{-2}\psi(n)), \text{uniformly in } i.$$

5 Proof of Theorem

Case 1 ($1 < q < \infty$)

Since $t_n^i(x)$ denote the transformations of the Fourier series f , we have

$$\begin{aligned} t_n^i(x) &= \sum_{k=0}^{\infty} a_{n,k}(i) s_k(x) \\ &= \sum_{k=0}^{\infty} a_{n,k}(i) \left[\frac{1}{\pi} \int_0^\pi \phi_x(u) D_k(u) du + f(x) \right] (\text{by (1.2)}) \end{aligned} \quad (5.1)$$

$$\begin{aligned}
 &= \frac{1}{\pi} \sum_{k=0}^{\infty} a_{n,k}(i) \int_0^{\pi} \phi_x(u) D_k(u) du + \sum_{k=0}^{\infty} a_{n,k}(i) f(x) \\
 &= \frac{1}{\pi} \int_0^{\pi} \left(\sum_{k=0}^{\infty} a_{n,k}(i) D_k(u) \right) \phi_x(u) du + f(x) \sum_{k=0}^{\infty} a_{n,k}(i)
 \end{aligned}$$

Now,

$$T_n^i(x) = \frac{1}{\pi} \int_0^{\pi} \phi_x(u) K_n^i(u) du \quad (5.2)$$

where we write

$$T_n^i(x) = t_n^i(x) - f(x). \quad (5.3)$$

We first consider the case $1 < q < \infty$.

We have for $p \geq 1$ and $0 \leq \beta < \alpha < 2$, by use of Besov norm defined in (2.18) for $B_q^\beta(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} = \|f\|_p + \|w_k(f, \cdot)\|_{\alpha, q} \quad (5.4)$$

$$\|T_n^i(\cdot)\|_{B_q^\beta(L_p)} = \|T_n^i(\cdot)\|_p + \|w_k(T_n^i, \cdot)\|_{\beta, q} \quad (5.5)$$

Applying Lemma 1(iii) in equation (5.2), we have

$$\begin{aligned}
 \|T_n^i(\cdot)\|_p &\leq \frac{1}{\pi} \int_0^{\pi} \|\phi(u)\|_p |K_n^i(u)| du \\
 &\leq \frac{1}{\pi} \int_0^{\pi} 2w_k(f, u)_p |K_n^i(u)| du \\
 &= \frac{2}{\pi} \int_0^{\pi} |K_n^i(u)| w_k(f, u)_p du
 \end{aligned}$$

Applying Hölder's inequality, we have

$$\|T_n^i(\cdot)\|_p \leq \frac{2}{\pi} \left\{ \int_0^{\pi} \left(|K_n^i(u)| u^{\frac{\alpha+1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \left\{ \int_0^{\pi} \left(\frac{w_k(f, u)_p}{u^{\frac{\alpha+1}{q}}} \right)^q du \right\}^{\frac{1}{q}}$$

By definition of Besov Space, we have

$$\begin{aligned}
 \|T_n^i(\cdot)\|_p &\leq O(1) \left\{ \int_0^{\pi} \left(|K_n^i(u)| u^{\frac{\alpha+1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(1) \left[\left\{ \int_0^{\mu_n} \left(|K_n^i(u)| u^{\frac{\alpha+1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_{\mu_n}^{\pi} \left(|K_n^i(u)| u^{\frac{\alpha+1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \right] \\
 &= O(1)[I + J], \quad (\text{say})
 \end{aligned} \quad (5.6)$$

By using Lemma 4(a) in I of (5.6), we have

$$\begin{aligned}
 I &= \left\{ \int_0^{\frac{\pi}{\mu_n}} (|K_n^i(u)| u^{\frac{\alpha+1}{q}})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\mu_n) \left\{ \int_0^{\frac{\pi}{\mu_n}} u^{(\alpha+\frac{1}{q}) \cdot \frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\mu_n) \left\{ \int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha+1)-1} du \right\}^{1-\frac{1}{q}} \\
 &= O\left(\frac{1}{\mu_n^\alpha}\right), \text{ uniformly in } i. \tag{5.7}
 \end{aligned}$$

Applying Lemma 4(b) in J of (5.6), we have

$$\begin{aligned}
 J &= \left\{ \int_{\frac{\pi}{\mu_n}}^{\pi} \left(|K_n^i(u)| u^{\frac{\alpha+1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \int_{\frac{\pi}{\mu_n}}^{\pi} \left(u^{\frac{\alpha+1}{q}-2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^{n-1} \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} \left(u^{\frac{\alpha+1}{q}-2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^n \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{(\alpha+\frac{1}{q}-2) \cdot \frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^n \frac{\mu_{k+1} - \mu_k}{\mu_k^2 \mu_k^{\frac{q}{q-1}(\alpha+\frac{1}{q}-2)}} \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{\frac{1}{q-1}}}{\mu_k^{\frac{(\alpha-1)}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \tag{5.8}
 \end{aligned}$$

Using (5.7) and (5.8) and we have from (5.6),

$$\| T_n^i(\cdot) \|_p = O\left(\frac{1}{\mu_n^\alpha}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{\frac{1}{q}}}{\mu_k^{(\alpha-\frac{1}{q})}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.9)$$

By using Besov space, we have

$$\begin{aligned} \| w_k(T_n^i, \cdot) \|_{\beta, q} &= \left\{ \int_0^\pi \left(t^{-\beta} w_k(T_n^i, t) \right)_p^q \frac{dt}{t} \right\}^{\frac{1}{q}} \\ &= \int_0^\pi \left\{ \left(\frac{w_k(T_n^i, t)}{t^\beta} \right)_p^q \frac{dt}{t} \right\}^{\frac{1}{q}} \end{aligned}$$

From definition of $w_k(T_n^i, t)_p$, we have

$$\begin{aligned} w_k(T_n^i, t)_p &= \| T_n^i(\cdot, t) \|_p \\ &\leq \left\{ \int_0^\pi \left(\frac{\| T_n^i(\cdot, t) \|_p}{t^\beta} \right)^q \frac{dt}{t} \right\}^{\frac{1}{q}} \\ &= \left[\int_0^\pi \left\{ \int_0^\pi |T_n^i(x, t)|^p dx \right\}_p^q \frac{dt}{t^{\beta q+1}} \right]^{\frac{1}{q}} \\ &= \left[\int_0^\pi \left\{ \int_0^\pi \left| \int_0^\pi \phi(x, t, u) K_n^i(u) du \right|^p dx \right\}_p^q \frac{dt}{t^{\beta q+1}} \right]^{\frac{1}{q}} \end{aligned}$$

By repeated application of generalized Minkowski's inequality, we have

$$\begin{aligned} \| w_k(T_n^i, \cdot) \|_{\beta, p} &\leq \frac{1}{\pi} \left[\int_0^\pi \left\{ \int_0^\pi \left(\int_0^\pi |\phi(x, t, u)|^p |K_n^i(u)|^p dx \right)^{\frac{1}{p}} du \right\}_p^q \frac{dt}{t^{\beta q+1}} \right]^{\frac{1}{q}} \\ &= \frac{1}{\pi} \left[\int_0^\pi \left\{ \int_0^\pi |K_n^i(u)| \|\phi(\cdot, t, u)\|_p du \right\}_p^q \frac{dt}{t^{\beta q+1}} \right]^{\frac{1}{q}} \\ &\leq \frac{1}{\pi} \int_0^\pi |K_n^i(u)| du \left(\int_0^\pi \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right)^{\frac{1}{q}} \\ &= \frac{1}{\pi} \int_0^\pi |K_n^i(u)| du \left\{ \left(\int_0^u + \int_u^\pi \right) \frac{\|\phi(\cdot, t, u)\|_p^q dt}{t^{\beta q}} \frac{1}{t} \right\}^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\pi} \int_0^\pi |K_n^i(u)| du \left\{ \int_0^u \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right\}^{\frac{1}{q}} + \frac{1}{\pi} \int_0^\pi |K_n^i(u)| du \left\{ \int_u^\pi \frac{\|\phi(\cdot, t, u)\|_p^q}{t^{\beta q}} \frac{dt}{t} \right\}^{\frac{1}{q}} \\
 &= O(1) \left[\left\{ \int_0^\pi (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_0^\pi \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \right] \\
 &\quad (\text{using Lemma 2}) \\
 &= O(1)[I' + J'], \quad (\text{say}) \tag{5.10}
 \end{aligned}$$

$$\begin{aligned}
 I' &= \left\{ \int_0^\pi (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= \left\{ \left(\int_{0/\mu_n}^{\pi/\mu_n} (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right)^{1-\frac{1}{q}} \right. \\
 &\leq \left\{ \int_0^{\pi/\mu_n} (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_{\pi/\mu_n}^\pi (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= I'_1 + I'_2, \quad (\text{say}) \tag{5.11}
 \end{aligned}$$

Applying Lemma 4(a) in I'_1 , we have

$$\begin{aligned}
 I'_1 &= \left\{ \int_0^{\pi/\mu_n} (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\mu_n) \left\{ \int_0^{\pi/\mu_n} u^{\alpha-\beta(\frac{q}{q-1})} du \right\}^{1-\frac{1}{q}} \\
 &= O(\mu_n) \left\{ \int_0^{\pi/\mu_n} u^{\frac{q}{q-1}(\alpha-\beta+1-\frac{1}{q})-1} du \right\}^{1-\frac{1}{q}} \\
 &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right), \text{ uniformly} \tag{5.12}
 \end{aligned}$$

Applying Lemma 4(b) in I'_2 , we have

$$I'_2 = \left\{ \int_{\pi/\mu_n}^\pi (|K_n^i(u)| u^{\alpha-\beta})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$

$$\begin{aligned}
 &= O(\psi(n)) \left\{ \int_{\frac{\pi}{\mu_n}}^{\pi} (u^{\alpha-\beta-2})^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^{n-1} \int_{\frac{\mu_k}{\pi}}^{\frac{\pi}{\mu_{k+1}}} u^{(\alpha-\beta-2)\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^n \int_{\frac{\mu_k}{\pi}}^{\frac{\pi}{\mu_{k+1}}} u^{(\alpha-\beta-2)\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i.
 \end{aligned}$$

Let $h(u) = (u^{\alpha-\beta-2})^{\frac{q}{q-1}}$ and $H(u)$ be a primitive of $h(u)$, then

$$\begin{aligned}
 \int_{\frac{\mu_k}{\pi}}^{\frac{\pi}{\mu_{k+1}}} (u^{\alpha-\beta-2})^{\frac{q}{q-1}} du &= \int_{\frac{\mu_k}{\pi}}^{\frac{\pi}{\mu_{k+1}}} h(u) du \\
 &= H\left(\frac{\pi}{\mu_k}\right) - H\left(\frac{\pi}{\mu_{k+1}}\right) \\
 &= \left(\frac{\pi}{\mu_k} - \frac{\pi}{\mu_{k+1}} \right) h(c), \text{ for some } \frac{\pi}{\mu_{k+1}} < c < \frac{\pi}{\mu_k} \\
 &= O(1) \frac{(\mu_{k+1} - \mu_k)}{\mu_k^2} \left(\frac{1}{\mu_k^{\alpha-\beta-2}} \right)^{\frac{q}{q-1}} \\
 &= O(1) \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}}
 \end{aligned}$$

$$I_2' = O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.13)$$

From (5.13), (5.12) and (5.11), we have

$$I' = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.14)$$

$$\begin{aligned}
 J' &= \left\{ \int_0^\pi \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= \left\{ \left(\int_{\mu_n}^{\frac{\pi}{\mu_n}} + \int_{\frac{\pi}{\mu_n}}^\pi \right) \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &\leq \left\{ \int_0^{\frac{\pi}{\mu_n}} \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} + \left\{ \int_{\frac{\pi}{\mu_n}}^\pi \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= (J_1^1 + J_2^1), \quad (\text{say}) \tag{5.15}
 \end{aligned}$$

Applying Lemma 4(a) in J_1^1 , we have

$$\begin{aligned}
 J_1^1 &= \left\{ \int_0^{\frac{\pi}{\mu_n}} \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O\left(\frac{1}{\mu_n}\right) \left\{ \int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha-\beta+\frac{1}{q})} du \right\}^{1-\frac{1}{q}} \\
 &= O\left(\frac{1}{\mu_n}\right) \left\{ \int_0^{\frac{\pi}{\mu_n}} u^{\frac{q}{q-1}(\alpha-\beta+1)-1} du \right\}^{1-\frac{1}{q}} \\
 &= O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right), \text{ uniformly} \tag{5.16}
 \end{aligned}$$

Applying Lemma 4(b) in J_2^1 , we have

$$\begin{aligned}
 J_2^1 &= \left\{ \int_{\frac{\pi}{\mu_n}}^\pi \left(|K_n^i(u)| u^{\alpha-\beta+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta-2+\frac{1}{q}} du \right\}^{1-\frac{1}{q}} \\
 &= O(\psi(n)) \left\{ \sum_{k=1}^{n-1} \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-\beta-2+\frac{1}{q}} du \right\}^{1-\frac{1}{q}}
 \end{aligned}$$

$$= O(\psi(n)) \left\{ \sum_{k=1}^n \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} \left(u^{\alpha-\beta-2+\frac{1}{q}} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i.$$

Proceeding as in I_2^1 , we have

$$J_2^1 = O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.17)$$

From (5.15), (5.16), (5.17), we have

$$J^1 = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.18)$$

From (5.10), (5.14) and (5.18), we have

$$\begin{aligned} \|w_k(T_{n,\cdot}^i)\|_{\beta,q} &= O(1)(I' + J') \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} + O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) \\ &\quad + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{\alpha-\beta-\frac{1}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.19) \end{aligned}$$

From (5.5), (5.9) and (5.19), we have

$$\|T_n^i(\cdot)\|_{B_q^\beta(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta-\frac{1}{q}}}\right) + O(\psi(n)) \left\{ \sum_{k=1}^n \left(\frac{(\mu_{k+1} - \mu_k)^{1-\frac{1}{q}}}{\mu_k^{(\alpha-\beta-\frac{2}{q})}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}, \text{ uniformly in } i. \quad (5.20)$$

This complete the proof of Case 1.

Case 2 ($q = \infty$)

Now, we consider the case $q = \infty$

$$\| T_n^i(\cdot) \|_{B_\infty^\beta(L_p)} = \| T_n^i(\cdot) \|_p + \| w_k(T_n^i, \cdot) \|_{\beta, \infty} \quad (5.21)$$

We know $T_n^i(x) = \frac{1}{\pi} \int_0^\pi \phi_x(u) K_n^i(u) du$.

Applying Lemma 1(iii), we have

$$\begin{aligned} \| T_n^i(\cdot) \|_p &\leq \frac{1}{\pi} \int_0^\pi \| \phi_x(u) \|_p |K_n^i(u)| du \\ &\leq \frac{2}{\pi} \int_0^\pi |K_n^i(u)| w_k(f, u)_p du \\ &= O(1) \int_0^\pi |K_n^i(u)| u^\alpha du \quad (\text{by the hypothesis}) \\ &= O(1) \left[\int_0^{\mu_n} |K_n^i(u)| u^\alpha du + \int_{\mu_n}^\pi |K_n^i(u)| u^\alpha du \right] \\ &= O(1)[I'' + J''], \quad (\text{say}) \end{aligned} \quad (5.22)$$

Applying Lemma 4(a) in I'' , we have

$$\begin{aligned} I'' &= \int_0^{\mu_n} |K_n^i(u)| u^\alpha du \\ &= O(\mu_n) \int_0^{\mu_n} u^\alpha du \\ &= O\left(\frac{1}{\mu_n^\alpha}\right), \text{ uniformly in } i. \end{aligned} \quad (5.23)$$

Applying Lemma 4(b) in J'' , we have

$$\begin{aligned} J'' &= \int_{\mu_n}^\pi |K_n^i(u)| u^\alpha du \\ &= O(\psi(n)) \int_{\mu_n}^\pi u^{\alpha-2} du \\ &= O(\psi(n)) \sum_{k=1}^{n-1} \int_{\mu_{k+1}}^{\mu_k} u^{\alpha-2} du \\ &= O(\psi(n)) \sum_{k=1}^n \int_{\mu_{k+1}}^{\mu_k} u^{\alpha-2} du, \text{ uniformly in } i. \end{aligned}$$

Proceeding as in I_2' , we have

$$= O(\psi(n)) \sum_{k=1}^n \left(\frac{\mu_{k+1} - \mu_k}{\mu_k^\alpha} \right), \text{ uniformly in } i. \quad (5.24)$$

From (5.22), (5.23) and (5.24), we have

$$\| T_n^i(\cdot) \|_p = O\left(\frac{1}{\mu_n^\alpha}\right) + O(\psi(n)) \sum_{k=1}^n \left(\frac{\mu_{k+1} - \mu_k}{\mu_k^\alpha} \right), \text{ uniformly in } i. \quad (5.25)$$

Again,

$$\begin{aligned} \|w_k(T_n^i, \cdot)\|_{\beta,q} &= \sup_{t>0} \frac{\|T_n^i(\cdot, t)\|_p}{t^\beta} \\ &= \sup_{t>0} \frac{t^{-\beta}}{\pi} \left[\int_0^\pi \left| \int_0^\pi \phi(x, t, u) K_n^i(u) du \right|^p dx \right]^{\frac{1}{p}} \end{aligned}$$

Applying generalised Minkowski's inequality, we have

$$\begin{aligned} \|w_k(T_n^i, \cdot)\|_{\beta,q} &= \sup_{t>0} \frac{t^{-\beta}}{\pi} \int_0^\pi du \left\{ \int_0^\pi |\phi(x, t, u)|^p |K_n^i(u)|^p dx \right\}^{\frac{1}{p}} \\ &= \sup_{t>0} \frac{t^{-\beta}}{\pi} \int_0^\pi |K_n^i(u)| \|\phi(\cdot, t, u)\|_p du \\ &\leq \frac{1}{\pi} \int_0^\pi |K_n^i(u)| du \sup_{t>0} t^{-\beta} \|\phi(\cdot, t, u)\|_p \end{aligned} \quad (5.26)$$

Using Lemma 3, we have

$$\begin{aligned} \|w_k(T_n^i, \cdot)\|_{\beta,\infty} &\leq O(1) \int_0^\pi u^{\alpha-\beta} |K_n^i(u)| du \\ &= O(1) \left(\int_0^{\frac{\pi}{\mu_n}} + \int_{\frac{\pi}{\mu_n}}^\pi \right) u^{\alpha-\beta} |K_n^i(u)| du \\ &= O(1) \left[\int_0^{\frac{\pi}{\mu_n}} u^{\alpha-\beta} |K_n^i(u)| du + \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta} |K_n^i(u)| du \right] \\ &= O(1)[I^{III} + J^{III}], \quad (\text{say}) \end{aligned} \quad (5.27)$$

Using Lemma 4(a) in I^{III} , we have

$$\begin{aligned} I^{III} &= \int_0^{\frac{\pi}{\mu_n}} |K_n^i(u)| u^{\alpha-\beta} du \\ &= O(\mu_n) \int_0^{\frac{\pi}{\mu_n}} u^{\alpha-\beta} du \\ &= O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right), \text{ uniformly in } i. \end{aligned} \quad (5.28)$$

Using Lemma 4(b) in J^{III} , we have

$$\begin{aligned} J^{III} &= \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta} |K_n^i(u)| du \\ &= O(\psi(n)) \int_{\frac{\pi}{\mu_n}}^\pi u^{\alpha-\beta-2} du \\ &= O(\psi(n)) \sum_{k=1}^{n-1} \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-\beta-2} du \\ &= O(\psi(n)) \sum_{k=1}^n \int_{\frac{\mu_k}{\mu_{k+1}}}^{\frac{\pi}{\mu_k}} u^{\alpha-\beta-2} du \end{aligned}$$

$$= O(\psi(n)) \sum_{k=1}^n \left(\frac{\mu_{k+1} - \mu_k}{\mu_k^{\alpha-\beta}} \right), \text{ uniformly in } i. \quad (5.29)$$

From (5.28), (5.29) and (5.27), we have

$$\| w_k(T_n^i, \cdot) \|_{\beta, \infty} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n)) \sum_{k=1}^n \left(\frac{\mu_{k+1} - \mu_k}{\mu_k^{\alpha-\beta}} \right), \text{ uniformly in } i. \quad (5.30)$$

From (5.21),(5.25) and (5.30), we have

$$\| T_n^i(\cdot) \|_{B_\infty^\beta(L_p)} = O\left(\frac{1}{\mu_n^{\alpha-\beta}}\right) + O(\psi(n)) \sum_{k=1}^n \left(\frac{\mu_{k+1} - \mu_k}{\mu_k^{\alpha-\beta}} \right), \text{ uniformly in } i. \quad (5.31)$$

This completes the Case 2.

Combining the Case 1 and Case 2, we obtain the proof of the theorem.

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References

- [1] Das, G.: Sublinear funcional and a class of Conservative Matrices, Bull. of the Institute of Mathematics Academia Sinica **15**(1987) 89-106.
- [2] Das, G., Ghosh, T. and Ray, B.K: Degree of Approximation of function by their Fourier series in the generalized Holder metric, proc. Indian Acad. Sci. (Math.Sci) **106**(1996) 139-153.
- [3] Devore Ronald A. Lorentz, G.: Constructive approximation, Springer- Verlay, Berlin Heidelberg New York, 1993.
- [4] King, J.P. : Almost summable sequence, Proc.Amer. Math.Soc.,**17**(1966) 1219-1225.
- [5] Lorentz, G.G. : A contribution to the theory of divergent sequences, Acta Math., **80**(1948),167-190.
- [6] Prossdorf, S.: Zur Konvergenz der Fourier richen Holder stetiger Funktionen math.Nachar, **69**(1975),7-14.
- [7] Stieglitz, Fine Verallgemeinerung des Begriff der Jeponical, **18** (1973) 53-70.
- [8] Wojtaszczyk, P.: A Mathematical Introduction to Wavelets, London Mathematical Society students texts **37**, Cambridge University Press, New York, 1997.
- [9] Zygmund, A.: Smooth fuctions, Duke math. Jounal **12**(1945), 47-56.
- [10] Zygmund,A.: Trigonometric series vols I & II combined, Cambridge Univ. Press, New York, 1993.