

# Pretest Single Stage Shrinkage Estimator for the Shape Parameter of the Power Function Distribution

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## Abstract

This paper focuses on the estimation the shape parameter ( $\alpha$ ) of power function distribution if a prior knowledge ( $\alpha_0$ ) is available about the shape parameter as the initial value and when the scale parameter is known ( $\theta=1$ ) via pretest single stage shrinkage estimator (SSSE) and proposed an optimal acceptable Region (R) for testing this prior knowledge.

Expressions of the Bias, Mean Squared Error [MSE( $\cdot$ )] and Relative Efficiency [R.EFF( $\cdot$ )] for the proposed estimator were derived.

Numerical results about behavior performance of considered estimator are discussed via study the mentioned expressions. These numerical results displayed in annexed tables. Comparisons between the proposed estimator and the classical estimator as well as with some earlier studies were made to show the effectiveness and usefulness of the considered estimator.

**Keywords:** Power Function Distribution, Single Stage Shrinkage Estimator, Prior Knowledge, pre test region, Bias Ratio, Mean Squared Error and Relative Efficiency

## 1. INTRODUCTION

"The Power Function Distribution (PFD) is a flexible distribution as it is able to model the various types of data. It is usually used for the reliability analysis, life time and income distribution data. Meniconi & Barry (1996) compare the PFD with Exponential, Lognormal and Weibull distribution to measure the reliability of electrical components.

They conclude that the PFD is the best distribution to model such types of data. Similarly many probability models are also used to assess the pattern of the income distribution but these models are mathematically more complex to handle. The PFD on the other hand is quite handy in this regard. Ahsanullah & Kabir (1974), Meniconi & Barry (1996), Ali, Woo & Nadarajah (2005), Chang (2007), Sinha, Singh, Singh & Singh (2008) and

Tavangar (2011) define the characteristics of the PFD. Saran & Pandey (2004) estimate the parameters of PFD and they also characterize this distribution. Rahman, Roy & Baizid (2012).

Power function distribution is preferred over exponential, lognormal and Weibull because it provides a better fit for failure data and more appropriate information about reliability and hazard rates"; [1], [2], [3], [4], [5].

A continuous random variable X is said to have Power function distribution if its probability density function is given by

$$f(x, \alpha) = \begin{cases} \alpha x^{\alpha-1} & , 0 < x < 1 \\ 0 & , \text{o.w.} \end{cases} \dots(1)$$

Here,  $\alpha$  is the shape parameter and  $\theta$  is the scale parameter.

We denoted by PD( $\alpha, \theta$ ) to power function distribution with shape parameter  $\alpha$  and scale parameter  $\theta$

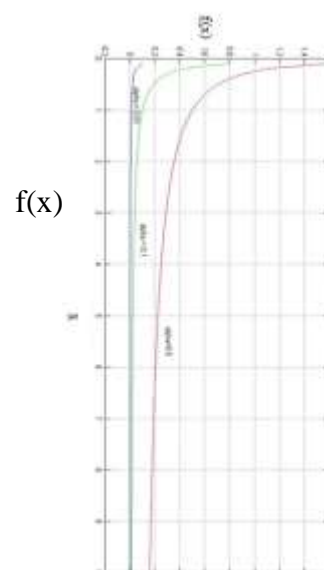


Figure (1): p.d.f of PD( $\alpha, \theta$ ), when  $\theta = 1$

In this paper we introduce the problem for estimating the unknown shape parameter ( $\alpha$ ) of Power function distribution with known scale parameter ( $\theta = 1$ ) when some prior knowledge ( $\alpha_0$ ) regarding true value ( $\alpha$ ) is available using preliminary test single stage shrinkage procedure.

Noted that, the prior knowledge regarding due reasons introduced by Thompson(1968)[8] as well as the classical estimator of  $\alpha$  ( $\hat{\alpha}$ ;MLE) and using shrinkage weight factor [ $\psi_1(\hat{\alpha})$ ],  $0 \leq \psi_1(\hat{\alpha}) \leq 1$  results the what is known as "shrinkage estimator", which though perhaps biased has smaller mean squared error [MSE] than that of  $\hat{\alpha}$ .

Thus, "Thompson-Type" shrinkage estimator will be

$$\psi_1(\hat{\alpha})\hat{\alpha} + (1 - \psi_1(\hat{\alpha}))\alpha_0 \dots (2)$$

Now, the preliminary test single stage shrinkage estimator (SSSE) introduced in this paper is a estimator of level of significance ( $\Delta$ ) for test the hypotheses  $H_0: \alpha = \alpha_0$  vs.  $H_A: \alpha \neq \alpha_0$ .

If  $H_0$  accepted we use the shrinkage estimator defined in (2).

However, if  $H_0$  rejected, we shall take another shrinkage estimator via different shrinkage weight factor  $\psi_2(\cdot)$ ;  $0 \leq \psi_2(\cdot) \leq 1$  and then using the following shrinkage estimator:

$$\psi_2(\hat{\alpha})\hat{\alpha} + (1 - \psi_2(\hat{\alpha}))\alpha_0 \dots (3)$$

Thus, the general form of preliminary test single stage shrinkage estimator (SSSE) will be:

$$\hat{\alpha} = \begin{cases} \psi_1(\hat{\alpha})\hat{\alpha} + (1 - \psi_1(\hat{\alpha}))\alpha_0 & , \text{if } \hat{\alpha} \in R \\ \psi_2(\hat{\alpha})\hat{\alpha} + (1 - \psi_2(\hat{\alpha}))\alpha_0 & , \text{if } \hat{\alpha} \notin R \end{cases} \dots (4)$$

where  $\psi_i(\hat{\alpha})$ ,  $0 \leq \psi_i(\hat{\alpha}) \leq 1$ ,  $i = 1, 2$  is a shrinkage weight factor specifying the belief of  $\hat{\alpha}$  and  $(1 - \psi_i(\hat{\alpha}))$  specifying the belief of  $\alpha_0$  and  $\psi_i(\hat{\alpha})$  may be a function of  $\hat{\alpha}$  or may be a constant (ad hoc basis), while (R) is a pretest region for acceptance the prior knowledge with level of significance  $\Delta$ .

Several authors have been studied preliminary test single stage shrinkage estimator (SSSE) defined in (4), see for example; [1], [6], [7], [8]

The aim of this paper is to estimate the shape parameter ( $\alpha$ ) of two parameters power distribution with known scale parameter ( $\theta = 1$ ) using proposed preliminary test (SSSE) defined in (5) via study the expressions of Bias, Mean squared error and Relative Efficiency of this estimator and display the numerical results for mentioned expressions in annexed tables. Also, study the performance of the consider estimator and make comparisons with the classical estimator as well as with some studies introduced by some authors.

## 2. MAXIMUM LIKELIHOOD ESTIMATOR OF $\alpha$

In this section, we consider the maximum likelihood estimator (MLE) of Power function distribution ( $\alpha, \theta$ ), see [1], [2].

Let  $x_1, x_2, \dots, x_n$  be a random sample of size  $n$  from PD( $\alpha, \theta$ ), then the log-likelihood function  $L(\alpha, \theta)$  can be written as:

$$L(\alpha) = n \ln(\alpha) + n\alpha \ln(\theta) + (\alpha - 1) \sum_{i=1}^n \ln x_i \dots (5)$$

In this paper, we assume that  $\theta$  is known ( $\theta = 1$ ).

The normal equation become

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln x_i = 0 \dots (6)$$

Thus, we obtain the MLE of  $\alpha$ , say  $\hat{\alpha}_{MLE}$  as below

$$\hat{\alpha}_{MLE} = - \frac{n}{\sum_{i=1}^n \ln(x_i)} \dots (7)$$

Let  $y = -\alpha \sum_{i=1}^n \ln x_i \sim G(n, 1)$

The distribution of  $\hat{\alpha}_{MLE}$  is the same as the distribution of  $\frac{n\alpha}{y}$  where  $y$  follows Gamma ( $n, 1$ ).

Therefore

$$\hat{\alpha}_{MLE} = \frac{n\alpha}{y} \dots (8)$$

## 3. PRELIMINARY TEST SINGLE STAGE SHRINKAGE ESTIMATOR (SSSE) $\hat{\alpha}$

In this section, we consider the preliminary test (SSSE) which is defined in (4) when  $\Psi_1(\hat{\alpha}) = 0$  and  $\Psi_2(\hat{\alpha}) = k$  for estimate the shape parameter  $\alpha$  of power function distribution when  $\theta = 1$ .

And R is a pretest region for testing the hypothesis  $H_0: \alpha = \alpha_0$  vs  $H_A: \alpha \neq \alpha_0$  with level of significance

( $\Delta$ ) using test statistic  $T(\hat{\alpha} / \alpha_0) = \frac{2n\alpha_0}{\hat{\alpha}}$ .

$$\text{i.e.; } R = \left[ \frac{2n\alpha_0}{b}, \frac{2n\alpha_0}{a} \right] \dots (9)$$

Where,  $a = (X_{1-\Delta/2, 2n}^2)$  and  $b = (X_{\Delta/2, 2n}^2)$ ,  $\dots (10)$

are respectively the lower and upper 100( $\Delta/2$ ) percentile point of chi-square distribution with degree of freedom  $2n$ .

The expression for Bias of  $\hat{\alpha}$  is

$$\text{Bias}(\hat{\alpha} / \alpha, R) = E(\hat{\alpha} - \alpha) = \int_R (\alpha_0 - \alpha) f(\hat{\alpha}) d\hat{\alpha} + \int_{\bar{R}} (k(\hat{\alpha} - \alpha_0) + (\alpha_0 - \alpha)) f(\hat{\alpha}) d\hat{\alpha}$$

Where,  $\bar{R}$  is the complement region of R in real space and  $f(\hat{\alpha})$  is a p.d.f. of  $\hat{\alpha}$  with the following form

$$f(\hat{\alpha}) = \begin{cases} \frac{\left[ \frac{n}{\hat{\alpha}} \alpha \right]^{n+1} e^{-\frac{n\alpha}{\hat{\alpha}}}}{\Gamma(n)\alpha} & \text{for } \hat{\alpha} > 0, \alpha > 0 \dots (11) \\ 0 & \text{o.w.} \end{cases}$$

We conclude,

$$\text{Bias}(\hat{\theta}/\alpha, R) = \alpha \{ (\zeta - 1) J_0(a^*, b^*) + \frac{k}{n-1} + (1-k)(\zeta - 1) + nk J_1(a^*, b^*) - k J_0(a^*, b^*) + (\zeta - 1)(1-k) J_0(a^*, b^*) \} \dots(12)$$

$$\text{where } J_1(a^*, b^*) = \int_{a^*}^{b^*} y^{-1} \frac{y^{n-1} e^{-y}}{\Gamma(n)} dy; 1 = 0, 1, 2 \dots(13)$$

$$\text{Also, } \zeta = \frac{\alpha_0}{\alpha}, y = \frac{(n-1)\alpha}{\hat{\alpha}}, a^* = \zeta^{-1} \cdot a \text{ and}$$

$$b^* = \zeta^{-1} \cdot b \dots(14)$$

The Bias ratio [B(·)] of  $\hat{\theta}$  is defined as below:-

$$B(\hat{\theta}) = \frac{\text{Bias}(\hat{\theta}/\alpha, R)}{\alpha} \dots(15)$$

The expression of Mean squared error (MSE) of  $\hat{\theta}$  given as

$$\begin{aligned} \text{MSE}(\hat{\theta}/\alpha, R) &= E(\hat{\theta} - \alpha)^2 \\ &= \alpha^2 \left\{ k^2 \frac{n+2}{(n-1)(n-2)} - \frac{2(\zeta-1)}{n-1} + (\zeta-1)^2 - n^2 J_2(a^*, b^*) \right. \\ &\quad \left. - 2n\zeta J_1(a^*, b^*) + \zeta^2 J_0(a^*, b^*) + 2k(\zeta-1) \left[ \frac{1}{n-1} - (\zeta-1) \right] + \right. \\ &\quad \left. (\zeta-1)^2 - 2kn(\zeta-1) \left[ J_1(a^*, b^*) + \frac{1}{n} J_0(a^*, b^*) \right] \right\} \dots(16) \end{aligned}$$

The Efficiency of  $\hat{\alpha}$  relative to the  $\hat{\alpha}$  denoted by R.Eff(  $\hat{\theta}/\alpha, R$ ) defined as

$$\text{R.Eff}(\hat{\theta}/\alpha, R) = \frac{\text{MSE}(\hat{\alpha})}{\text{MSE}(\hat{\theta}/\alpha, R)} \dots(17)$$

See for example; [7], [8].

#### 4. CONCLUSIONS AND NUMERICAL RESULTS

The computations of Relative Efficiency [R.Eff(·)] and Bias Ratio [B(·)] expression were used for the considered estimators  $\hat{\theta}$ . These computations were performed for the constants  $\Delta = 0.05, 0.01, 0.1$ ,  $n = 4, 6, 8, 10, 12, 16, 20, 30$ ,  $k = e^{-\frac{100}{n}}$  and  $\zeta = 0.25(0.25), 2$ . Some of these computations are displayed in tables (1) and (2) for some samples of these constants. The observation mentioned in the tables leads to the following results:

- i. The Relative Efficiency [R.Eff(·)] of  $\hat{\theta}$  are adversely proportional with small value of  $\Delta$ , i.e.  $\Delta = 0.01$  yield highest efficiency.
- ii. The Bias ratio [B(·)] are increasing function with increasing value of  $k$  and  $n$ .
- iii. The Relative Efficiency [R.Eff(·)] of  $\hat{\theta}$  has maximum value when  $\alpha = \alpha_0 (\zeta = 1)$ , for each  $k, n, \Delta$ , and decreasing otherwise ( $\zeta \neq 1$ ). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.

iv. Bias ratio [B(·)] of  $\hat{\theta}$  increases when  $\zeta$  increases.

v. Bias ratio [B(·)] of  $\hat{\theta}$  are reasonably small when  $\alpha = \alpha_0$  and increases otherwise for all  $n$  and  $\Delta$ . This property shown that the proposed estimator  $\hat{\theta}$  is very closely to unbiasedness especially when  $\alpha = \alpha_0$ .

vi. The Relative efficiency [R.Eff(·)] of  $\hat{\theta}$  decreases function with increases value of  $k$  and  $n$ , for each  $\Delta, \zeta$ . This property employ the role of the prior information for proposed shrinkage estimator via takes high weight for prior information which leads to maximum efficiency and reduce the cost of sample.

vii. The Effective Interval [the value of  $\zeta$  that makes R.Eff(·) greater than one] using proposed estimator  $\hat{\theta}$  is [0.5, 1.5]. Here the pretest criterion is very important for guarantee that prior information is very closely to the actual value and prevent it faraway from it, which get optimal effect of the considered estimator to obtain high efficiency.

viii. The considered estimator  $\hat{\theta}$  is better than the classical estimator especially when  $\alpha \approx \alpha_0$ , which is given the effective of  $\hat{\theta}$  when given an important weight of prior knowledge. And the augmentation of efficiency may be reach to tens times.

ix. The proposed estimator  $\hat{\theta}$  has smaller MSE than some existing estimators introduced by authors, see for examples [1].

#### 5. REFERENCES

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Table(1)

Shown the R.E.ff of  $\hat{\theta}$  w.r.t.  $\Delta$ , n and  $\zeta$  when

$$k = e^{-\frac{100}{n}}$$

$\Delta$	n	R.Eff	$\zeta$					
			0.25	0.75	1	1.25	1.75	2
0.05	4	R.Eff( )	1.0071055	8.5716	347.5272	2.6329	.883657	0.4989075
	8	R.Eff( )	0.349546	3.081120	238.989	2.569178	0.295571	0.166691
	16	R.Eff( )	.228255	1.913522	21.6614	1.05324	0.126732	.007161212
0.01	4	R.Eff( )	2.171433	1.694985	14.29253	.832049	.009874	.0055757
	8	R.Eff( )	1.0153918	9.1488148	286.32138	7.33851	0.879730	0.499158
	16	R.Eff( )	0.349549	3.179488	221.442999	2.5272137	295485	.16732071
0.05	16	R.Eff( )	228255	1864801	21.661428	1.112045	.1279409	.07266135
	20	R.Eff( )	217149	1.650754	12.19599	.9352259	.1003849	.0568135

Table (2)

Shown Bias Ratio B(·) of  $\hat{\theta}$  w.r.t.  $\Delta$ , n and  $\zeta$  when

$$k = e^{-\frac{100}{n}}$$

$\Delta$	n	Bias	$\zeta$					
			0.25	0.75	1	1.25	1.75	2
0.01	4	B(·)	(-.70286)	(-.237567)	(.000702)	(.293074)	(.751784)	0.4989075
	8	B(·)	(-.689732)	(-.2304131)	(.008859)	(.254026)	(.750838)	(.9998513)
	16	B(·)	.5550749	(-.1817786)	(0.019417)	(.258693)	(.7506336)	(0.998321)
0.05	4	B(·)	(-.4996574)	(-.164432)	(.0028292)	(.256508)	(.74991)	(.997582)
	8	B(·)	(-.7001389)	(-.0229575)	(.0129981)	(.02585977)	(.0753276)	(1.000337)
	16	B(·)	(-.6897299)	(-.022696)	(0.0119664)	(.0256090)	(.7508839)	(.99781011)
0.05	16	B(·)	(-.555074)	(-.183649)	.01941726	(.250876)	(.7466908)	(.98970729)
	20	B(·)	(-.49965)	(-.1660671)	(.015806)	(.0239371)	(.743093)	(0.98604029)