On the Degree Distance of Standard Graphs

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Abstract

In this paper, we have calculated the degree distance of Standard graphs.

1**. Introduction:**

Throughout this paper, by a graph, we mean a non- empty, finite, connected and simple one.

For standard notation, terminology and definitions, we refer Bondy and Murthy [1].

A graph invariant is a mapping on the concerned graph that does not depend on the labeling of its vertices. Such quantities are also known as topological indices. The Wiener Index [5] and the Zagreb index [4] are of that type.

Let G be a graph; V(G) and E(G) represent its vertex set and edge set respectively. For any v $\in$ V(G), $\deg_G(v)$ denotes the degree of v in G.

When there is only one graph under consideration, we omit the subscript 'G' anddenote V(G), E(G) etc. by V,E etc.

§2. Preliminaries

Dobrynin and Kocdetova [2] introduced a new graph invariant namely, 'degree distance', defined as follows:

Definition 2.1: Let G be a graph with vertex set V.

(i) The degree distance of a vertex u of the graph G, denoted by D' (u), is defined as

$$D'(u) = D(u)\deg(u), \text{ where}$$

$$D(u) = \sum_{v \in V} d(u,v).$$

(ii) The degree distance of G, denoted by D' (G), is defined as

$$D'(G) = \sum_{u \in V} D'(u)$$

$$= \sum_{u \in V} D(u)\deg(u)$$

$$= \sum_{u \in V(G)} \left\{ \sum_{v \in V} d(u,v) \right\} \{\deg(u)\}$$

$$= \frac{1}{2} \sum_{u,v \in V(G)} d(u,v)\{\deg(u) + \deg(v)\}.$$

When we denote $V(G) = \{v_1, v_2, \ldots, v_n\}$, then

$$D'(G) = \sum_{1 \le i < j \le n} d(v_i, v_j)\{\deg(v_i) + \deg(v_j)\}.$$

In this paper, we mainly concentrate on the calculation of the degree distance of standard graphs.

For elegance, we use the following definition and result.

Definition 2.2 [5]: The Wiener index of a graph G, denoted by W(G), with the vertex set $V=\{v_1, v_2, \ldots, v_n\}$ is defined as

$$W(G) = \sum_{1 \le i < j \le n} d(v_i, v_j).$$

This number is widely used in computational chemistry to measure some topological concepts and in the study of quantitative structure property relations.

Result 2.3 [3]

(a) For the complete graph K_n, $W(K_n) = n(n-1)/2$.

(b) For the cycle C_n

$$W(C_n) = \begin{cases} n^3 & \text{if n is even,} \\ \frac{1}{8}n(n^2-1) & \text{if n is odd.} \end{cases}$$

§3. Main Results

Theorem 3.1 For the complete graph K_n,

(a) $D'(v) = (n-1)^2$ for any vertex v of K_n

and

(b) $D'(K_n) = n(n-1)^2$.

Proof. If n = 1, then the result is trivial, since K_1 is an empty graph.

Let n be any integer ≥ 2 and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

Clearly, $\deg(v_i) = (n-1)$ for $i=1,2,\ldots,n$ and $d(v_i, v_j) = 1$ $(1 \le i \ne j \le n)$. For any vertex v_i of K_n,

$$D'(v_i) = \left\{\sum_{j=1}^{n} d\left(v_i, v_j\right)\right\} \deg(v_i) = \left\{\sum_{\substack{j=1 \\ j\neq i}}^{n} 1\right\} (n-1) \text{ (since } d\left(v_i, v_i\right)=0)$$

$$= (n-1)(n-1) = (n-1)^2.$$

$$D'(K_n) = \sum_{1\le i<j\le n} d(v_i, v_j)\{ deg\ (v_i) + deg\ (v_j)\}$$

$$= 2(n-1) \sum_{1\le i<j\le n} d(v_i, v_j)$$

$$= 2(n-1)\, W(K_n)$$

$$= 2(n-1)\, n(n-1)/2 \text{ (by Result (2.3)(a))}$$

$$= n(n-1)^2.$$

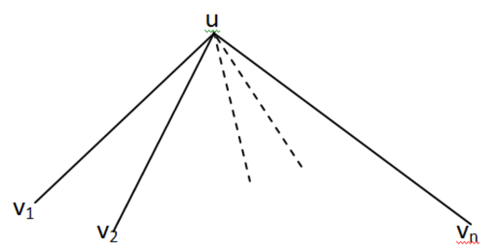
Theorem 3.2: For the star graph $K_{1,n}$ ($n \ge 2$) with vertex set $\{u, v_1, v_2, \ldots, v_n\}$ where deg (u) = n, deg (v_j) = 1, j = 1, 2, …, n,

(a) (i) $D'(u) = n^2$;

(ii) $D'(v_j) = (2n-1)$ (j = 1, 2, …, n).

(b) $D'(K_{1,n}) = n(3n-1)$.

Proof. The diagrammatic representation of $K_{1,n}$ is



Now d (u, v_j) = 1 for j = 1, 2, …, n and d(v_i, v_j) = 2 for $1\le i \ne j \le n$.

$$D'(u) = \left\{\sum_{j=1}^{n} 1\right\} n = n.n = n^2.$$

For j = 1, 2, …, n, since d(v_i, v_i) = 0,

$$D'(v_j) = \{ d(v_j, u) + \sum_{i=1}^{n} d(v_j, v_i) \} \deg(v_j)$$

$$= \{1 + \sum_{\substack{i=1 \\ i\neq j}}^{n} (2)\}(1)\} = \{1 + 2(n-1)\} = (2n-1).$$

$$D'(K_{1,n}) = \left[\sum_{j=1}^{n} \left\{d\left(u, v_j\right)\right\} \left\{deg\left(u\right) + deg\left(v_j\right)\right\}\right] +$$

$$\left[\sum_{1\le j<j'\le n} \left\{d\left(v_j, v_{j'}\right)\right\} \left\{deg\left(v_j\right) + deg\left(v_{j'}\right)\right\}\right]$$

$$= \left[\left(\sum_{j=1}^{n} 1\right)(n+1)\right] + \left[2\sum_{j=1}^{n-1} 1 \sum_{j'=j+1}^{n} 1\right](1+1)$$

$$= n\,(n + 1) + 4 \sum_{j=1}^{n-1}(n-j)$$

$$= n\,(n + 1) + 4 \left[n(n-1) - \sum_{j=1}^{n-1} j\right]$$

$$= n\,(n + 1) + 4 \left[n(n-1) - \frac{(n-1)n}{2}\right]$$

$= n\,(n + 1) + 2n\,(n – 1)$
$= n\,[\,(n+1) + 2(n – 1)]$
$= n\,(3n – 1).$

Theorem 3.3: For the complete Bipartite graph $K_{m,n}(m, n \geq 2)$ with bipartition (X, Y) where $X = \{ u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$,

(a) (i) $D'(u_i) = n\,(2m+ n – 2), i = 1, 2, \ldots, m,$

(ii) $D'(v_j) = m\,(2n+ m – 2), j = 1, 2, \ldots, n;$

and

(b) $D'(K_{m,n}) = mn\,[3(m+n) – 4].$

Proof. Under the given hypotheses,

$\deg (u_i) = n\ (i=1,2, \ldots, m), \deg (v_j) = m\ (j=1, 2, \ldots, n),$

$d (u_i, u_{i'}) = 2 = d(v_j, v_{j'})$ for $1\leq i \neq i' \leq m, 1\leq j \neq j' \leq n,$

$d (u_i, v_j) = 1$ for $1\leq i \leq m$ and $1\leq j \leq n.$

For i = 1, 2, …, m,

$$D'(u_i) = \left\{ \sum_{y \,\in\, V(K_{m,n})} d(u_i, y)\right\} \deg(u_i)$$

$$= n\left\{\sum_{i'=1}^{m} d(u_i,u_{i'}) + \sum_{j=1}^{n} d(u_i,v_j)\right\}$$

$$= n\left\{\sum_{\substack{i'=1\\ i'\neq i}}^{m} 2 + \sum_{j=1}^{n} 1\right\}$$

$= n\,\{2(m-1) + n\}$
$= n(2m+n-2).$

Similarly, for j = 1,2,…,n, we get that

$D'(v_j) = m(2n+m-2).$ (interchanging m and n in the above equality).

Now,

$$D'(K_{m,n}) = \sum_{1\le i<i'\le m} d(u_i,u_{i'})\{\deg(u_i)+\deg(u_{i'})\} +$$
$$\sum_{1\le j<j'\le n} d(\mathrm{v}_j,\mathrm{v}_{j'})\{\deg(\mathrm{v}_i)+\deg(\mathrm{v}_{i'})\} +$$
$$\sum_{i=1}^{m}\sum_{j=1}^{n} d(u_i,v_j)\{\deg(u_i)+\deg(v_j)\}$$
$$= \sum_{1\le i<i'\le m} 2(n+n) + \sum_{1\le j<j'\le n} 2(m+m) + \sum_{i=1}^{m}\sum_{j=1}^{n} 1(n+m)$$
$$= 4n\sum_{i=1}^{m-1}\sum_{i'=i+1}^{m} 1 + 4\mathrm{m}\sum_{j=1}^{n-1}\sum_{j'=j+1}^{n} 1 + (\mathrm{n+m})\sum_{i=1}^{m}\sum_{j=1}^{n} 1$$
$$= 4n\,\frac{m(m-1)}{2} + 4\mathrm{m}\,\frac{n(n-1)}{2} + (\mathrm{m+n})\mathrm{mn}$$
$= \mathrm{mn}(2\mathrm{m}-2 + 2\mathrm{n}-2 + \mathrm{m}+\mathrm{n})$
$= \mathrm{mn}\,[3(\mathrm{m}+\mathrm{n}) - 4]$.

Observation 3.4: With the convention that the summation taken over an empty set is zero, we can deduce **Th.(3.2)** from **Th.(3.3)** by taking m = 1.

Theorem 3.5: In the cycle C_n ($n \ge 3$), if v is any vertex of C_n,

a) $$D'(\mathrm{v}) = \begin{cases} \frac{n^2}{2} \text{ if n is even} \\ \frac{1}{2}(n^2-1) \text{ if n is odd.} \end{cases}$$

b) $$D'(C_n) = \begin{cases} 4n^3 \text{ if n is even,} \\ \frac{1}{2}(n^2-1) \text{ if n is odd.} \end{cases}$$

Proof: I f n = 3, $C_3 = K_3$ and this is already discussed.
Let $n \ge 4$ and $V(C_n) = \{ v_1, v_2, \ldots, v_n\}$.
Clearly ,$\deg(v_i) = 2$ for I = 1,2,…,n.
Since the figure is the same for all rotations follows that $D'(v_i) = D'(v_1)$ for all i.

Case (i): Let n be even, n = 2m (say).

Now $$D'(\mathrm{v}_i) = D'(\mathrm{v}_1) = \left\{\sum_{j=1}^{2m} d(v_1,v_j)\right\}\deg(v_1)$$
$$= 2\left\{\sum_{j=1}^{m} d(v_1,v_j) + \mathrm{d}(\mathrm{v}_1,v_{m+1}) + \sum_{j=m+2}^{2m} d(v_1,v_j)\right\}$$
$$= 2\left[(1+2+\ldots+m-1)+m+\{(m-1)+(m-2)+\ldots+1\}\right]$$
$$= 2\left[2(1+2+\ldots+m-1)+m\right]$$

$$= 2\left[2\frac{(m-1)\,\mathrm{m}}{2} + m\right]$$
$$= 2m^2$$
$$= 2\left(\frac{n}{2}\right)^2$$
$$= \frac{n^2}{2}.$$

Case (ii): Let n be odd and n = 2m+1.

$$Now\ \mathrm{D}'(\mathrm{v}_i) = \mathrm{D}'(\mathrm{v}_1)$$
$$= \left\{\sum_{j=1}^{2m+1} d(v_1, v_j)\right\}\deg(v_1)$$
$$=2\left\{\sum_{j=1}^{m+1} d(v_1, v_j) + \sum_{j=m+2}^{2m+1} d(v_1, v_j)\right\}$$
$$=2\{(1+2+\ldots+\mathrm{m}) + \{\mathrm{m}+(\mathrm{m}+1)+\ldots+1\}\}$$
$$= 2\left\{2.\frac{m(m+1)}{2}\right\}$$
$$= 2\mathrm{m}(\mathrm{m}+1)$$
$$= 2\frac{(n-1)}{2}\frac{(n+1)}{2}$$
$$= \frac{1}{2}(n^2-1).$$

$$Also\ \mathrm{D}'(\mathrm{C}_n) = \sum_{1\le i<j\le n} d(v_i, v_j)\{d(v_i) + d(v_j)\}$$
$$= \sum_{1\le i<j\le n} d(v_i, v_j)(2+2)$$
$$= 4\sum_{1\le i<j\le n} d(v_i, v_j)$$
$$= \begin{cases} 4\mathrm{n}^3 & \text{if n is even,} \\ \frac{1}{2}\ \mathrm{n}(\mathrm{n}^2\text{-}1) & \text{if n is odd} \end{cases}$$
$$= 4\ \mathrm{W}(\mathrm{C}_n)$$

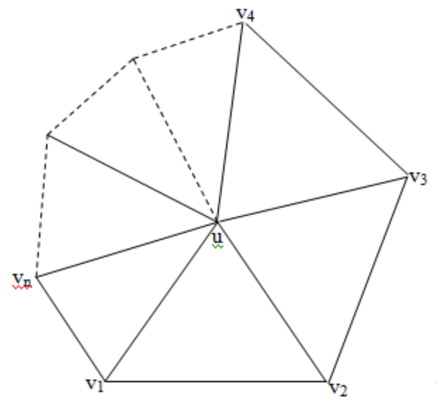
(by **Result 2.3 (b))**

Theorem 3.6: For any wheel, K_1 v C_n, n ≥ 3,

$$a)\ \mathrm{D}'(\mathrm{v}) = \begin{cases} \mathrm{n}^2 & \text{if v is the central vertex,} \\ 3(2\mathrm{n}\text{-}3) & \text{if v is any vertex of the cycle } \mathrm{C}_n. \end{cases}$$

$$b)\ \ \mathrm{D}'(\mathrm{K}_1 \vee \mathrm{C}_n) = \mathrm{n}(\mathrm{n}+3) + \mathrm{W}(\mathrm{C}_n).$$

Proof: Let $V(K_1\ VC_n) = \{u, v_1, v_2, \ldots, v_n\}$, where u is the central vertex
The diagrammatic representation of K_1 v C_n is



So $deg(u) = n$, $deg(v_i) = 3$ for i = 1,2,…,n.
Now
$d(u,v_i) = 1$ for all i and

$$d(v_i,v_j)=\begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 2 & \text{if } v_i \text{ is not adjacent to } v_j \end{cases}$$

(with the convention $v_0 = v_n$ and $v_{n+1} = v_1$)

$$Now\ D'(u)=\left\{\sum_{j=1}^{n} d(u,v_j)\right\}\deg(u)$$

$$= n\left[\sum_{j=1}^{n} 1\right]$$

$$= n^2.$$

$$D'(v_i) = \left\{d(u,v_i)+\sum_{j=1, j\neq i}^{n} d(v_i,v_j)\right\}\deg(v_i)$$

$$= 3\left\{1+d(v_i,v_{i-1})d(v_i,v_{i+1})+\sum_{j=1, j\notin\{i-1,i,i+1\}}^{n} d(u_i,v_j)\right\}$$

$$= 3\{1+1+1+2(n-3)\}$$

$$= 3(3+2n-6)$$

$$=3(2n-3).$$

$$D'(K_1\vee C_n)=\frac{1}{2}\sum_{x,y\in V(K_1 \vee C_n)}\left[d(x,y).\{\deg(x)+\deg(y)\}\right]$$

$$= \sum_{i=1}^{n}\left[d(u,v_i)\{\deg(u)+\deg(v_i)\}\right]+\sum_{1\le i<j\le n} d(v_i,v_j).\{d(v_i)+d(v_j)\}$$

$$= \sum_{i=1}^{n} 1\}(n+3) + \sum_{1\le i<j\le n} d(v_i,v_j)(3+3)$$

$$=n(n+3)+6\ W(C_n) .$$

Theorem 3.7: For the path P_n ($n \geq 2$),

$D'(v) = \frac{n(n-1)}{2}$ if v is the end vertex of P_n.

$D'(v_i) = n^2+(1-2i)n+2i(i-1)$, if v_i is the $(i+1)^{th}$ vertex from the starting vertex.

$D'(P_n) = (3n^2-7n+4)+2\sum_{i=4}^{n}(i-3)(i-2)$.

(with the convention $\sum_{i=r}^{s}\ldots = 0$ if s < r, we get that $D'(P_3)=10$)

Proof : If n = 2, $P_2 = K_2$ and this is already discussed.

Let $n \geq 3$ and $V(P_n) = \{v_1, v_2,\ldots,v_n\}$.

We observe that the end vertices of the path, namely v_1 and v_n have the same properties with respect to the distances from other vertices.

$$\text{So, } D'(v_n) = D'(v_1) = \left\{\sum_{j=1}^{n} d(v_1, v_n)\right\}.\deg(v_1)$$
$$= 1(1+2+\ldots+(n-1))$$
$$= \frac{(n-1)n}{2}.$$

$$\text{For } 2 \leq i \leq (n-1), \quad D'(v_i) = \left\{\sum_{j=1}^{n} d(v_i, vj)\right\}.\deg(v_1)$$
$$= 2\left\{\sum_{j=1}^{i-1} d(v_j, v_i) + \sum_{j=i+1}^{n} d(v_j, v_i)\right\}$$
$$= 2\left[\{(i-1)+(i-2)+\ldots+1\}+\{1+2+\ldots+(n-i)\}\right]$$
$$= 2\left[\frac{i(i-1)}{2}+\frac{(n-i)(n-i+1)}{2}\right]$$
$$= \left[i^2-i+(n-i)^2+(n-i)\right]$$
$$= \left[n^2+(1-2i)n+2i(i-1)\right].$$

Clearly $D'(P_3) = d(v_1,v_2)\{\deg(v_1)+\deg v_2\}+d(v_1,v_3)\{\deg(v_1+\deg v_3)+d(v_2,v_3)\{\deg v_2+\deg v_3\}$

$= 1(1+2)2(1+2) + 2(1+1)+1(2+1)=10$.

Let $n \geq 4$. Now

$$D'(P_n) = \sum_{1\le i<j\le n} d(v_i, v_j)\left\{\deg(v_i) + \deg(v_j)\right\}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d(v_i, v_j)\left\{\deg(v_i) + \deg(v_j)\right\}$$

$$= \left[\sum_{j=2}^{n} d(v_i, v_j)\left\{\deg(v_i) + \deg(v_j)\right\} \right] +$$

$$\left[\sum_{j=3}^{n} d(v_2, v_j)\left\{\left\{\deg(v_2) + \deg(v_j)\right\}\right\} \right] +$$

$$\left[\sum_{j=n-1}^{n} d(v_{n-2}, v_j)\left\{\deg(v_{n-2}) + \deg(v_j)\right\} \right] +$$

$$\left[d(v_{n-1}, v_j)\left\{\deg(v_{n-2}) + \deg(v_j)\right\} \right]$$

$$= \left[d(v_1, \mathrm{v}_2)\left\{\deg(v_1) + \deg(v_2)\right\} \right] + d(v_1, v_3)\left\{\deg(v_1) + \deg(v_3)\right\} +$$

$$\ldots\ldots\ldots + \mathrm{d}(\mathrm{v}_1, v_{n-1})\left\{\deg(v_1) + \deg(v_{n-1})\right\} + d(v_1, v_n)\left\{\deg(v_1) + \deg(v_n)\right\}$$

$$+[\mathrm{d}(\mathrm{v}_2, v_3)\left\{\deg(v_2) + \deg(v_3)\right\} + \mathrm{d}(\mathrm{v}_2, v_4)\left\{\deg(v_2) + \deg(v_4)\right\}$$

$$+\ldots\ldots\ldots\ldots + \mathrm{d}(\mathrm{v}_2, v_n)\left\{\deg(v_2) + \deg(v_n)\right\}] + \ldots\ldots\ldots +$$

$$\left[d(v_{n-2}, v_{n-1})\left\{\deg(v_{n-2}) + \deg(v_{n-1})\right\} \right] + d(v_{n-2}, v_n)\left\{\deg(v_{n-2}) + \deg(v_n)\right\} +$$

$$d(v_{n-1}, v_n)\left\{\deg(v_{n-1}) + \deg(v_n)\right\}$$

=[1(1+2)+2(1+2)+…+(n-2)(1+2)+(n-1)(1+1)] +

[1(2+2)+2(2+2)+…+(n-3)(2+2)+(n-2)(2+1)]+

………………………………………………..+

[1(2+2)+2(2+1)] + [1(2+1)].

=2(n – 1) + 3{1+2+3+…+(n – 2)} + ----

--- + 3(n – 2) +4{1+2+---+(n – 3)}

--- + 3(n – 2) +4{1+2+---+(n – 3)}

--- --- --- --- --- ---

--- --- --- --- --- ---

--- --- --- --- --- ---

--- + 3(2) --- --- --- + 4{1}

--- + 3(1) --- --- --- ---

$$= 2(n-1) + 3\,[2(1+2+3+\ldots+(n-2)] + 4 \sum_{i=4}^{n} (i-3)\,(i-2)/2$$

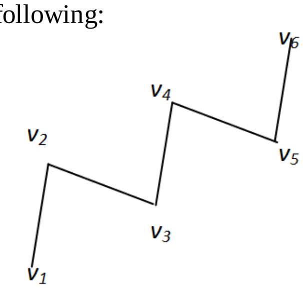
$= 2(n-1) + 3\,[2(1+2+3+\ldots+(n-2)] + 4\sum_{i=4}^{n}(i-3)(i-2)/2$

$=2(n-1) + 3(n-2)(n-1) + 2\sum_{i=4}^{n}(i-3)(i-2).$

This completes the proof of the theorem.

Now, we calculate $D'(P_6)$ just by using the definition.

The diagrammatic representation of P_6 is the following:



$D'(P_6) = \sum_{1\le i<j\le 6}^{n} d(v_i, v_j)\ \{\deg(v_i)+\deg(v_j)\}$

$=\sum_{i=1}^{5}\sum_{j=i+1}^{6} d(v_i,v_j\ \{\deg(v_i)+\deg(v_j)\}$

$=\sum_{j=2}^{6} d(v_i,v_j)\ \{\deg(v_1)+\deg(v_j)\} + \sum_{j=3}^{6} d(v_i,v_j)\ \{\deg(v_2)+\deg(v_j)\}$

$+\sum_{j=4}^{6} d(v_3,v_j)\ \{\deg(v_3)+\deg(v_j)\} + \sum_{j=5}^{6} d(v_4,v_j)\ \{\deg(v_3)+\deg(v_j)\}$

$+\ d(v_5,v_6)\ \{\deg(v_5)+\deg(v_6)\}$

$= [d(v_1,v_2)\ \{\deg(v_1)+\deg(v_2)\} + d(v_1,v_3)\ \{\deg(v_1)+\deg(v_3)\} + d(v_1,v_3)\ \{\deg(v_1)+\deg(v_3)\}$

$+\ d(v_1,v_5)\ \{\deg(v_1)+\deg(v_5)\} + d(v_1,v_6)\ \{\deg(v_1)+\deg(v_6)\}] +$

$[d(v_2,v_3)\ \{\deg(v_2)+\deg(v_3)\} + d(v_2,v_4)\ \{\deg(v_2)+\deg(v_4)\} + d(v_2,v_5)\ \{\deg(v_2)+\deg(v_5)\}$

$+\ d(v_2,v_6)\ \{\deg(v_2)+\deg(v_6)\} +$

$[d(v_3,v_4)\ \{\deg(v_3)+\deg(v_4)\} + d(v_3,v_5)\ \{\deg(v_3)+\deg(v_5)\} + d(v_3,v_6)\ \{\deg(v_3)+\deg(v_6)\} +$

$+[d(v_4,v_5)\ \{\deg(v_4)+\deg(v_5)\} + d(v_4,v_6)\ \{\deg(v_4)+\deg(v_6)\} + d(v_5,v_6)\ \{\deg(v_5)+\deg(v_6)\}$

= [1(1+2) + 2(1+2) + 3(1+2) + 4(1+2) + 5(1+1)]+
[1(2+2) + 2(2+2) + 3(2+2) + 4(2+2)+
[1(2+2) + 2(2+2) + 3(2+1)]+
[1(2+2) + 2(2+1)]+
[1(2+1)]
=[1{2(1+2) + 3(2+2)}] + [2{2(1+2) + 2(2+2)}] +[3{2(1+2) + 1(2+2)}] +[4{2(1+2)}] +[5{(1+1)}]
= 1(6 + 12) + 2(6+8) + 3(6+4) + 4(6) + 5(2)
= 18 + 28 + 30 + 24 + 10 = 110.

From, **Th.(3.7)**

$D'(P_6) = 2(6-1)+3(6-2)+2\sum_{i=4}^{6}(i-3)(i-2)$

= 10 + 60 + 2(2+6+12)
= 70+40 = 110

Definition 3.8: Let n_1, n_2 be integers, each ≥ 2. Then, B_{n_1,n_2} is the (n_1, n_2)-bistar obtained from the two disjoint graphs K_{1,n_2} and K_{1,n_2} by joining the centre vertices by an edge (it has (n_1+n_2+2) vertices and (n_1+n_2+1) edges (Observe that $B_{1,1} = P_4$ and $B_{1,2}$ ($B_{2,1}$) are bipartite graphs).

Theorem 3.9: For the graph $B_{n_1,n_2}(n_1,n_2 \geq 2)$

$$D'(B_{n_1,n_2}) = 3(n_1^2 + n_2^2) + 10n_1n_2 + 5(n_1+n_2) + 2.$$

Proof: The diagrammatic representation of B_{n_1,n_2} is the following:

Let V(K_{1,n_1})= {u_0, u_1, u_2,…, u_{n_1} } and V(K_{1,n_2}) = {v_0, v_1, v_2,…, v_{n_2} }, where u_0, v_0 are the entre vertices.

$$V\left(B_{n_1,n_2}\right) = V\left(K_{1,n_1}\right) \mathrm{U} V\left(K_{1,n_2}\right)$$

So, $\deg(u_0) = (n_1+1)$, $\deg(v_0)=(n_2+1)$, $\deg(u_i) = 1 = \deg(v_j)$, $i = 1,2,\ldots,n_1$;

$j = 1,2,\ldots, n_2$. $d(u_0, v_j) = 2$ ($j=1,\ldots, ni$), $d(u_i, v_j) = 3$ ($i = 1,\ldots, n_1$; $j = 1,2, \ldots, n_2$).

Now,

$$D'(u_0) = \left\{ \sum_{x \in V(B_{n_1,n_2})} d(u_0,x) \right\} \deg(u_0)$$

$$= \left\{ \sum_{i=1}^{n_1} d(u_0,u_i) + d(u_0,v_0) + \sum_{j=1}^{n_2} d(u_0,v_j) \right\} (n_1+1)$$

$$= (\mathrm{n}_1+1) \left\{ \left(\sum_{i=1}^{n_1} 1 \right) + 1 + \left(\sum_{j=1}^{n_2} 2 \right) \right\}$$

$$= (\mathrm{n}_1+1)(n_1+1+2n_2)$$

$$= (\mathrm{n}_1+1)(n_1+2n_2+1)$$

Similarly, by interchanging n_1 and n_2 in D' (u_0), we get that

$D'(v_0) = (n_2 + 1)(n_2 + 2n_2+1)$.

For i = 1,2,…,n

$$D'(u_i) = \left\{ \sum_{x \in V(B_{n_1,n_2})} d(u_i, x) \right\} \deg(u_i)$$

$$=(1)\left\{ d(u_i,u_0) + \sum_{i'=1}^{n_1} d(u_0,u_{i'}) + d(u_i,v_0) + \sum_{j=1}^{n_2} d(u_i,v_j) \right\}(n_1+1)$$

$$= \left\{ 1 + 2\left(\sum_{\substack{i'=1 \\ i' \neq i}}^{n_1} 1 \right) + 2 + \left(\sum_{j=1}^{n_2} 3 \right) \right\}$$

$$=1+2(n_1-1+1)+3\mathrm{n}_2$$

$$=(1+2n_1+3n_2)$$

Similarly, by interchanging n_1 and n_2 in $D^1(u_i)$, we get that

D' $(v_j) = (1 + 2n_2 + 3n_1)$ for j -1,2, …,n_2.

Now

$$D'(B_{n_1,n_2}) = \frac{1}{2} \sum_{x,y \vee (B_{n_1,n_2})} (\deg x + \deg y) d(x,y)$$

$$= \sum_{i=1}^{n_1} d(u_0,u_i)\{\deg(u_0) + \deg(u_i)\} + d(u_0,v_0)\{\deg(u_0) + \deg(v_0)\}$$

$$+ \sum_{j=1}^{n_2} d(u_0,v_j)\{\deg(u_o) + \deg(v_j)\} + \sum_{i=1}^{n_1} d(u_i,v_0)\{\deg(u_i) + \deg(v_0)\}$$

$$+ \sum_{j=1}^{n_2} d(v_0,v_j)\{\deg(v_o) + \deg(v_j)\} + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} d(u_i,v_j)\{\deg(u_i) + \deg(v_j)\}$$

$$= \left(\sum_{i=1}^{n_1} 1 \right)(n_1+1+1) + (1)(n_1+1+n_2+1) + \left(\sum_{j=1}^{n_2} 2 \right)(n_1+1+1)$$

$$+ \left(\sum_{i=1}^{n_1} 2 \right)(1+n_2+1) + \left(\sum_{j=1}^{n_2} 1 \right)(n_2+1+1)$$

$$+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} (3)\{1+1\} + 2\sum_{i=1}^{n_1-1} \sum_{i'+1}^{n_1} (1+1) + 2\sum_{j=1}^{n_2-1} \sum_{j'+1}^{n_2} (1+1)$$

$$= n_1(n_1+2) + (n_1+n_2+2) + 2n_2(n_1+2) + 2n_1(n_2+2)$$

$$+ n_2(n_2+2) + 6n_1n_2 + 4\sum_{i=1}^{n_1-1} \left(\sum_{i'=i+1}^{n_1} 1 \right) + 4\sum_{j=1}^{n_2-1} \left(\sum_{j'=j+1}^{n_2} 1 \right)$$

$$= (n_1^{\,2} + n_2^{\,2} + 10n_1n_2 + 7n_1 + 7n_2 + 2) + 4\frac{(n_1-1)n_1}{2} + 4\frac{(n_2-1)n_2}{2}$$

$$= (n_1^2 + n_2^2 + 10n_1n_2 + 7n_1 + 7n_2 + 2) + 2(n_1^2 - n_1 + n_2^2 - n_2)$$
$$= 3(n_1^2 + n_2^2) + 10n_1n_2 + 5(n_1 + n_2) + 2.$$

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