

Existence of a Positive Solution for Nonlinear Differential Equations

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Abstract: In this paper, we prove the existence result for nonlinear differential equation without using the monotonicity of upper and lower solutions.

Keywords: Fractional Differential Equation (FDE), Upper and Lower Control Functions

Definition 2.1: The positive solution means $u \in U$ such that $u(t) > 0$, $0 < t \leq T$ and $x(0) = 0$.

Definition 2.2: Let $t_1, t_2 \in \mathbb{R}^+$ such that $t_2 > t_1$. For any $u \in [t_1, t_2]$, we define the upper control function

$$X(t, u) = \sup \{ f(t, a) - g(t, a) : t_1 \leq a \leq u \}.$$

Similarly, lower-control function

$$Y(t, u) = \inf \{ f(t, a) - g(t, a) : u \leq a \leq t_2 \}.$$

Obviously, $X(t, u)$ and $Y(t, u)$ are monotonous non-decreasing on the argument u and

$$X(t, u) \leq f(t, u) - g(t, u) \leq Y(t, u)$$

Definition 2.3[1]: The Riemann Liouville fractional integral of order α is defined as

$$I^\alpha f(t, u(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) ds. \quad (1)$$

Definition 2.3[1] The Caputo fractional derivative of order α ($0 < \alpha < 1$) is defined as

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds \quad (2)$$

$$D^\alpha u(t) = I^{\alpha-2} u^{(2)}, u \in U.$$

We assume that $f: J \times U \rightarrow U$ is a continuous function such that the (1) exists for any order $0 < \alpha \leq 2$ and (2) exists for any order $1 < \alpha \leq 2$.

Consider the following nonlinear fractional differential equation

$$D^\alpha u(t) = f(t, u(t)) - g(t, u(t)), \quad 0 < t \leq 1 \quad (3)$$

$$u(0) = 0, u'(0) = x > 0$$

I. INTRODUCTION

If the non-integer order derivative appears in an differential equation then it is called as Fractional differential equation. It is realized widely that in many situations fractional derivative based models are much better than integer order models. Being nonlocal in nature, the fractional derivative provides an excellent tool for the understanding of memory and hereditary properties of various materials and processes. In last few decades the theory achieves great interest [5,1]. Several papers on positivity of solutions of FDE are developed by [7,2]. Recently D. Delbosco and L. Rodino [6] proved the existence of the solutions to FDE using Banach and Schauder fixed point theorems; Zhang [4] investigated the existence and uniqueness of positive solution using the method of upper and lower solution and Cone fixed point theorem; Nanware [3] investigated existence result for FDE involving difference of two function. In all these works the nonlinear function must be monotone. Recently FDE with nonmonotone function its positivity of solution is given in [9].

In this paper, we apply method given by [9] to FDE with difference of two functions.

II. BASIC DEFINITIONS

Let $U = C(J)$, $J = [0, T]$ be the Banach space of all real-valued continuous functions defined on the compact interval J , with the maximum norm.

Define the subspace

$$S = \{ u \in U : u(t) \geq 0, t \in J \} \text{ of } U.$$

Where $1 < \alpha \leq 2$, D^α is the Caputo fractional derivative of order α .

The equivalent fractional Volterra integral equations of (3) is

$$u(t) = xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \{f(s, u(s)) - g(s, u(s))\} ds \quad (4)$$

The transform equation (2.4) to be applicable to schauder fixed point, we define an operator $\phi: A \rightarrow A$ by

$$(\psi u)t = xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \{f(s, x(s)) - g(s, x(s))\} ds \quad t \in J. \quad (5)$$

Where the figured fixed point has to satisfy the identity operator equation $\psi u = u$.

III.EXISTENCE THEOREM:

We consider following set of assumptions needed for the next results

(H₁) Let $v(t), w(t) \in A$ such that $t_1 \leq v(t) \leq w(t) \leq t_2$ and

$$D^\alpha v(t) \geq X(t, v(t)),$$

$$D^\alpha w(t) \leq Y(t, w(t))$$

For any $t \in J$.

We consider the existence of solution of the FDE (3).

Theorem 3.1 : Assume that (H₁) is satisfied, then the FDE (5) has atleast one solution $u \in U$ satisfying $v(t) \leq u(t) \leq w(t)$, $t \in J$.

Proof : Let $D = \{u \in S : v(t) \leq u(t) \leq w(t); t \in J\}$

with the norm $\|u\| = \max_{t \in J} |u(t)|$, then we have $\|u\| \leq t_2$.

Hence, D is a convex, bounded and closed subset of the Banach space U . Moreover, the continuity of f implies the continuity of the operator ψ on D defined by (1). Now, if $u \in D$, there exists a positive constant d such that

$$\max\{f(t, u(t)) - g(t, u(t)) : t \in J, u(t) \leq t_2\} < d.$$

Then

$$\begin{aligned} |(\psi u)t| &= xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \{f(s, u(s)) - g(s, u(s))\} ds \\ &\leq x + \frac{dt^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$$\text{Thus,} \quad \|\phi u\| \leq x + \frac{d}{\Gamma(\alpha+1)}$$

Hence, $\psi(D)$ is uniformly bounded.

Next, we prove the equicontinuity of ψ .

Let $u \in D, \epsilon > 0, \delta > 0$ and $0 \leq z \leq y \leq 1$ such that $|y - x| < \delta$.

$$\text{If } \delta = \min \left\{ 1, \frac{\epsilon \Gamma(\alpha+1)}{2\sqrt{(\alpha+1)+2d}}, \left(\frac{\epsilon \Gamma(\alpha+1)}{4d} \right)^{\frac{1}{\alpha}} \right\},$$

Then

$$\begin{aligned} |(\psi u)(y) - (\psi u)(z)| &\leq x(z-y) + \\ &\left| \int_0^z \frac{1}{\Gamma(\alpha)} (z-s)^{\alpha-1} \{f(s, u(s)) - g(s, u(s))\} ds \right. \\ &\quad \left. - \int_0^y \frac{1}{\Gamma(\alpha)} (y-s)^{\alpha-1} \{f(s, u(s)) - g(s, u(s))\} ds \right| \\ &\leq x(z-y) + \\ &\left| \int_0^z \frac{1}{\Gamma(\alpha)} ((z-s)^{\alpha-1} - (y-s)^{\alpha-1}) \{f(s, u(s)) - g(s, u(s))\} ds \right| \\ &\quad + \int_z^y \frac{1}{\Gamma(\alpha)} (y-s)^{\alpha-1} \{f(s, u(s)) - g(s, u(s))\} ds \\ &\leq x(z-y) + \frac{d}{\Gamma(\alpha+1)} (y^\alpha - z^\alpha + \\ &\quad 2((y-z)^\alpha)) \\ &\leq \left(x + \frac{2d}{\Gamma(\alpha+1)} \right) \delta + \frac{2d\delta^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

$< \epsilon$.

Therefore, $\psi(D)$ is equicontinuous. The Arzele-Ascoli Theorem implies that $\psi : S \rightarrow S$ is compact. Then by applying Schauder fixed point we prove $\psi(D) \subseteq D$. Let $u \in D$, then by hypothesis, we have

$$\begin{aligned} (\psi u)(t) &= xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \{f(s, u(s)) \\ &\quad - g(s, u(s))\} ds \\ &\leq xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} X(s, u(s)) ds \\ &\leq xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} X(s, v(s)) ds \\ &\leq v(t) \end{aligned}$$

And

$$\begin{aligned} (\psi u)(t) &= xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \{f(s, u(s)) \\ &\quad - g(s, u(s))\} ds \\ &\geq xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} Y(s, u(s)) ds \\ &\geq xt + \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} Y(s, w(s)) ds \\ &\geq w(t) \end{aligned}$$

Hence, $v(t) \leq (\psi u)(t) \leq w(t)$, $t \in J$, that is, $\psi(D) \subseteq D$. According to Schauder fixed point theorem, the operator ψ has atleast one fixed point $u \in D$. Therefore, the FDE (5) has atleast one positive solution $u \in U$ and $v(t) \leq u(t) \leq w(t)$, $t \in J$.

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