Integration of certain products associated with the multivariable Aleph-function and a general class of polynomials of several variables

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ABSTRACT

In the present document we evaluate two finite integrals involving the product of Aleph-function of one variable, the multivariable Aleph-function and a general class of polynomials of several variables. On account of the most general nature of the functions involved herein a very large number of known and new integrals involved simpler special functions and orthogonal polynomials follow as particular cases of our main integrals.

Keywords : multivariable Aleph-function, Aleph-function, class of multivariable polynomials, hypergeometric function, finite integral,. I-function of several variables, Aleph-function of two variables.

2010 Mathematics Subject Classification. 33C99, 33C60, 44A20

1.Introduction and preliminaries.

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph_{P_i,Q_i,c_i;r'}^{M,N} \left(z \left| \begin{array}{c} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r'} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r'} \end{array} \right. \right) = \frac{1}{2\pi\omega} \int_L \Omega_{P_i,Q_i,c_i;r'}^{M,N}(s) z^{-s} \mathrm{d}s \quad (1.1)$$

for all z different to 0 and

$$\Omega_{P_i,Q_i;r'}^{M,N}(s) = \frac{\prod_{j=1}^M \Gamma(b_j + B_j s) \prod_{j=1}^N \Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r'} c_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} + A_{ji} s) \prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} - B_{ji} s)} \quad (1.2)$$

With $|argz| < \dfrac{1}{2}\pi\Omega$ where $\Omega = \displaystyle\sum_{j=1}^M \beta_j + \sum_{j=1}^N \alpha_j - c_i(\sum_{j=M+1}^{Q_i} \beta_{ji} + \sum_{j=N+1}^{P_i} \alpha_{ji}) > 0, i = 1, \cdots, r'$

For convergence conditions and other details of Aleph-function , see Südland et al [8]. The serie representation of Aleph-function is given by Chaurasia et al [3].

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(z) = \sum_{G=1}^M \sum_{g=0}^\infty \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(s)}{B_G g!} z^{-s} \quad (1.3)$$

With $s = \eta_{G,g} = \dfrac{b_G + g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega_{P_i,Q_i,c_i;r'}^{M,N}(s)$ is given in (1.2) $\quad (1.4)$

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!}$$

$$A[N_1, K_1; \cdots ; N_s, K_s] y_1^{K_1} \cdots y_s^{K_s} \quad (1.5)$$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_s, K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!} A[N_1, K_1; \cdots ; N_s, K_s] \qquad (1.6)$$

The Aleph-function of several variables generalize the multivariable I-function defined by Sharma and Ahmad [4] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}$
$$\left(\begin{array}{c|c} z_1 \\ . \\ . \\ . \\ z_r \end{array} \right.$$

$[(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]$:
.... $,[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}]$:

$[(c_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ; \; ; [(c_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}]$
$[(d_j^{(1)}), \delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;; [(d_j^{(r)}), \delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}]$
$$\left. \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, ds_1 \cdots ds_r \qquad (1.7)$$

with $\omega = \sqrt{-1}$

For more details, see Ayant [1]. The reals numbers τ_i are positives for $i = 1, \cdots, R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \cdots, R^{(k)}$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i(k)}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i(k)}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R , i^{(k)} = 1, \cdots, R^{(k)} \qquad (1.8)$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function. We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \to \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \; ; \; V = m_1, n_1; \cdots; m_r, n_r \tag{1.9}$$

$$\mathsf{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.10}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.11}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.12}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots, \{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.13}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.14}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots, z_r) = \aleph_{U:W}^{0,\mathsf{n}:V} \left(\begin{array}{c} z_1 \\ . \\ . \\ . \\ z_r \end{array} \middle| \begin{array}{c} A : C \\ . \; . \; . \\ B : D \end{array} \right) \tag{1.15}$$

2. Required integral

We have the following results, see Chaurasia [2].

Lemme 1

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho} \, {}_2F_1\left[u, v; \frac{u+v+1}{2}; \frac{hx}{hx+k(1-x)}\right] \mathrm{d}x =$$

$$= \frac{\pi 2^{1-2\rho}(hk)^{-\rho}\Gamma(\rho)\Gamma\left(\rho + \frac{1-u-v}{2}\right)\Gamma\left(\frac{u+v+1}{2}\right)}{\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)\Gamma\left(\rho + \frac{1-u}{2}\right)\Gamma\left(\rho + \frac{1-v}{2}\right)}, Re(\rho) > 0, Re\left(\rho + \frac{1-u+v}{2}\right) > 0 \tag{2.1}$$

and $hx + k(1-x) \neq 0$, $h \neq 0, k \neq 0$

Lemme 2

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho-1} \, {}_2F_1\left[u-1, -u; v; \frac{hx}{hx+k(1-x)}\right] \mathrm{d}x =$$

$$= \frac{\pi 2^{1-2\rho}\Gamma(\rho)\Gamma(v)\Gamma(1+\rho-v)}{\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{1-u+v}{2}\right)\Gamma\left(\rho + \frac{u-v+1}{2}\right)\Gamma\left(\rho - \frac{u+v}{2}\right)} , Re(\rho) > 0, Re(v) > 0 \tag{2.2}$$

and $hx + k(1-x) \neq 0$, $h \neq 0, k \neq 0$

3. Main integrals

The following integrals have been establish in this section.

Let $X = \dfrac{hkx(1-x)}{(hx + k(1-x))^2}$ (3.1)

Theorem 1

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx + k(1-x)\}^{-2\rho}\, {}_2F_1\left[u, v; \frac{u+v+1}{2}; \frac{hx}{hx+k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^{\sigma})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \text{y}_1 X^{\mu_1} \\ \cdots \\ \cdots \\ \text{y}_s X^{\mu_s} \end{pmatrix} \aleph_{U:W}^{0,\mathfrak{n}:V}\begin{pmatrix} \text{z}_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ \text{z}_r X^{\sigma_r} \end{pmatrix} \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma\left(\frac{u+v+1}{2}\right)}{(hk)^{\rho}\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}a_1(4hk)^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s}K_i\mu_i)}$$

$$z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\aleph_{U_{20}:W}^{0,\mathfrak{n}+2:V}\begin{pmatrix} \frac{z_1}{(4hk)^{\sigma_1}} & \left| \left(\text{-}\rho + \frac{u+v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r}K_i\mu_i : \sigma_1,\cdots,\sigma_r\right), \right. \\ \cdots & \\ \cdots & \cdots \\ \frac{z_r}{(4hk)^{\sigma_r}} & \left(\text{-}\rho + \frac{u+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r}K_i\mu_i : \sigma_1,\cdots,\sigma_r\right), \end{pmatrix}$$

$$\left.\begin{array}{l} \left(1\text{-}\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r}K_i\mu_i : \sigma_1,\cdots,\sigma_r\right), A : C \\ \cdots \\ \left(\text{-}\rho + \frac{v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r}K_i\mu_i : \sigma_1,\cdots,\sigma_r\right), B : D \end{array}\right)$$ (3.2)

where $U_{22} = p_i + 2, q_i + 2, \tau_i; R$

Provided that

a) $min\{\sigma, \mu_i, \sigma_j\} > 0, i = 1,\cdots,s; j = 1,\cdots,r, Re\left(\dfrac{1-u-v}{2}\right) > 0$

b) $Re\left[\rho + \sigma\min\limits_{1\leqslant j\leqslant M}\dfrac{b_j}{B_j} + \sum\limits_{i=1}^{r}\sigma_i\min\limits_{1\leqslant j\leqslant m_i}\dfrac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$

c) $|argz_k| < \dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

d) $|argx| < \dfrac{1}{2}\pi\Omega$ where $\Omega = \sum\limits_{j=1}^{M}\beta_j + \sum\limits_{j=1}^{N}\alpha_j - c_i\left(\sum\limits_{j=M+1}^{Q_i}\beta_{ji} + \sum\limits_{j=N+1}^{P_i}\alpha_{ji}\right) > 0$

e) $hx + k(1-x) \neq 0$

Theorem 2

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\left\{hx+k(1-x)\right\}^{-2\rho+v-1}\,_2F_1\left[u-1,-u;v;\frac{hx}{hx+k(1-x)}\right]\aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^\sigma)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1X^{\mu_1} \\ \cdots \\ \cdots \\ y_sX^{\mu_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1X^{\sigma_1} \\ \cdots \\ \cdots \\ z_rX^{\sigma_r}\end{array}\right)\mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma(v)}{\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{v+1-u}{2}\right)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}a_1\,4^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s}K_i\mu_i)}$$

$$z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\aleph_{U20:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} \frac{z_1}{4^{\sigma_1}} & (1\text{-}\rho-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\cdots,\sigma_r), \\ \cdots & \cdots \\ \cdots & \\ \frac{z_r}{4^{\sigma_r}} & (\text{-}\rho+\frac{1-u+v}{2}-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\cdots,\sigma_r), \end{array}\right.$$

$$\left.\begin{array}{c} (v\text{-}\rho-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\cdots,\sigma_r),A:C \\ \cdots \\ (1\text{-}\rho+\frac{u+v}{2}-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\cdots,\sigma_r),B:D \end{array}\right) \tag{3.3}$$

where $U_{22}=p_i+2,q_i+2,\tau_i;R$

Provided that

a) $min\{\sigma,\mu_i,\sigma_j\}>0, i=1,\cdots,s; j=1,\cdots,r, Re(v)>0$

b) $Re\left[\rho+\sigma\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>0$

c)$|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

d) $|argx|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

e) $hx+k(1-x)\neq 0$

Proof

To prove (3.1), expressing the Aleph-function of one variable in series with the help of (1.3), the general class of polynomials of several variables in series with the help of (1.5), and the Aleph-function of r variables in MellinBarnes contour integral with the help of (1.7), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of the lemme 1. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

To prove (3.2), we use the similar method and use the lemme 2.

4. Multivariable I-function

If $\tau_i, \tau_{i(1)}, \cdots, \tau_{i(r)} \to 1$, the Aleph-function of several variables degenere to the I-function of several variables. The following two finite integrals have been derived in this section for multivariable I-functions defined by Sharma et al [4].

Corollary 1

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho} \, {}_2F_1\left[u, v; \frac{u+v+1}{2}; \frac{hx}{hx+k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^\sigma)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s}\end{array}\right) I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_r X^{\sigma_r}\end{array}\right) \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma\left(\frac{u+v+1}{2}\right)}{(hk)^\rho \Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)} \sum_{G=1}^{M} \sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G \, g!} \, a_1 \, (4hk)^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s} K_i\mu_i)}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} I_{U20:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \frac{z_1}{(4hk)^{\sigma_1}} \\ \cdots \\ \cdots \\ \frac{z_r}{(4hk)^{\sigma_r}} \end{array}\middle|\begin{array}{c} (\text{-}\rho + \frac{u+v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1,\cdots,\sigma_r), \\ \cdots \\ (\text{-}\rho + \frac{u+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1,\cdots,\sigma_r), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1,\cdots,\sigma_r), A:C \\ \cdots \\ (\text{-}\rho + \frac{v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1,\cdots,\sigma_r), B:D \end{array}\right) \tag{4.1}$$

under the same notations and conditions that (3.2) with $\tau_i, \tau_{i(1)}, \cdots, \tau_{i(r)} \to 1$

Corollary 2

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho+v-1} \, {}_2F_1\left[u-1, -u; v; \frac{hx}{hx+k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^\sigma)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s}\end{array}\right) I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_r X^{\sigma_r}\end{array}\right) \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma(v)}{\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{v+1-u}{2}\right)} \sum_{G=1}^{M} \sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G \, g!} \, a_1 \, 4^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s} K_i\mu_i)}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} I_{U_{20}:W}^{0,\mathfrak{n}+2:V} \left(\begin{array}{c} \frac{z_1}{4^{\sigma_1}} \\ \cdots \\ \cdots \\ \frac{z_r}{4^{\sigma_r}} \end{array} \middle| \begin{array}{c} (1\text{-}\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \cdots, \sigma_r), \\ \cdots \\ (\text{-}\rho + \frac{1-u+v}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \cdots, \sigma_r), \end{array} \right.$$

$$\left. \begin{array}{c} (\text{v-}\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \cdots, \sigma_r), A : C \\ \cdots \\ (1\text{-}\rho + \frac{u+v}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \cdots, \sigma_r), B : D \end{array} \right) \tag{3.3}$$

under the same notations and conditions that (3.3) with $\tau_i, \tau_{i^{(1)}}, \cdots, \tau_{i^{(r)}} \to 1$

5. Aleph-function of two variables

If $r = 2$, we obtain the Aleph-function of two variables defined by K.Sharma [6], and we have the following integrals.

Corollary 3

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho} \, {}_2F_1\left[u, v; \frac{u+v+1}{2}; \frac{hx}{hx+k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^\sigma)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s} \end{array} \right) \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_2 X^{\sigma_2} \end{array} \right) \mathrm{d}x = \frac{\pi 2^{1-2\rho} \Gamma\left(\frac{u+v+1}{2}\right)}{(hk)^\rho \Gamma\left(\frac{u+1}{2}\right) \Gamma\left(\frac{v+1}{2}\right)} \sum_{G=1}^{M} \sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} a_1 (4hk)^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s} K_i\mu_i)}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} \aleph_{U_{20}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c} \frac{z_1}{(4hk)^{\sigma_1}} \\ \cdots \\ \cdots \\ \frac{z_r}{(4hk)^{\sigma_r}} \end{array} \middle| \begin{array}{c} (\text{-}\rho + \frac{u+v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \\ \cdots \\ (\text{-}\rho + \frac{u+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \end{array} \right.$$

$$\left. \begin{array}{c} (1\text{-}\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), A : C \\ \cdots \\ (\text{-}\rho + \frac{v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), B : D \end{array} \right) \tag{5.1}$$

under the same notations and conditions that (3.2) with $r = 2$

Corollary 4

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} \{hx + k(1-x)\}^{-2\rho+v-1} \, {}_2F_1\left[u-1, -u; v; \frac{hx}{hx+k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^\sigma)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s} \end{array}\right) \aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_2 X^{\sigma_2} \end{array}\right) \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma(v)}{\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{v+1-u}{2}\right)} \sum_{G=1}^{M}\sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}a_1\,4^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s}K_i\mu_i)}$$

$$z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\aleph_{U_{20}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c} \frac{z_1}{4^{\sigma_1}} & (1\text{-}\rho-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\sigma_2), \\ \cdots & \cdots \\ \cdots & \\ \frac{z_r}{4^{\sigma_r}} & (\text{-}\rho+\frac{1-u+v}{2}-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\sigma_2), \end{array}\right.$$

$$\left.\begin{array}{c} (v\text{-}\rho-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\sigma_2),A:C \\ \cdots \\ (1\text{-}\rho+\frac{u+v}{2}-\sigma\eta_{G,g}-\sum_{i=1}^{r}K_i\mu_i:\sigma_1,\sigma_2),B:D \end{array}\right) \tag{5.2}$$

under the same notations and conditions that (3.3) with $r = 2$

6. I-function of two variables

If $\tau_i, \tau_i', \tau_i'' \to 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [5] and we obtain the same formulas with the I-function of two variables.

Corollary 5

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx+k(1-x)\}^{-2\rho}\,_2F_1\left[u,v;\frac{u+v+1}{2};\frac{hx}{hx+k(1-x)}\right]\aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^{\sigma})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s} \end{array}\right) I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_2 X^{\sigma_2} \end{array}\right) \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma\left(\frac{u+v+1}{2}\right)}{(hk)^{\rho}\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{v+1}{2}\right)} \sum_{G=1}^{M}\sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}a_1\,(4hk)^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s}K_i\mu_i)}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} I_{U20:W}^{0,\mathfrak{n}+2:V} \left(\begin{array}{c} \frac{z_1}{(4hk)^{\sigma_1}} \\ \cdots \\ \cdots \\ \frac{z_r}{(4hk)^{\sigma_r}} \end{array} \middle| \begin{array}{c} (-\rho + \frac{u+v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \\ \cdots \\ (-\rho + \frac{u+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \end{array} \right.$$

$$\left. \begin{array}{c} (-\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), A : C \\ \cdots \\ (-\rho + \frac{v+1}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), B : D \end{array} \right) \tag{6.1}$$

under the same notations and conditions that (3.2) with $r = 2$

Corollary 6

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}\{hx + k(1-x)\}^{-2\rho+v-1} \, {}_2F_1\left[u-1, -u; v; \frac{hx}{hx + k(1-x)}\right] \aleph_{P_i,Q_i,\mathfrak{c}_i;r'}^{M,N}(zX^{\sigma})$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s} \left(\begin{array}{c} y_1 X^{\mu_1} \\ \cdots \\ \cdots \\ y_s X^{\mu_s} \end{array} \right) I_{U:W}^{0,\mathfrak{n}:V} \left(\begin{array}{c} z_1 X^{\sigma_1} \\ \cdots \\ \cdots \\ z_2 X^{\sigma_2} \end{array} \right) \mathrm{d}x = \frac{\pi 2^{1-2\rho}\Gamma(v)}{\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{v+1-u}{2}\right)} \sum_{G=1}^{M} \sum_{g=0}^{\infty}$$

$$\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} a_1 \, 4^{-(\sigma\eta_{G,g}+\sum_{i=1}^{s} K_i\mu_i)}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} I_{U20:W}^{0,\mathfrak{n}+2:V} \left(\begin{array}{c} \frac{z_1}{4^{\sigma_1}} \\ \cdots \\ \cdots \\ \frac{z_r}{4^{\sigma_r}} \end{array} \middle| \begin{array}{c} (1-\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \\ \cdots \\ (-\rho + \frac{1-u+v}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), \end{array} \right.$$

$$\left. \begin{array}{c} (v-\rho - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), A : C \\ \cdots \\ (1-\rho + \frac{u+v}{2} - \sigma\eta_{G,g} - \sum_{i=1}^{r} K_i\mu_i : \sigma_1, \sigma_2), B : D \end{array} \right) \tag{6.2}$$

under the same notations and conditions that (3.3) with $r = 2$

Remark : Specializing the parameters, we obtain the results of Chaurasia [2].

7. Conclusion

In this paper we have evaluated two general finite integrals involving the multivariable Aleph-function, a class of polynomials of several variables and a Aleph-function of one variable and apolynomial system.The integral established in this paper is of very general nature as it contains multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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