# Exact Solutions of Some Nonlinear Partial Differential Equations via Extended $\left(\frac{G'}{G}\right)$-Expansion Method

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**Abstract**: *In this paper, we apply the*extended $\left(\frac{G'}{G}\right)$-expansion method*for solving the* Burger's equation, the Korteweg-*de Vries-Burgers* (KdV) equation and the Lax' fifth-order (Lax5)equation. With *the aid of themathematical software Maple, some exact solutions for withequations are successfully.*

**Keywords**: *The*extended $\left(\frac{G'}{G}\right)$-expansion method, *exact solutions, some nonlinear partial differential equations.*

## 1. Introduction

Mathematical modeling of many real phenomena leads to a non-linear ordinary or partial differential equations in various fields of physics and engineering. There are some methods to obtain approximate or exact solutions of these kinds of equations, such as:the extended tanh-function method[1-4],the sub-equation method [5,6], the Bäcklund transform method [7], the Exp-function method [8-17], the simple equation method[18-19], the extended multiple Riccati equation [20],,the Jacobi elliptic function expansion method [21-25], the modified extended tanh with fractional Riccati equation[26-32],the Fractional sub-equation method[33-36], the sine-cosine method [37-39], the $\left(\frac{G'}{G}\right)$-expansion method[40-44],the $\left(\frac{G'}{G},\frac{1}{G}\right)$-expansion method [45-47],the modified simple equation method [48-50],the Kudryashov method [51-53],and so on. In this paper we have considered the followingNPDEs

**(I)- Burger's equation**

$$u_t - u_{xx} - auu_x = 0\,. \qquad (1.1)$$

**(II)- Generalized Burgers-Kdv Equation**

$$u_t + pu^m u_x + qu_{xx} - ru_{xxx} = 0. \quad (1.2)$$

**(III)- Lax' Fifth-Order (Lax5) Equation**

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0. (1.3)$$

This paper is arranged as follows: In Section 2, we give the description for main steps of the extended $\left(\frac{G'}{G}\right)$-expansion method. In Section 3, we apply this method to finding exact solutions for the equationswhich we stated above.

## 2. Description Of Extended $\left(\frac{G'}{G}\right)$-Expansion Method

Consider the following nonlinear evolution equation, say in the two independent variables x, t

$$P(u, D_t u, D_x u, D_x^2 u, D_x^3 u, \dots) = 0.(2.1)$$

Where Pis a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method: [40-44]

**Step1**. Using the wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \qquad (2.2)$$

where cis a constant to be determined later. Then equation (2.1) becomes a nonlinear ordinary differential equation

$$Q(u, u', u'', u''', \dots) = 0, \qquad (2.3)$$

whereQis a polynomial of u and its derivatives and the superscripts indicate the ordinary derivatives with respect to $\xi$. If possible, we should integrate Eq. (2.3) term by term one or more times.

**Step2.** Suppose the solutions of Eq. (2.3) can be expressed as a polynomial of $\left(\frac{G'}{G}\right)$ in the form

$$u(\xi) = \sum_{i=-M}^{M} a_i \left(\frac{G'}{G}\right)^i, \qquad (2.4)$$

where $a_i (i = 0, 1\dots,M)$ in Eq. (2.4) are constants to be determined later. The positive integer M in Eq. (2.4) can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq. (2.3). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = M$ which gives rise to the degree of other expressions as follows:

$$\begin{cases} D\left[\frac{d^q u}{d\xi^q}\right] = M \\ D\left[u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right] = Mp + s(q+M). \end{cases} \quad (2.5)$$

Therefore can get the value of M in Eq. (2.4)

If M is equal to a fractional or negative number we can take the following transformations: [54]

1- When $M = \frac{q}{p}$(where $M = \frac{q}{p}$ is a fraction in lowest terms), we let

$$u(\xi) = v^{\frac{q}{p}}(\xi). \quad (2.6)$$

Substituting Eq. (2.6) into Eq. (2.3) and then determine the value of M in new Eq. (2.3).

2- When M is a negative integer, we let

$$u(\xi) = v^M(\xi). \quad (2.7)$$

Substituting Eq. (2.7) into Eq. (2.3) and return to determine the value of M once again.

The function $G = G(\xi)$ in Eq. (2.4) satisfies the following second order linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (2.8)$$

where $\lambda$ and $\mu$ are real constants to be determined.

**Step3**. Substituting Eq. (2.4) along with Eq. (2.8), into Eq. (2.3), collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, and then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for $a_i, c, \mu, \lambda, k$. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant$(\Omega = \lambda^2 - 4\mu)$, the general solutions of Eq. (2.8) are as follows:

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\Omega}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\Omega\mu}}{2}\xi\right)}{c_1\cosh\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}\xi\right)}\right) - \frac{\lambda^2}{2}, \Omega > 0, \\ \frac{\sqrt{\Omega}}{2}\left(\frac{c_1\sin\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{\Omega}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{\Omega}}{2}\xi\right)}\right) - \frac{\lambda^2}{2}, \ \Omega < 0, \\ \frac{c_2}{c_1+\xi c_2} - \frac{\lambda^2}{2}, \Omega = 0, \end{cases} \quad (2.9)$$

where$c_1; c_2$are arbitrary constants. Then substituting $a_i, c, \mu, \lambda,$ and k along with Eq. (2.9) into Eq. (2.4), we get the solutions of Eq. (2.1).

## 3. Applications

### 3.1-Exact solutions of the Burger's equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - au\frac{\partial u}{\partial x} = 0\,. \quad (3.1)$$

In [55], the author solved Eq. (3.1) by the tanh- coth method and established some exact solutions for it.

Now we will apply the extended $\left(\frac{G'}{G}\right)$-Expansion Method to Eq. (3.1). To begin with, suppose the $u(x,t) = u(\xi), \xi = x - ct$,where c is an arbitrary constant to be determined later, to convert the Eq. (3.1) into the following nonlinear (ODE)

$$c\frac{du}{d\xi} + \frac{d^2u}{d\xi^2} + au\frac{du}{d\xi} = 0, \quad 0 < \alpha < 1.(3.2)$$

Integrating (3.2) once with respect to $\xi$and neglecting the constant of integration, we have

$$cu + \frac{du}{d\xi} + a\frac{u^2}{2} = 0. \quad (3.3)$$

Balancing ( $\frac{du}{d\xi}$ ) with $(u^2)$, we obtain $(M = 1)$. Thus Eq. (3.3) becomes

$$u(\xi) = a_{-1}\left(\frac{G'}{G}\right)^{-1} + a_0 + a_1\left(\frac{G'}{G}\right).(3.4)$$

Using Eq. (3.4) along with Eq. (2.8), we derive**:**

$$\frac{du}{d\xi} = \mu a_{-1}\left(\frac{G'}{G}\right)^{-2} + \lambda a_{-1}\left(\frac{G'}{G}\right)^{-1}$$

$$+(a_{-1} - \mu a_1)\left(\frac{G'}{G}\right)^0 - \lambda a_1\left(\frac{G'}{G}\right)^1 - a_1\left(\frac{G'}{G}\right)^2. (3.5)$$

Substituting Eq. (3.4) and Eq. (3.5) into Eq. (3.3), collecting the coefficients of powers of $\left(\frac{G'}{G}\right)$ and setting them to zero, we obtain the following system of algebraic equations involving the parameters $a_i, (i = 0, 1), \lambda, \mu$and c as follows:

$$\left(\frac{G'}{G}\right)^{-2} : \frac{1}{2}aa_0^2 + \mu a_{-1} = 0,$$

$$\left(\frac{G'}{G}\right)^{-1} : ca_{-1} + \lambda a_{-1} + aa_0a_{-1}$$

$$\left(\frac{G'}{G}\right)^0 : aa_1a_{-1} + a_{-1} + ca_0 - \mu a_1 + \frac{1}{2}aa_0^2 = 0,$$

$$\left(\frac{G'}{G}\right)^1 : aa_0a_1 - \lambda a_1 + ca_1 = 0,$$

$$\left(\frac{G'}{G}\right)^2 : -a_1 + \frac{1}{2}aa_1^2 = 0.$$

Solving this system by Maple, we have the following two sets solutions

$$S_1 = \left\{a_{-1} = 0, a_0 = \frac{\lambda \pm \sqrt{\lambda^2 - 4\mu}}{a}, a_1 = \frac{2}{a}, c = \mp\sqrt{\lambda^2 - 4\mu}\right\}$$

$$S_2 = \left\{a_{-1} = \frac{2\mu}{a}, a_0 = \frac{\pm 4\sqrt{-\mu}}{a}, a_1 = \frac{2}{a}, c = \mp\sqrt{-4\mu}, \lambda = 0\right\}.$$

Substituting the solution set $S_1$ along with Eq. (2.9) into Eq. (3.4), we have the solutions of Eq. (3.1) as follows:

When $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_1(x,t) = \frac{\lambda \pm \sqrt{\Omega}}{a} + \frac{2}{a}\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right), \quad (3.6)$$

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.6), we get

$$u_{1,1}(x,t) = \pm\frac{2\sqrt{-\mu}}{a} + \frac{2\sqrt{-\mu}}{a}\tanh\left(\sqrt{-\mu}(x-ct)\right), (3.7)$$

while, if we setting $c_2 = 0, \lambda$ and $c_2 \neq 0$ in Eq. (3.6), we get

$$u_{1,2}(x,t) = \pm\frac{2\sqrt{-\mu}}{a} + \frac{2\sqrt{-\mu}}{a}\coth\left(\sqrt{-\mu}(x-ct)\right). (3.8)$$

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_2(x,t) = \frac{\lambda \pm \sqrt{\Omega}}{a} + \frac{2}{a}\left(\frac{\sqrt{\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{--\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right) (3.9)$$

In particular, by setting $c_2 = 0, \lambda = 0$ and $c_1 \neq 0$ in Eq. (3.9), we get

$$u_{2,1}(x,t) = \pm\frac{2\sqrt{-\mu}}{a} - \frac{2\sqrt{-\mu}}{a}\tan\left(\sqrt{-\mu}(x-ct)\right), (3.10)$$

while, if we setting $c_1 = 0, \lambda = 0$ and $c_2 \neq 0$ in Eq. (3.9), we get

$$u_{2,2}(x,t) = \pm\frac{2\sqrt{-\mu}}{a} + \frac{2\sqrt{-\mu}}{a}\cot\left(\sqrt{-\mu}(x-ct)\right). (3.11)$$

Similarly, For $S_2$:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_2(x,t) = \pm\frac{4\sqrt{-\mu}}{a} + \frac{2}{a}\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)$$
$$-\frac{2\mu}{a}\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right). (3.12)$$

In particular, by setting $c_2 = 0, \lambda = 0, \mu < 0$ *and* $c_1 \neq 0$ in Eq. (3.12), we get

$$u_{2,1}(x,t) = \pm\frac{4\sqrt{-\mu}}{a} + \frac{2\sqrt{-\mu}}{a}\tanh\left(\sqrt{-\mu}(x-ct)\right) + \frac{2\sqrt{-\mu}}{a}\coth\left(\sqrt{-\mu}(x-ct)\right), \quad (3.13)$$

When $\lambda^2 - 4\mu < 0$,, we obtain the trigonometric functions travelling wave solutions:

$$u_2(x,t) = \pm\frac{4\sqrt{-\mu}}{a} + \frac{2}{a}\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right),$$
$$-\frac{2\mu}{a}\left(\frac{\sqrt{\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right).$$

(3.14)

In particular, by setting $c_1 = 0, \lambda = 0, \mu > 0$ *and* $c_2 \neq 0$ in Eq. (3.14), we get

$$u_{2,2}(x,t) = \pm\frac{4\sqrt{-\mu}}{a} - \frac{2\sqrt{\mu}}{a}\tan\left(\sqrt{\mu}(x-ct)\right) + \frac{2\sqrt{\mu}}{a}\cot\left(\sqrt{\mu}(x-ct)\right). \quad (3.15)$$

### 3.2-Exact Solutions of the Generalized Burgers-Kdv Equation

$$u_t + pu^m u_x + qu_{xx} - ru_{xxx} = 0. \quad (3.16)$$

where $p, q$ and $r$ are real constants, while $m \in \mathbb{Q}$.This equation incorporates the KdV equation $(m = 1, q = 0)$, Modified KdV equation $(m = 2, q = 0)$, generalized KdV equation $(q = 0)$, Burgers equation $(m = 1, r = 0)$, modified Burgers equation $(m = 2, r = 0)$, generalized Burgers equation $(r = 0)$, and the modified Burgers-KdV equation $(m = 2)$, which are integrable. These equations are widely used in such fields as solid-states physics, plasma physics, fluid physics and quantum field theory. In [56], the authors solved Eq. (3.16) by extended tanh method and established some exact solutions for it. Now we will apply the extended $\left(\frac{G'}{G}\right)$-Expansion Method to solve Eq. (3.16). To begin with, suppose that

$$u(x,t) = u(\xi), \xi = x - ct, (3.17)$$

where $c$is an arbitrary constant to be determined later, the equations above converted into the following ODE

$$-cu_\xi + pu^m u_\xi + qu_{\xi\xi} - ru_{\xi\xi\xi} = 0. (3.18)$$

By Integrating Eq. (3.18) and setting constant integration to zero, we get

$$-cu + \frac{p}{m+1}u^{m+1} + qu' - ru'' = 0. \quad (3.19)$$

Balancing $u''$ with $u^{m+1}$ gives $M = \frac{2}{m}$.To obtain a closed form analytic solution, the parameter M should be an integer. To achieve this goal we use a transformation formula $u(\xi) = v^{\frac{2}{m}}(\xi)$. This Eq. (3.19) becomes

$$-cv^2 + \frac{p}{m+1}v^4 + \frac{2q}{m}vv' - \frac{2r}{m}vv'' - \frac{2r(2-m)}{m^2}(v')^2 = 0. (3.20)$$

Balancing $vv''$ with $v^4$ gives $M = 1$.Consequently, Eq. (3.20) has the formula solution:

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + b_{-1}\left(\frac{G'}{G}\right)^{-1}. (3.21)$$

Using Eq. (3.21) along with Eq. (2.8), we derive:

$$u' = \mu a_{-1}\left(\frac{G'}{G}\right)^{-2} + \lambda a_{-1}\left(\frac{G'}{G}\right)^{-1} + (a_{-1} - \mu a_1)\left(\frac{G'}{G}\right)^{0} - \lambda a_1\left(\frac{G'}{G}\right)^{1} - a_1\left(\frac{G'}{G}\right)^{2}, \quad (3.22)$$

$$u'' = 2\mu^2 a_{-1}\left(\frac{G'}{G}\right)^{-3} + 3\lambda\mu a_{-1}\left(\frac{G'}{G}\right)^{-2} + (a_{-1}\lambda^2 + 2\mu a_1)\left(\frac{G'}{G}\right)^{-1} + (\lambda a_{-1} + a_1\lambda\mu)\left(\frac{G'}{G}\right)^{0} + (2a_1\mu + a_1\lambda^2)\left(\frac{G'}{G}\right)^{1} + 3a_1\lambda\left(\frac{G'}{G}\right)^{2} + 2a_1\left(\frac{G'}{G}\right)^{3} \quad (3.23)$$

Substituting Eq. (3.21), Eq. (3.22) and Eq. (3.23) into Eq. (3.20), collecting the coefficients of powers of $\left(\frac{G'}{G}\right)^{i}$ and setting them to zero, we obtain the following system of algebraic equations involving the parameters $a_i$, $b_i$ and $c$ as follows:

$$-2rm^2a_0a_1\lambda\mu - 2rma_0a_1\lambda\mu - 2rma_1^2\mu^2 + 2rm^2a_1^2\mu^2 + 8ra_1a_{-1}\lambda^2 + 16ra_1a_{-1}\mu + 2qma_0a_{-1} + 6pm^2a_1^2a_{-1}^2 + 2qm^2a_0a_{-1} - 2cm^3a_1a_{-1} - 2cm^2a_1a_{-1} + 2rm^2a_{-1}^2 - 2rma_{-1}^2 - cm^3a_0^2 - 4ra_1^2\mu^2 + pm^2a_0^4 - cm^2a_0^2 - 2qma_0a_1\mu - 2rm^2a_0a_{-1}\lambda - 8rm^2a_1a_{-1}\lambda^2 - 2rma_0a - 2qm^2a_0a_1\mu + 12pm^2a_0^2a_1a_{-1} - 16rm^2a_1a_{-1}\mu - 4ra_{-}^2 = 0,$$

$$-8ra_{-1}^2\lambda + 2qm^2a_{-1}^2 - 2cm^3a_0a_{-1} + 4pm^2a_0^3a_{-1} - 16rm^2a_1a_{-1}\lambda\mu + 12pm^2a_0a_1a_{-1}^2 - 6rma_{-1}^2\lambda + 2qma_0a_{-1}\lambda + 16ra_1a_{-1}\lambda\mu - 4rm^2a_0a_{-1}\mu - 2cm^2a_0a_{-1} - 4rma_0a_{-1}\mu + 2qm^2a_0a_{-1}\lambda - 2rma_0a_{-1}\lambda^2 - 2rm^2a_0a_{-1}\lambda^2 + 2qma_{-1}^2 + 2rm^2a_{-1}^2\lambda = 0,$$

$$8ra_1a_{-1}\mu^2 - 6rma_0a_{-1}\lambda\mu - 4ra_{-1}^2\lambda^2 + 2qma_{-1}^2\lambda - 8ra_{-1}^2\mu + 2qm^2a_0a_{-1}\mu - 6rm^2a_0a_{-1}\lambda\mu - 4rma_{-1}^2\lambda^2 - 8rma_{-1}^2\mu + 6pm^2a_0^2a_{-1}^2 - cm^3a_{-1}^2 - cm^2a_{-1}^2 + 4pm^2a_1a_{-1}^3 - 8rm^2a_1a_{-1}\mu^2 + 2qma_0a_{-1}\mu + 2qm^2a_{-1}^2\lambda = 0,$$

$$4pm^2a_0^3a_1 - 2qma_1^2\mu - 4rm^2a_0a_1\mu - 2qma_0a_1\lambda - 2cm^2a_0a_1 - 2rm^2a_0a_1\lambda^2 - 2rma_0a_1\lambda^2 - 16rm^2a_1a_{-1}\lambda + 2rm^2a_1^2\lambda\mu + 16ra_1a_{-1}\lambda - 4rma_0a_1\mu - 6rma_1^2\lambda\mu + 12pm^2a_0a_1^2a_{-1} - 2qm^2a_1^2\mu - 8ra_1^2\lambda\mu - 2qm^2a_0a_1\lambda - 2cm^3a_0a_1 = 0,$$

$$-8rma_1^2\mu - 4rma_1^2\lambda^2 - 2qm^2a_1^2\lambda + 4pm^2a_1^3a_{-1} - cm^3a_1^2 + 8ra_{-1}a_1 - 8rm^2a_{-1}a_1 - 2qm^2a_0a_1 - 6rm^2a_0a_1\lambda - 8ra_1^2\mu - 6rma_0a_1\lambda - cm^2a_1^2 - 4ra_1^2\lambda^2 + 6pm^2a_0^2a_1^2 - 2qma_0a_1 - 2qma_1^2\lambda = 0,$$

$$-2rm^2a_{-1}^2\mu^2 - 6rma_{-1}^2\mu^2 + pm^2a_{-1}^4 - 4ra_{-1}^2\mu^2 = 0,\ 4pm^2a_0a_1^3 - 2qma_1^2 - 4rm^2a_0a_1 - 10rma_1^2\lambda - 4rma_0a_1 - 8ra_1^2\lambda - 2qm^2a_1^2 - 2rm^2a_1^2\lambda = 0,$$

$$-4rma_0a_{-1}\mu^2 - 10rma_{-1}^2\lambda\mu - 2rm^2a_{-1}^2\lambda\mu + 2qm^2a_{-1}^2\mu - 4rm^2a_0a_{-1}\mu^2 - 8ra_{-1}^2\lambda\mu + 2qma_{-1}^2\mu + 4pm^2a_0a_{-1}^3 = 0,$$

$$-2rm^2a_1^2 - 6rma_1^2 - 4ra_1^2 + pm^2a_1^4 = 0.$$

Solving this system by Maple, we have the following sets solutions

$$S_1 = \left\{a_{-1} = 0,\ a_0 = \frac{(qm + r\lambda m + 4r\lambda)\sqrt{2pr(m^2+3m+2)}}{2pmr(m+4)},\ a_1 = \frac{\sqrt{2pr(m^2+3m+2)}}{pm},\ \lambda = \lambda,\ \mu = -\frac{(q^2m^2 - r^2m^2\lambda^2 - 8r^2m\lambda^2 - 16\lambda^2r^2)}{r^2(m^2+8m+16)},\ c = \frac{2(m+2)q^2}{(m+4)^2r}\right\},$$

$$S_2 = \left\{a_{-1} = \frac{q^2m\sqrt{2pr(m^2+3m+2)}}{16pr^2(m^2+8m+16)},\ a_0 = \frac{(qm + r\lambda m + 4r\lambda)\sqrt{2pr(m^2+3m+2)}}{2pmr(m+4)},\ a_1 = \frac{\sqrt{2pr(m^2+3m+2)}}{pm},\ \lambda = \lambda,\ \mu = -\frac{(q^2m^2)}{16r^2(m^2+8m+16)},\ c = \frac{2q^2(m+2)}{r(m^2+8m+16)}\right\}.$$

Substituting the solution set $S_1$ along with Eq. (2.9) into Eq. (3.21), we have the following solutions of Eq. (3.16) as follows:

When $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_1(x,t) = \left[\pm a_0 \pm a_1\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)\right]^{\frac{2}{m}}, \quad (3.24)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.23), we get

$$u_{1,1}(x,t) = \pm\frac{q}{m+4}\sqrt{\frac{m^2+3m+2}{2pr}}\left(1 + \tanh\left(\pm\frac{mq}{2r(m+4)}(x-ct)\right)\right)^{\frac{2}{m}}. \quad (3.25)$$

while, if $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.23), we get

$$u_{1,2}(x,t) = \pm\frac{q}{m+4}\sqrt{\frac{m^2+3m+2}{2pr}}\left(1 + \coth\left(\pm\frac{mq}{2r(m+4)}(x-ct)\right)\right)^{\frac{2}{m}}. \quad (3.26)$$

Where $c = \frac{2(m+2)q^2}{(m+4)^2r}$.

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric functions travelling wave solutions:

$$u_2(x,t) =$$

$$\left[\pm a_0 \pm a_1\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)\right]^{\frac{2}{m}} \quad (3.27)$$

For $S_2$:

When $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_3(x,t) =$$

$$\left[\pm a_0 \pm a_1\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right),\right.$$

$$\left.\pm a_{-1}\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^{-1}\right]^{\frac{2}{m}}. \quad (3.28)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.27), we get

$$u_{3,1}(x,t) = \pm\Pi\left(2+\tanh\left(\pm\frac{mq}{2r(m+4)}(x-ct)\right)\right.$$
$$\left.+\coth\left(\pm\frac{mq}{2r(m+4)}(x-ct)\right)\right)^{\frac{2}{m}} \quad (3.29)$$

When $\Omega = \lambda^2 - 4\mu < 0$, we obtain the trigonometric functions travelling wave solutions

$$u_4(x,t) =$$

$$\left[\pm a_0 \pm a_1\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right),\right.$$

$$\left.\pm a_{-1}\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^{-1}\right]^{\frac{2}{m}}. \quad (3.30)$$

Where $\Pi = \frac{q}{m+4}\sqrt{\frac{m^2+3m+2}{8pr}}$.

### 3.3 -Exact Solutions ofLax' Fifth-Order [(Lax5) Equation

$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0$.(3.32)

This equation solved by [57] by Adomian decomposition method, [58] by extended tanh method, modified [59] by Hirota's bilinear method and established some exact solutions for it. Now we will apply the extended $\left(\frac{G'}{G}\right)$-Expansion Method to solve Eq. (3.16). To begin with, suppose that

$$u(x,t) = u(\xi), \xi = x - ct,$$

where cis a constant to be determined later.

Substituting Eq. (3.31), we get the following (ODE)

$$-cu' + u''''' + 10uu''' + 20u'u'' + 30u^2u' = 0. \quad (3.33)$$

Balancing $u'''''$ with $u'u''$, in the Eq. (3.33), we have $M = 2$. Thus Eq. (3.33) has the formula solution:

$$u(\xi) = a_{-2}\left(\frac{G'}{G}\right)^{-2} + a_{-1}\left(\frac{G'}{G}\right)^{-1} + a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2. \quad (3.34)$$

Using Eq. (3.34) along with Eq. (2.8), we derive:

$$\frac{d}{d\xi}u(\xi) = -2a_2Y^3 + (-a_1 - 2a_2\lambda)Y^2 + (-a_1\lambda - 2a_2\mu)Y + a_{-1} - a_1\mu + \frac{a_{-1}\lambda + 2a_{-2}}{Y} + \frac{2a_{-2}\lambda + a_{-1}\mu}{Y^2} + \frac{2a_{-2}\mu}{Y^3},$$

$$\frac{d^2}{d\xi^2}u(\xi) = 6a_2Y^4 + (10a_2\lambda + 2a_1)Y^3 + (3a_1\lambda + 4a_2\lambda^2 + 8a_2\mu)Y^2 + (a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu)Y + a_1\lambda\mu + 2a_2\mu^2 + a_{-1}\lambda + 2a_{-2} + \frac{6a_{-2}\lambda + 2a_{-1}\mu + a_{-1}\lambda^2}{Y} + \frac{4a_{-2}\lambda^2 + 3a_{-1}\lambda\mu + 8a_{-2}\mu}{Y^2} + \frac{2a_{-1}\mu^2 + 10a_{-2}\lambda\mu}{Y^3} + \frac{6a_{-2}\mu^2}{Y^4},$$

$$\frac{d^3}{d\xi^3}u(\xi) = -24a_2Y^5 + (-6a_1 - 54a_2\lambda)Y^4 + (-40a_2\mu - 38a_2\lambda^2 - 12a_1\lambda)Y^3 + (-8a_2\lambda^3 - 52a_2\lambda\mu - 7a_1\lambda^2 - 8a_1\mu)Y^2 + (-14a_2\lambda^2\mu - 8a_1\lambda\mu - 16a_2\mu^2 - a_1\lambda^3)Y - 6a_2\lambda\mu^2 - a_1\lambda^2\mu + a_{-1}\lambda^2 - 2a_1\mu^2 + 6a_{-2}\lambda + 2a_{-1}\mu + \frac{14a_{-2}\lambda^2 + 16a_{-2}\mu + a_{-1}\lambda^3 + 8a_{-1}\lambda\mu}{Y} + \frac{8a_{-1}\mu^2 + 7a_{-1}\lambda^2\mu + 52a_{-2}\lambda\mu + 8a_{-2}\lambda^3}{Y^2} + \frac{12a_{-1}\lambda\mu^2 + 38a_{-2}\lambda^2\mu + 40a_{-2}\mu^2}{Y^3} + \frac{6a_{-1}\mu^3 + 54a_{-2}\lambda\mu^2}{Y^4} + \frac{24a_{-2}\mu^3}{Y^5},$$

$$\begin{aligned}\frac{d^4}{d\xi^4}u(\xi) &= 120\,a_2\,Y^6 + \left(336\,a_2\,\lambda + 24\,a_1\right)Y^5 + \left(330\,a_2\,\lambda^2 + 240\,a_2\,\mu + 60\,a_1\,\lambda\right)Y^4 + \left(50\,a_1\,\lambda^2 + 130\,a_2\,\lambda^3 + 440\,a_2\,\lambda\,\mu + 40\,a_1\,\mu\right)Y^3 \\ &+ \left(232\,a_2\,\lambda^2\,\mu + 60\,a_1\,\lambda\,\mu + 15\,a_1\,\lambda^3 + 16\,a_2\,\lambda^4 + 136\,a_2\,\mu^2\right)Y^2 + \left(16\,a_1\,\mu^2 + 22\,a_1\,\lambda^2\,\mu + a_1\,\lambda^4 + 120\,a_2\,\lambda\,\mu^2 + 30\,a_2\,\lambda^3\,\mu\right)Y \\ &+ 14\,a_2\,\lambda^2\,\mu^2 + 14\,a_{-2}\,\lambda^2 + a_{-1}\,\lambda^3 + a_1\,\lambda^3\,\mu + 16\,a_{-2}\,\mu + 16\,a_2\,\mu^3 + 8\,a_{-1}\,\lambda\,\mu + 8\,a_1\,\lambda\,\mu^2 \\ &+ \frac{30\,a_{-2}\,\lambda^3 + 22\,a_{-1}\,\lambda^2\,\mu + a_{-1}\,\lambda^4 + 16\,a_{-1}\,\mu^2 + 120\,a_{-2}\,\lambda\,\mu}{Y} \\ &+ \frac{1}{Y^2}\left(15\,a_{-1}\,\lambda^3\,\mu + 232\,a_{-2}\,\lambda^2\,\mu + 136\,a_{-2}\,\mu^2 + 16\,a_{-2}\,\lambda^4 + 60\,a_{-1}\,\lambda\,\mu^2\right) \\ &+ \frac{440\,a_{-2}\,\lambda\,\mu^2 + 40\,a_{-1}\,\mu^3 + 130\,a_{-2}\,\lambda^3\,\mu + 50\,a_{-1}\,\lambda^2\,\mu^2}{Y^3} + \frac{60\,a_{-1}\,\lambda\,\mu^3 + 330\,a_{-2}\,\lambda^2\,\mu^2 + 240\,a_{-2}\,\mu^3}{Y^4} \\ &+ \frac{24\,a_{-1}\,\mu^4 + 336\,a_{-2}\,\lambda\,\mu^3}{Y^5} + \frac{120\,a_{-2}\,\mu^4}{Y^6},\end{aligned}$$

$$\begin{aligned}\frac{d^5}{d\xi^5}u(\xi) &= -720\,a_2\,Y^7 + \left(-2400\,a_2\,\lambda - 120\,a_1\right)Y^6 + \left(-1680\,a_2\,\mu - 3000\,a_2\,\lambda^2 - 360\,a_1\,\lambda\right)Y^5 + \left(-1710\,a_2\,\lambda^3 - 240\,a_1\,\mu - 3960\,a_2\,\lambda\,\mu - 390\,a_1\,\lambda^2\right)Y^4 \\ &+ \left(-480\,a_1\,\lambda\,\mu - 3104\,a_2\,\lambda^2\,\mu - 422\,a_2\,\lambda^4 - 180\,a_1\,\lambda^3 - 1232\,a_2\,\mu^2\right)Y^3 \\ &+ \left(-136\,a_1\,\mu^2 - 32\,a_2\,\lambda^5 - 292\,a_1\,\lambda^2\,\mu - 31\,a_1\,\lambda^4 - 1712\,a_2\,\lambda\,\mu^2 - 884\,a_2\,\lambda^3\,\mu\right)Y^2 \\ &+ \left(-a_1\,\lambda^5 - 52\,a_1\,\lambda^3\,\mu - 272\,a_2\,\mu^3 - 584\,a_2\,\lambda^2\,\mu^2 - 62\,a_2\,\lambda^4\,\mu - 136\,a_1\,\lambda\,\mu^2\right)Y + 16\,a_{-1}\,\mu^2 \\ &+ a_{-1}\,\lambda^4 + 120\,a_{-2}\,\lambda\,\mu - 120\,a_2\,\lambda\,\mu^3 + 22\,a_{-1}\,\lambda^2\,\mu - 16\,a_1\,\mu^3 + 30\,a_{-2}\,\lambda^3 - a_1\,\lambda^4\,\mu - 22\,a_1\,\lambda^2\,\mu^2 - 30\,a_2\,\lambda^3\,\mu^2 \\ &+ \frac{1}{Y}\left(584\,a_{-2}\,\lambda^2\,\mu + 136\,a_{-1}\,\lambda\,\mu^2 + 272\,a_{-2}\,\mu^2 + a_{-1}\,\lambda^5 + 62\,a_{-2}\,\lambda^4 + 52\,a_{-1}\,\lambda^3\,\mu\right) \\ &+ \frac{1}{Y^2}\left(136\,a_{-1}\,\mu^3 + 32\,a_{-2}\,\lambda^5 + 31\,a_{-1}\,\lambda^4\,\mu + 1712\,a_{-2}\,\lambda\,\mu^2 + 292\,a_{-1}\,\lambda^2\,\mu^2 + 884\,a_{-2}\,\lambda^3\,\mu\right) \\ &+ \frac{1}{Y^3}\left(480\,a_{-1}\,\lambda\,\mu^3 + 1232\,a_{-2}\,\mu^3 + 422\,a_{-2}\,\lambda^4\,\mu + 180\,a_{-1}\,\lambda^3\,\mu^2 + 3104\,a_{-2}\,\lambda^2\,\mu^2\right) \\ &+ \frac{240\,a_{-1}\,\mu^4 + 1710\,a_{-2}\,\lambda^3\,\mu^2 + 3960\,a_{-2}\,\lambda\,\mu^3 + 390\,a_{-1}\,\lambda^2\,\mu^3}{Y^4} \\ &+ \frac{1680\,a_{-2}\,\mu^4 + 360\,a_{-1}\,\lambda\,\mu^4 + 3000\,a_{-2}\,\lambda^2\,\mu^3}{Y^5} + \frac{2400\,a_{-2}\,\lambda\,\mu^4 + 120\,a_{-1}\,\mu^5}{Y^6} + \frac{720\,a_{-2}\,\mu^5}{Y^7}.\end{aligned}$$

Where $Y = \left(\frac{G'}{G}\right)$.

Substituting the function $u$ and its derivatives into the Eq. (3.33), setting the coefficients of $\left(\frac{G'}{G}\right)$, $(i = 0, 1, 2, 3, 4)$ to zero, we obtain the following system of algebraic equations:

$$\left(\frac{G'}{G}\right)^{-7} : 720\,a_{-2}\,\mu^5 + 480\,a_{-2}^2\,\mu^3 + 60\,a_{-2}^3\,\mu = 0,$$

$$\begin{aligned}\left(\frac{G}{G}\right)^{-6} &: 150\,a_{-1}\,a_{-2}^2\,\mu + 2400\,a_{-2}\,\lambda\,\mu^4 + 1180\,a_{-2}^2\,\lambda\,\mu^2 \\ &+ 500\,a_{-1}\,a_{-2}\,\mu^3 + 60\,a_{-2}^3\,\lambda + 120\,a_{-1}\,\mu^5 = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{-5} &: 240\,a_0\,a_{-2}\,\mu^3 + 60\,a_{-2}^3 + 1680\,a_{-2}\,\mu^4 + 150\,a_{-1}\,a_{-2}^2\,\lambda \\ &+ 100\,a_{-1}^2\,\mu^3 + 1180\,a_{-1}\,a_{-2}\,\lambda\,\mu^2 + 120\,a_{-1}^2\,a_{-2}\,\mu + 120\,a_0\,a_{-2}^2\,\mu \\ &+ 3000\,a_{-2}\,\lambda^2\,\mu^3 + 940\,a_{-2}^2\,\lambda^2\,\mu + 960\,a_{-2}^2\,\mu^2 + 360\,a_{-1}\,\lambda\,\mu^4 = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{-4} &: 120\,a_0\,a_{-2}^2\,\lambda + 120\,a_{-1}^2\,a_{-2}\,\lambda + 1480\,a_{-2}^2\,\lambda\,\mu + 240\,a_{-1}\,\mu^4 \\ &+ 890\,a_{-1}\,a_{-2}\,\lambda^2\,\mu + 920\,a_{-1}\,a_{-2}\,\mu^2 + 60\,a_0\,a_{-1}\,\mu^3 + 220\,a_{-1}^2\,\lambda\,\mu^2 \\ &+ 180\,a_0\,a_{-2}\,a_{-1}\,\mu + 240\,a_{-2}^2\,\lambda^3 + 150\,a_{-1}\,a_{-2}^2 + 3960\,a_{-2}\,\lambda\,\mu^3 \\ &+ 30\,a_{-1}^3\,\mu + 540\,a_0\,a_{-2}\,\lambda\,\mu^2 + 390\,a_{-1}\,\lambda^2\,\mu^3 + 1710\,a_{-2}\,\lambda^3\,\mu^2 \\ &+ 120\,a_1\,a_{-2}\,\mu^3 + 90\,a_1\,a_{-2}^2\,\mu = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{-3} &: 540\,a_{-2}^2\,\lambda^2 + 1232\,a_{-2}\,\mu^3 + 120\,a_{-1}^2\,a_{-2} + 120\,a_0\,a_{-2}^2 + 30 \\ &a_{-1}^3\,\lambda + 60\,a_0\,a_{-1}^2\,\mu - 2\,c\,a_{-2}\,\mu + 80\,a_2\,a_{-2}\,\mu^3 + 90\,a_1\,a_{-2}^2\,\lambda \\ &+ 210\,a_{-1}\,a_{-2}\,\lambda^3 + 422\,a_{-2}\,\lambda^4\,\mu + 20\,a_1\,a_{-1}\,\mu^3 + 180\,a_{-1}\,\lambda^3\,\mu^2 \\ &+ 400\,a_0\,a_{-2}\,\mu^2 + 60\,a_0^2\,a_{-2}\,\mu + 480\,a_{-1}\,\lambda\,\mu^3 + 3104\,a_{-2}\,\lambda^2\,\mu^2 \\ &+ 60\,a_2\,a_{-2}^2\,\mu + 160\,a_{-1}^2\,\mu^2 + 560\,a_{-2}^2\,\mu + 120\,a_{-1}\,a_{-2}\,a_1\,\mu \\ &+ 180\,a_0\,a_{-2}\,a_{-1}\,\lambda + 260\,a_1\,a_{-2}\,\lambda\,\mu^2 + 1320\,a_{-1}\,a_{-2}\,\lambda\,\mu \\ &+ 380\,a_0\,a_{-2}\,\lambda^2\,\mu + 120\,a_0\,a_{-1}\,\lambda\,\mu^2 + 150\,a_{-1}^2\,\lambda^2\,\mu = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^{-2} &: 520\,a_0\,a_{-2}\,\lambda\,\mu + 80\,a_0\,a_{-2}\,\lambda^3 + 180\,a_1\,a_{-2}\,\mu^2 \\ &+ 1712\,a_{-2}\,\lambda\,\mu^2 + 200\,a_{-1}^2\,\lambda\,\mu + 430\,a_{-1}\,a_{-2}\,\lambda^2 - c\,a_{-1}\,\mu \\ &+ 80\,a_0\,a_{-1}\,\mu^2 + 20\,a_2\,a_{-1}\,\mu^3 + 60\,a_0^2\,a_{-2}\,\lambda + 30\,a_0^2\,a_{-1}\,\mu \\ &+ 292\,a_{-1}\,\lambda^2\,\mu^2 + 31\,a_{-1}\,\lambda^4\,\mu + 460\,a_{-1}\,a_{-2}\,\mu + 60\,a_0\,a_{-1}^2\,\lambda \\ &+ 30\,a_1\,a_{-1}^2\,\mu + 60\,a_2\,a_{-2}^2\,\lambda + 380\,a_{-2}^2\,\lambda + 136\,a_{-1}\,\mu^3 + 90\,a_1\,a_{-2}^2 \\ &- 2\,c\,a_{-2}\,\lambda + 180\,a_0\,a_{-2}\,a_{-1} + 884\,a_{-2}\,\lambda^3\,\mu + 30\,a_{-1}^3 \\ &+ 70\,a_0\,a_{-1}\,\lambda^2\,\mu + 60\,a_0\,a_{-2}\,a_1\,\mu + 60\,a_2\,a_{-2}\,a_{-1}\,\mu \\ &+ 40\,a_1\,a_{-1}\,\lambda\,\mu^2 + 120\,a_{-1}\,a_{-2}\,a_1\,\lambda + 160\,a_2\,a_{-2}\,\lambda\,\mu^2 \\ &+ 170\,a_1\,a_{-2}\,\lambda^2\,\mu + 30\,a_{-1}^2\,\lambda^3 + 32\,a_{-2}\,\lambda^5 = 0,\end{aligned}$$

$$\left(\frac{G'}{G}\right)^{-1}: 62a_{-2}\lambda^4 + 272a_{-2}\mu^2 + 60a_0a_{-1}^2 + 60a_2a_{-2}^2 + 60a_0^2a_{-2} + a_{-1}\lambda^5 + 60a_{-1}^2\mu + 30a_1a_{-1}^2\lambda + 140a_0a_{-2}\lambda^2 + 260a_{-1}a_{-2}\lambda + 52a_{-1}\lambda^3\mu + 584a_{-2}\lambda^2\mu + 30a_1a_{-2}\lambda^3 + 30a_0^2a_{-1}\lambda + 80a_2a_{-2}\mu^2 + 120a_{-1}a_{-2}a_1 + 10a_0a_{-1}\lambda^3 + 20a_1a_{-1}\mu^2 + 160a_0a_{-2}\mu - ca_{-1}\lambda + 136a_{-1}\lambda\mu^2 + 50a_{-1}^2\lambda^2 + 20a_2a_{-1}\lambda\mu^2 + 80a_2a_{-2}\lambda^2\mu + 80a_{-2}^2 + 200a_1a_{-2}\lambda\mu + 80a_0a_{-1}\lambda\mu + 60a_2a_{-2}a_{-1}\lambda + 60a_0a_{-2}a_1\lambda + 20a_1a_{-1}\lambda^2\mu - 2ca_{-2} = 0,$$

$$\left(\frac{G'}{G}\right)^{0}: a_{-1}\lambda^4 + 30a_1a_{-1}^2 - 10a_0a_1\lambda^2\mu + 30a_0^2a_{-1} + 40a_{-1}a_{-2} - 30a_2a_{-1}\lambda^2\mu + 40a_1a_{-2}\mu - 120a_2\lambda\mu^3 - 20a_1^2\mu^2\lambda - a_1\lambda^4\mu + ca_1\mu + 22a_{-1}\lambda^2\mu + 120a_{-2}\lambda\mu - 22a_1\lambda^2\mu^2 - 40a_2a_{-1}\mu^2 + 30a_1a_{-2}\lambda^2 - 30a_2\lambda^3\mu^2 - 60a_0a_2\lambda\mu^2 + 60a_2a_{-2}a_{-1} - 60a_2a_{-2}a_1\mu - 30a_1^2a_{-1}\mu + 60a_0a_{-2}a_1 - 30a_0^2a_1\mu + 16a_{-1}\mu^2 + 30a_{-2}\lambda^3 + 10a_0a_{-1}\lambda^2 - 20a_0a_1\mu^2 + 20a_0a_{-1}\mu + 60a_0a_{-2}\lambda - 40a_1\mu^3a_2 - 60a_0a_2a_{-1}\mu + 20a_{-1}^2\lambda - ca_{-1} - 16a_1\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{1}: -a_1\lambda^5 - 80a_2^2\mu^3 - 272a_2\mu^3 - 62a_2\lambda^4\mu - 80a_2a_{-2}\lambda^2 - 20a_1a_{-2}\lambda + ca_1\lambda - 20a_1a_{-1}\lambda^2 - 52a_1\lambda^3\mu - 136a_1\lambda\mu^2 - 30a_1^2a_{-1}\lambda - 60a_2^2a_{-2}\mu - 60a_0a_1^2\mu + 2ca_2\mu - 20a_1a_{-1}\mu - 30a_0^2a_1\lambda - 80a_2a_{-2}\mu - 60a_0^2a_2\mu - 50a_1^2\lambda^2\mu - 10a_0a_1\lambda^3 - 30a_2a_{-1}\lambda^3 - 160a_0a_2\mu^2 - 60a_2a_{-2}a_1\lambda - 584a_2\lambda^2\mu^2 - 60a_1^2\mu^2 - 60a_0a_2a_{-1}\lambda - 140a_0a_2\lambda^2\mu - 80a_0a_1\lambda\mu - 260a_1a_2\lambda\mu^2 - 200a_2a_{-1}\lambda\mu - 120a_1a_2a_{-1}\mu = 0,$$

$$\left(\frac{G'}{G}\right)^{2}: -30a_1^2a_{-1} - 520a_0a_2\lambda\mu - 430a_1a_2\lambda^2\mu - 120a_1a_2a_{-1}\lambda - 180a_0a_2a_1\mu - 30a_0^2a_1 - 60a_0a_2a_{-1} - 170a_2a_{-1}\lambda^2 - 160a_2a_{-2}\lambda - 1712a_2\lambda\mu^2 - 292a_1\lambda^2\mu - 60a_2a_{-2}a_1 + 2ca_2\lambda - 90a_2^2a_{-1}\mu - 80a_0a_1\mu - 460a_1a_2\mu^2 - 380a_2^2\lambda\mu^2 - 884a_2\lambda^3\mu - 200a_1^2\lambda\mu - 30a_1^3\mu - 30a_1^2\lambda^3 - 70a_0a_1\lambda^2 - 32a_2\lambda^5 - 40a_1a_{-1}\lambda - 80a_0a_2\lambda^3 - 60a_0^2a_2\lambda - 60a_0a_1^2\lambda - 60a_2^2a_{-2}\lambda + ca_1 - 180a_2a_{-1}\mu - 20a_1a_{-2} - 31a_1\lambda^4 - 136a_1\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{3}: -260a_2a_{-1}\lambda - 90a_2^2a_{-1}\lambda - 20a_1a_{-1} - 1320a_1a_2\lambda\mu - 180a_0a_2a_1\lambda - 80a_2a_{-2} - 60a_0^2a_2 - 60a_2^2a_{-2} - 30a_1^3\lambda - 160a_1^2\mu - 3104a_2\lambda^2\mu - 180a_1\lambda^3 - 150a_1^2\lambda^2 - 560a_2^2\mu^2 - 422a_2\lambda^4 + 2ca_2 - 60a_0a_1^2 - 1232a_2\mu^2 - 380a_0a_2\lambda^2 - 120a_0a_2^2\mu - 480a_1\lambda\mu - 120a_0a_1\lambda - 540a_2^2\lambda^2\mu - 400a_0a_2\mu - 120a_1^2a_2\mu - 210a_1a_2\lambda^3 - 120a_1a_2a_{-1} = 0,$$

$$\left(\frac{G'}{G}\right)^{4}: -3960a_2\lambda\mu - 120a_0a_2^2\lambda - 1480a_2^2\lambda\mu - 120a_1^2a_2\lambda - 920a_1a_2\mu - 180a_0a_2a_1 - 1710a_2\lambda^3 - 30a_1^3 - 150a_1a_2^2\mu - 90a_2^2a_{-1} - 120a_2a_{-1} - 60a_0a_1 - 890a_1a_2\lambda^2 - 540a_0a_2\lambda - 390a_1\lambda^2 - 240a_1\mu - 220a_1^2\lambda - 240a_2^2\lambda^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{5}: -940a_2^2\lambda^2 - 1180a_1a_2\lambda - 120a_0a_2^2 - 3000a_2\lambda^2 - 360a_1\lambda - 240a_0a_2 - 120a_1^2a_2 - 60a_2^3\mu - 150a_1a_2^2\lambda - 1680a_2\mu - 960a_2^2\mu - 100a_1^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{6}: -60a_2^3\lambda - 2400a_2\lambda - 150a_1a_2^2 - 1180a_2^2\lambda - 120a_1 - 500a_1a_2 = 0,$$

$$\left(\frac{G'}{G}\right)^{7}: -60a_2^3 - 720a_2 - 480a_2^2 = 0.$$

Solving this system by Maple, we have the following sets solutions

$$S_1 = \left\{a_{-2} = 0, a_{-1} = 0, a_0 = -\frac{\lambda^2}{2} - 4\mu, a_1 = -6\lambda, a_2 = -6, c = 56\mu^2 + \frac{7\lambda^4}{2} - 28\lambda^2\mu, \mu = \mu, \lambda = \lambda\right\}$$

$$S_2 = \left\{a_{-2} = 0, a_{-1} = 0, a_0 = -\frac{1}{6}\lambda^2 - \frac{4}{3}\mu \pm \frac{\sqrt{40\lambda^2\mu - 80\mu^2 - 5\lambda^4 + 30c}}{30}, a_1 = -2\lambda, a_2 = -2, c = c, \mu = \mu, \lambda = \lambda\right\}$$

$$S_3 = \left\{a_{-2} = -6\mu^2, a_{-1} = -6\lambda\mu, a_0 = -\frac{\lambda^2}{2} - 4\mu, a_1 = 0, a_2 = 0, c = 56\mu^2 + \frac{7\lambda^4}{2} - 28\lambda^2\mu, \mu = \mu, \lambda = \lambda\right\}$$

$$S_4 = \left\{a_{-2} = -2\mu^2, a_{-1} = -2\lambda\mu, a_0 = \frac{\lambda^2}{6} - \frac{4}{3} \pm \frac{1}{30}\sqrt{\Theta}, a_1 = 0, a_2 = 0, c = c, \mu = \mu, \lambda = \lambda\right\},$$

Where

$$\Theta = -5\lambda^4 + 400\lambda^2 + 1600 - 1680\mu^2 - 360\lambda^2\mu + 30c$$

Substituting the solution set $S_1$ along with Eq. (2.9) into Eq. (3.34), we have the solutions following of Eq. (3.31)

When $\Omega = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_1(x,t) = -\frac{\lambda^2}{2} - 4\mu -6\lambda\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right) -6\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^2, \quad (3.35)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.35), we get

$$u_{1,1}(x,t)\,\frac{(4\mu-\lambda^2)}{2}\left(1-3\mathrm{sech}^2\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right). \quad (3.36)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.35), we get

$$u_{1,2}(x,t) = \frac{(4\mu-\lambda^2)}{2}\left(1+3\mathrm{csch}^2\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right). \quad (3.37)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_2(x,t) = -\frac{\lambda^2}{2} - 4\mu -6\lambda\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right) -6\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{-\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^2 \quad (3.38)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.38), we get

$$u_{2,1}(x,t) = \frac{(4\mu-\lambda^2)}{2}\left(1-3\sec^2\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right). \quad (3.39)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.39), we get

$$u_{2,2}(x,t) = \frac{(4\mu-\lambda^2)}{2}\left(1-3\csc^2\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right). \quad (3.40)$$

When $\lambda^2 - 4\mu = 0$, we have

$$u_3(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\left(\frac{c_2}{c_1+(x-ct)c_2}-\frac{\lambda}{2}\right) -6\left(\frac{c_2}{c_1+(x-ct)c_2}-\frac{\lambda}{2}\right)^2. \quad (3.41)$$

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.41), we get

$$u_{3,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\left(\frac{1}{(x-ct)}-\frac{\lambda}{2}\right) - 6\left(\frac{c_2}{(x-ct)}-\frac{\lambda}{2}\right)^2 -6\left(\frac{c_2}{(x-ct)}-\frac{\lambda}{2}\right)^2 \quad (3.42)$$

For $S_2$:

where $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_4(x,t) = a_0 -2\lambda\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right) -2\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)+c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^2, \quad (3.43)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.43), we get

$$u_{4,1}(x,t) = \pm\frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30} + \frac{(4\mu-\lambda^2)}{6}\left(1-3\mathrm{sech}^2\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right), \quad (3.44)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.43), we get

$$u_{4,2}(x,t) = \pm\frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30} + \frac{(4\mu-\lambda^2)}{6}\left(1+3\mathrm{csch}^2\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right), \quad (3.45)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_5(x,t) = a_0 -2\lambda\left(\frac{\sqrt{\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right) -2\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^2, \quad (3.46)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.46), we get

$$u_{5,1}(x,t) = \pm\frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30} + \frac{(4\mu-\lambda^2)}{6}\left(1-3\sec^2\left(\frac{\sqrt{-\Omega}}{2}(x-ct, \quad (3.47)\right.\right.$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.46), we get

$$u_{5,2}(x,t) = \pm\frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30} + \frac{(4\mu-\lambda^2)}{6}\left(1-3\csc^2\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right), \quad (3.48)$$

When $\lambda^2 - 4\mu = 0$, we have the solutions:

$$u_6(x,t) = a_0 - 2\lambda\left(\frac{c_2}{c_1+(x-ct)c_2}-\frac{\lambda}{2}\right) -2\left(\frac{c_2}{c_1+(x-ct)c_2}-\frac{\lambda}{2}\right)^2. \quad (3.49)$$

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.49), we get

$$u_{6,1}(x,t) = \frac{1}{30\xi^2}(10\lambda^2(x-ct)^2 - 40\mu(x-ct)^2 - 60 \pm \frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30}) \quad (3.50)$$

Where $a_0 = -\frac{4\mu}{3} - \frac{\lambda^2}{6} \pm \frac{\sqrt{-80\mu^2+40\lambda^2\mu-5\lambda^4+30c}}{30}$

For $S_3$:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_7(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-1} - 6\mu^2\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-2}, \quad (3.51)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.51), we get

$$u_{7,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-1} - \left(\frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-2}, \quad (3.52)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.51), we get

$$u_{7,2}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{\Omega}}{2}\coth\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-1} - \left(\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-2}. \quad (3.53)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solutions:

$$u_8(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) + c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) + c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-1} - 6\mu^2\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) + c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) + c_2\sin\left(\frac{-\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-2}, \quad (3.54)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.54), we get

$$u_{8,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{-\Omega}}{2}\tan\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-1} - \left(\frac{\sqrt{\lambda^2-4\mu}}{2}\tan\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-2}, \quad (3.55)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.54), we get

$$u_{8,2}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{-\Omega}}{2}\cot\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-1} - \left(\frac{\sqrt{\lambda^2-4\mu}}{2}\cot\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-2}. \quad (3.56)$$

When $\lambda^2 - 4\mu = 0$, we have the solutions:

$$u_9(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{c_2}{c_1 + (x-ct)c_2} - \frac{\lambda}{2}\right)^{-1} - 6\mu^2\left(\frac{c_2}{c_1+(x-ct)c_2} - \frac{\lambda}{2}\right)^{-2}, \quad (3.57)$$

In particular, by setting $c_1 = 0$ and $c_2 \neq 0$ in Eq. (3.57), we get

$$u_{9,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{1}{(x-ct)} - \frac{\lambda}{2}\right)^{-1} - 6\mu^2\left(\frac{1}{(x-ct)} - \frac{\lambda}{2}\right)^{-2}. \quad (3.58)$$

For $S_4$:

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solutions:

$$u_{10}(x,t) = a_0 - 2\lambda\mu\left(\frac{\sqrt{\Omega}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-1} - 2\mu^2\left(\frac{\sqrt{4\mu-\lambda^2}\left(c_1\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cosh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) + c_2\sinh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)\right)} - \frac{\lambda}{2}\right)^{-2}, \quad (3.59)$$

In particular, by setting $c_2 = 0$ and $c_1 \neq 0$ in Eq. (3.59), we get

$$u_{10,1}(x,t) = -\frac{\lambda^2}{2} - 4\mu - 6\lambda\mu\left(\frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-1} - \left(\frac{\sqrt{\Omega}}{2}\tanh\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right) - \frac{1}{2}\lambda\right)^{-2}, \quad (3.60)$$

while, if $c_1 = 0, c_2 \neq 0$ in Eq. (3.59), we get

$$u_{10,2}(x,t) = -\frac{\lambda^2}{2} - 4\mu$$

$$-6\lambda\mu\left(\frac{\sqrt{\Omega}}{2}\coth\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)-\frac{1}{2}\lambda\right)^{-1}-\left(\frac{\sqrt{\Omega}}{2}\coth\left(\frac{\sqrt{\Omega}}{2}(x-ct)\right)-12\lambda-2.\right.\quad(3.61)$$

When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solutions:

$$u_{11}(x,t)=a_0-2\lambda\mu\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^{-1}-2\mu^2\left(\frac{\sqrt{-\Omega}\left(-c_1\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}{2\left(c_1\cos\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)+c_2\sin\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)\right)}-\frac{\lambda}{2}\right)^{-2},\quad(3.62)$$

In particular, by setting $c_2=0$ and $c_1\neq 0$ in Eq. (3.62), we get

$$u_{11,1}(x,t)=-\frac{\lambda^2}{2}-4\mu-6\lambda\mu\left(\frac{\sqrt{-\Omega}}{2}\tan\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)-\frac{1}{2}\lambda\right)^{-1}-\left(\frac{\sqrt{\Omega}}{2}\tan\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)-12\lambda-2,\right.\quad(3.63)$$

while, if $c_1=0, c_2\neq 0$ in Eq. (3.62), we get

$$u_{11,2}(x,t)=-\frac{\lambda^2}{2}-4\mu-6\lambda\mu\left(\frac{\sqrt{-\Omega}}{2}\cot\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)-\frac{1}{2}\lambda\right)^{-1}-\left(\frac{\sqrt{-\Omega}}{2}\cot\left(\frac{\sqrt{-\Omega}}{2}(x-ct)\right)-\frac{1}{2}\lambda\right)^{-2}.\quad(3.64)$$

When $\lambda^2-4\mu=0$, we have

$$u_{12}(x,t)=a_0-2\lambda\mu\left(\frac{c_2}{c_1+c_2(x-ct)}-\frac{1}{2}\lambda\right)^{-1}-2\mu^2\left(\frac{c_2}{c_1+c_2(x-ct)}-\frac{1}{2}\lambda\right)^{-2},\quad(3.65)$$

In particular, by setting $c_1=0$ and $c_2\neq 0$ in Eq. (2.65), we get

$$u_{12,1}(x,t)=a_0-2\lambda\mu\left(\frac{1}{(x-ct)}-\frac{1}{2}\lambda\right)^{-1}-2\mu^2\left(\frac{1}{(x-ct)}-\frac{1}{2}\lambda\right)^{-2}.\quad(3.66)$$

## 5. Conclusions

In this paper, we successfully use the extended $\left(\frac{G'}{G}\right)$-Expansion Methodto solve some non-linear partial differential equations**.** This method is reliable and efficient**.** By comparing the results of subsection **(3.1)** with the results of [55], we conclude that the results: (3.7),(3.8) and (3.13) are in agreement whit the results: (71), (72) and (73) of [55], respectively, when $\mu=\frac{-c^2}{4}$. this shows that our results are more general. Comparing the results of subsection **(3.2)** with the results of [56],we conclude that the results: (3.24),(3.25) and (3.28) are agreement with the results of [56].This shows that our results are more general. The solutions obtained in subsection **(3.3)** have not been reported in the literature so far. According theresultsof sub-sec. **(3.1)** and sub-sec. **(3.2)**, we conclude that the$\left(\frac{G'}{G}\right)$-Expansion Method is more effective and general than of extended tanh method.

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