Semi $^{\#}$ generalized closed sets in Topological Spaces

S.Saranya[1] and Dr.K.Bageerathi[2]

[1&2]Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur, (T N), INDIA

Abstract

In this paper a new class of generalized closed sets, namely $s^{\#}$g-closed sets is introduced in topological spaces. We prove that this class lies between the class of sg-closed sets and the class of gs-closed sets. Also we find some basic properties and characterizations of $s^{\#}$g-closed sets.

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1. Introduction

The study of generalized closed sets in topological space was initiated by Levine in [8]. Biswas[5], Njasted[15], Mashhour[12], Robert[19], Bhattacharya[4], Arya and Nour[1], Maki, Devi and Balachandran[10, 11], Sheik John[19], Pushpalatha and Anitha[17], Gnanachandra and Velmurugan[7], Veerakumar[20] introduced and investigated semi closed, α-open and α-closed, pre-open, semi*-open, sg-closed, gs-closed, gp-closed, αg-closed, g*s-closed, s*g-closed, w-closed, g*-closed respectively.

In this paper we introduce a new class of sets called $s^{\#}$g-closed sets which is properly placed in between the class of gs-closed sets and the class of sg-closed sets. We give characterizations of $s^{\#}$g-closed sets also investigate many fundamental properties of $s^{\#}$g-closed set.

2.Preliminaries

Throughout this paper (X, τ) represents a topological space on which no separation axiom is assumed unless otherwise mentioned. For a subset A of a topological space X, cl(A) and int(A) denote the closure of A and the interior of A respectively. (X, τ) will be replaced by X if there is no changes of confusion. We recall the following definitions and results.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be generalized closed [8] (briefly g-closed) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is an open in (X, τ).

Definition 2.2. Let (X, τ) be a topological space and A $\subseteq$ X. . The generalized closure of A [6], denoted by cl*(A) and is defined by the intersection of all g-closed sets containing A and generalized interior of A[6], denoted by int*(A) and is defined by union of all g-open sets contained in A.

Definition 2.3. Let (X, τ) be a topological space. A subset A of the space X is said to be

1. semi-open [9] if A $\subseteq$ cl(int(A)) and semi-closed [3] if int(cl(A)) $\subseteq$ A.
2. α-open [13] if A $\subseteq$ int(cl(int(A))) and α-closed if cl(int(cl(A))) $\subseteq$ A.
3. pre-open [14] if A $\subseteq$ int(cl(A)) and pre-closed if cl(int(A)) $\subseteq$ A.
4. semi*-open [16] if A $\subseteq$ cl*(int(A)) and semi*-closed if int*(cl(A)) $\subseteq$ A.

Definition 2.4. Let (X, τ) be a topological space and A$\subseteq$ X. The semi-closure of A[4], denoted by scl(A) and is defined by the intersection of all semi-closed sets containing A.

Definition 2.5. Let (X, τ) be a topological space. A subset A of X is said to be

1. semi-generalized closed [4] (briefly sg-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a semi-open in (X, τ).
2. generalized semi-closed [1] (briefly gs-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is an open in (X, τ).
3. α-generalized closed [10] (briefly αg-closed) if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is an open in (X, τ).
4. g*s-closed set [15] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a gs-open in (X, τ).

5. semi*generalized closed [7] (briefly semi*g-closed) if s*cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a semi*-open in (X, τ).
6. w-closed [19] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a semi-open in (X, τ).
7. *g-closed [20] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a w-open in (X, τ).
8. wg-closed [2] if cl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is an open in (X, τ).
9. wgα-closed [14] if αcl(int(A)) $\subseteq$ U whenever A $\subseteq$ U and U is α-open in (X, τ).
10. wα-closed [3] ifαcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a w-open in (X, τ).
11. gwα-closed [2] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is wα-open in (X, τ).

The complements of the above mentioned closed sets are their respective open sets.

Remark 2.6.

$$\begin{tikzcd}
 & \text{gs-closed} & \\
\text{closed} \arrow[r] & \text{semi*-closed} \arrow[u] \arrow[r] & \text{semi-closed}
\end{tikzcd}$$

Theorem 2.7.[16] Intersection of semi*-closed sets is semi*-closed set.

3. Semi $^{\#}$ generalized closed set

Definition 3.1. A subset A of a space (X, τ) is called semi $^{\#}$ generalized closed set (briefly, s$^{\#}$ g-closed) if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open in (X, τ).

Theorem 3.2. Every closed set is s$^{\#}$g-closed.

Proof: Let A be a closed set. Let A $\subseteq$ U, U is semi*-open. Since A is closed, cl(A) = A $\subseteq$ U. But scl(A) $\subseteq$ cl(A). Thus we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is a s$^{\#}$g-closed set.

Remark 3.3. The converse of the above theorem is not true, as seen from the following example.

Example 3.4. Let X = {a,b,c} and τ = {ϕ, X, {a}, {b}, {a,b}}. Consider A = {a}. A is not a closed set, However A is a s$^{\#}$g-closed set.

Theorem 3.5. Every semi-closed set is s$^{\#}$g-closed.

Proof: Let A be a semi-closed set. Let A $\subseteq$ U, U is semi*-open. Since A is semi-closed, scl(A) = A $\subseteq$ U. Therefore, A is s$^{\#}$g-closed set.

Remark 3.6. The converse of the above theorem is not true, as seen from the following example.

Example 3.7. Let X = {a, b, c} and τ = {ϕ, X, {a}, {b, c}}. Consider A = {b}. A is not a semi-closed set. However A is a s$^{\#}$g-closed set.

Theorem 3.8. Every semi*-closed set is s$^{\#}$g-closed.

Proof: Let A be a semi*-closed set. Let A $\subseteq$ U, U is semi*-open. Since A is semi*-closed set, int*(cl(A)) $\subseteq$ A. But scl (A) $\subseteq$ int*(cl(A)) $\subseteq$ A. Thus we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is s$^{\#}$g-closed set.

Remark 3.9. The converse of the above theorem is not true, as seen from the following example.

Example 3.10. Let X = {a, b, c} with τ = {ϕ, X, {a}, {b, c}}. Consider A = {b}. A is not a semi*-closed set. However A is a s$^{\#}$g-closed set.

Theorem 3.11. Every sg-closed set is s$^{\#}$g-closed.

Proof: Let A be a sg-closed set. Let A $\subseteq$ U, U is semi*-open. Then U is semi-open. Since A is sg-closed, scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is s$^{\#}$g-closed set.

Remark 3.12. The converse of the above theorem is not true, as seen from the following example.

Example 3.13. Let X = {a, b, c} with τ = {ϕ, X, {a}}. Here A = {a, c} is not a sg-closed set. Whereas A is a s$^{\#}$g-closed set.

Theorem 3.14. Every s$^{\#}$g-closed set is gs-closed.

Proof: Let A be a s$^{\#}$g-closed set. Let A $\subseteq$ U, U is open. Then U is semi*-open. Since A is s$^{\#}$g-closed, scl(A) $\subseteq$ U. Thus, we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open. Therefore, A is gs-closed set.

Remark 3.15. The converse of the above theorem is not true, as seen from the following example.

Example 3.16. Let X = {a, b, c} with τ = {ϕ, X, {a}, {a, b}}. Consider A = {a, c}. A is not a s$^{\#}$g-closed set. However A is a gs-closed set.

Theorem 3.17. Every w-closed set is $s^{\#}$g-closed set.

Proof: Let A be a w-closed set. Let A $\subseteq$ U, U is semi*-open. Then U is semi-open. Since A is w-closed, cl(A) $\subseteq$ U. But scl(A) $\subseteq$ cl(A). Thus, we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is $s^{\#}$g-closed.

Remark 3.18. The converse of the above theorem is not true, as seen from the following example.

Example 3.19. Let X = {a, b, c} with τ = {ϕ, X, {a}, {c}, {a, c}}. Here A = {a} is not a w-closed set, whereas A is a $s^{\#}$g-closed set.

Theorem 3.20. Every g*s-closed set is $s^{\#}$g-closed set.

Proof: Let A be a g*s-closed set. Let A $\subseteq$ U, U is semi*-open. Then U is gs-open. Since A is g*s-closed, scl(A) $\subseteq$ U. Thus, we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is $s^{\#}$g-closed.

Remark 3.21. The converse of the above theorem need not be true, as seen from the following example.

Example 3.22. Let X = {a, b, c} with τ = {ϕ, X, {a}}. Consider A = {a, b}. A is not a g*s- closed set. However A is a $s^{\#}$g-closed set.

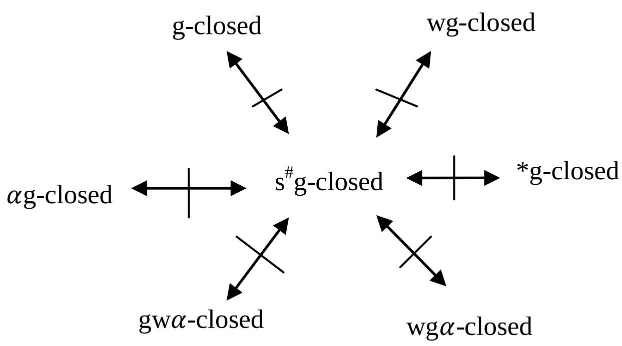
Theorem 3.23. Every semi*g-closed set is $s^{\#}$g-closed set.

Proof: Let A be a semi*g-closed set. Let A $\subseteq$ U, U is semi*-open. Since A is semi* g-closed, s*cl(A) $\subseteq$ U. But scl(A) $\subseteq$ s*cl(A). Thus, we have scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi*-open. Therefore, A is a $s^{\#}$g-closed.

Remark 3.24. The converse of the above theorem is not true, as seen from the following example.

Example 3.25. Let X = {a, b, c} with τ = {ϕ, X, {a}}. Consider A = {a, c}. A is not a semi*g-closed set. However A is a $s^{\#}$g-closed set.

Remark 3.26.



Remark 3.27. The following example shows that $s^{\#}$g-closed sets are independent from αg-closed set, g-closed set, wgα-closed set, gwα-closed set, wg-closed set and *g-closed set.

Example 3.28. Let X = {a, b, c} and Y = {a, b, c, d} be the topological spaces.

(i) Consider the topology τ = {ϕ, X, {a}, {a, b}}. Then the set {a, c} is an αg-closed set but not $s^{\#}$g-closed in (X, τ).

(ii) Consider the topology τ = {ϕ, Y, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. Then the sets {a}, {b}, {c}, {a, b}, {b, c} are $s^{\#}$g-closed sets but not an αg-closed in (X, τ).

(iii) Consider the topology τ = {ϕ, X, {a}, {a, b}}. Then the set {b} is $s^{\#}$g-closed set but not g-closed, also the set {a, c} is g-closed set but not a $s^{\#}$g-closed in (X, τ).

(iv) Consider the topology τ = {ϕ, Y, {c}, {a, b}, {a, b, c}}.Then the set {b} is a wgα-closed and gwα-closed but not $s^{\#}$g-closed set also {c}, {a, b} are $s^{\#}$g-closed sets but not a wgα-closed and gwα-closed.

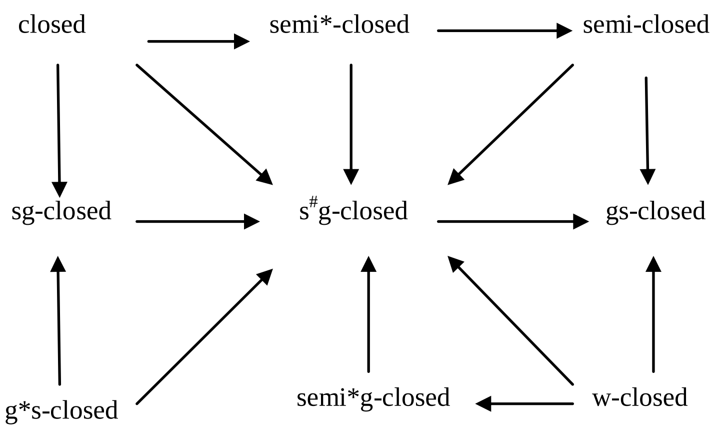
(v) Consider the topology τ = {ϕ, X, {a}, {a, b}}. Then the set {a, c} is wg-closed set but not a $s^{\#}$g-closed in (X, τ).

(vi) Consider the topology τ = {ϕ, X, {a}, {b}, {a, b}}. Then the sets {a}, {b} are $s^{\#}$g-closed set but not a wg-closed in (X, τ).

(vii) Consider the topology τ = {ϕ, X, {a}, {b}, {a, b}}. Then the sets {a}, {b} are $s^{\#}$g-closed but not a *g-closed in (X, τ).

(viii) Consider the topology τ = {ϕ, X, {a}, {a, b}}. Then the sets {a, b}, {a, c} are *g- closed but not a $s^{\#}$g-closed in (X, τ).

Remark 3.29. The above discussions are summarized in the following implications



4.Characterisation

Theorem 4.1. Let A ⊆ X. If A is $s^{\#}$g-closed in (X, τ), then scl(A)\A does not contain any non empty semi*-closed set in (X, τ).

Proof: Let F be any semi*-closed set such that F ⊆ scl(A)\A. Then A ⊆ X\F and X\F is semi*-open in (X, τ). Since A is $s^{\#}$g-closed in X, scl(A) ⊆ X\F, F ⊆ X\scl(A). Thus F ⊆ (scl(A)\A) ∩ (X\scl(A))= ϕ.

Theorem 4.2. Let A be any $s^{\#}$g-closed set in (X, τ). Then A is semi-closed in (X, τ) iff scl(A)\A is semi*-closed.

Proof: Necessity: Since A is semi-closed set in (X, τ), scl(A) = A. Then scl(A)\A= ϕ, which is a semi*-closed set in (X, τ). Sufficiency: Since A is $s^{\#}$g-closed set in (X, τ), by above theorem, scl(A)\A does not contains any non-empty semi*-closed set. Therefore, scl(A)\A = ϕ. Hence scl(A) = A. Thus A is semi-closed set in (X, τ).

Theorem 4.3. Let A be any $s^{\#}$g-closed set in (X, τ). If A ⊆ B ⊆ scl(A), then B is also a $s^{\#}$g-closed set.

Proof: Let B ⊆ U where U is semi*-open in (X, τ). Then A ⊆ U. Also since A is $s^{\#}$g-closed, scl(A) ⊆ U. Since B ⊆ scl(A), scl(B) ⊆ scl(scl(A)) = scl(A) ⊆ U. This implies, scl(B) ⊆ U. Thus B is a $s^{\#}$g-closed set.

Remark 4.4. Intersection of any two $s^{\#}$g-closed sets in (X, τ) need not be a $s^{\#}$g-closed set, as seen from the following example.

Example 4.5. Let X = {a, b, c} with τ = {ϕ, X, {a}}. Consider {a, b} and {a, c} are $s^{\#}$g-closed sets. But their intersection {a} is not a $s^{\#}$g-closed set.

Theorem 4.6. If A is $s^{\#}$g-closed set in X and B is closed set in X, then A ∩ B is $s^{\#}$g-closed.

Proof: Let U be semi*-open such that A ∩ B ⊂ U. Then U ∪ (X-B) is semi*-open containing A. Since A is $s^{\#}$g-closed, scl(A) ⊆ U ∪ (X-B). Now scl(A∩B) ⊆ scl(A) ∩ scl(B) ⊆ scl(A) ∩ cl(A) = scl(A) ∩ B ⊆ (U∩(X-B)) ∩ B = U ∩ B ⊆ U. Thus we have scl(A) ⊆ U, U is semi*-open and A ∩ B ⊆ U. Therefore A ∩ B is $s^{\#}$g-closed

Remark 4.7. Union of any two $s^{\#}$g-closed sets in (X, τ) need not be a $s^{\#}$g-closed set, as seen from the following example.

Example 4.8. Let X = {a, b, c} with τ = {ϕ, X, {a}, {b}, {a, b}}. Consider {a} and {b} are $s^{\#}$g-closed sets. But their union {a, b} is not a $s^{\#}$g-closed set.

Theorem 4.9. Let A and B be $s^{\#}$g-closed sets in (X, τ) such that cl(A) = scl(A) and cl(B) = scl(B), then A ∪ B is $s^{\#}$g-closed.

Proof: Let A∪B ⊂ U, where U is semi*-open. Then A ⊂ U and B ⊂U. Since A and B are $s^{\#}$g-closed, scl(A) ⊂ U and scl(B) ⊂ U. Now cl (A∪B) = cl(A) ∪ cl(B) = scl(A) ∪ scl(B) ⊂ U. But scl(A∪B) ⊂ cl(A∪B). So, scl (A∪B) ⊂ cl (A∪B) ⊂ U. Therefore scl (A∪B) ⊂ U whenever A∪B ⊂ U, U is semi*-open. Hence A∪B is $s^{\#}$g-closed.

Theorem 4.10. For every element x in a space X, X – {x} is $s^{\#}$g-closed or semi*-open.

Proof: Suppose X–{x} is not semi*-open. Then X is the only semi*-open set containing

X– {x}. This implies scl(X–{x}) ⊆ X. Hence X–{x} is $s^{\#}$g-closed.

Theorem 4.11. If A is both semi*-open and $s^{\#}$g-closed set in X, then A is semi-closed set.

Proof: Since A is semi*-open and $s^{\#}$g-closed in X, scl(A) ⊆ A. But always A ⊆ scl(A). Therefore, A = scl(A). Hence A is a semi-closed set.

Theorem 4.12. If A ⊆ Y ⊆ X and A is $s^{\#}$g-closed in X then A is $s^{\#}$g-closed relative to Y.

Proof: Given that A ⊆ Y ⊆ X and A is a $s^{\#}$g-closed set in X. To prove that A is $s^{\#}$g-closed set relative to Y. Let us assume that A ⊆ Y ∩ U, where U is semi*-open in X. Since A is $s^{\#}$g- closed set, A ⊆ U. This implies scl(A) ⊆ U. It follows that Y ∩ scl(A) ⊆ Y ∩ U. That is, A is $s^{\#}$g- closed set relative to Y.

Theorem 4.13. Every subset is $s^{\#}$g-closed in X iff every semi*-open set is semi-closed.

Proof: Necessity: Let A be a semi*-open in X. Then by hypothesis A is $s^{\#}$g-closed in X. By theorem 4.12, A is a semi-closed set. Sufficiency: Let A be a subset of X and U a semi*-open set such that A ⊆ U. Then by hypothesis, U is semi-closed. This implies that scl(A) ⊆ scl(U) = U. Hence A is a $s^{\#}$g-closed set.

Conclusion

The present chapter has introduced a new concept called $s^{\#}$g-closed set in topological spaces. It also analyzed some of the properties. The implication shows the relationship between $s^{\#}$g-closed sets and the other existing sets.

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