A Note on Convolution and Composite Convolution Operators

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Abstract. *In this note we obtain the condition for convolution and composite convolution operators to be bounded and Hermition .We also find that only the compact composite convolution operator is the Zero operator.*

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Introduction. For p = 1, 2, let $\lambda^p(z)$ denote the space of pth summable sequence of complex numbers. If p = 2 then $\lambda^2(z)$ is Hilbert space under the inner product

$$\langle f,g\rangle=\sum_{n=-\infty}^{\infty} f_n \bar{g}_n \text{ and for } p=1, \lambda^1(z),$$

is a Banach space under the norm

$$\|x\|=\sum_{n=-\infty}^{\infty} |x_n|.$$

If $\phi\in\lambda^1(z)$, $f\in\lambda^2(z)$ then we form the convolution product $f*\phi$ which is defined by

$$(f*\phi)(m)=\sum_{n=-\infty}^{\infty} f(n)\phi(m-n).$$

If $T:Z\to Z$ is a mapping such that the transformation $C_{T,\phi}:\lambda^2(z)\to\lambda^2(z)$ defined by $(C_{T,\phi}f)=(f*\phi)oT$ is bounded we call $C_{T,\phi}$ a composite convolution operator induced by the pair (ϕ,T). In case T(z) = z for all $z\in Z$. We write $C_{T,\phi}=C_\phi$ which is known as convolution operator.

In this paper we study the convolution and composite convolution operators. The Hermition ,Bounded and Compact convolution operators are characterized. For literature concerning Composite operators and Convolution operators , we refer to singand komal[11], komal and gupta [2],komal and sharma [3], kumar [4], Nordgren [6], Ridge [7], gupta and komal [5], singh , gupta and komal [8].

2. Bounded convolution operators:

In this, section the convolution operators to be bounded and hermition operators be studied

Theorem 2.1 Let $\phi\in\lambda^2(z)$be such that $\phi(m)=\tilde{\phi}(m)$. Then $C_\phi^*=C_{\phi^*}$.

Proof: For $f,g\in\lambda^2(z)$ .We have

$$\begin{aligned}
\langle C_\phi f,g\rangle &= \sum_{n=-\infty}^{\infty} C_\phi f(n)\overline{g(n)} \\
&= \sum_{n=-\infty}^{\infty} (f*\phi)(n)\overline{g(n)} \\
&= \sum_{n=-\infty}^{\infty}\Big(\sum_{m=-\infty}^{\infty} f(m)\phi(n-m)\,\overline{g(n)}\Big) \\
&= \sum_{m=-\infty}^{\infty} f(m)\sum_{n=-\infty}^{\infty}\overline{g(n)}\;\overline{\phi(m-n)} \\
&= \sum_{m=-\infty}^{\infty} f(m)\,\overline{g*\phi}\,(m)
\end{aligned}$$

$$= \quad < f, C_f^* g >.$$

Hence $C_f^* = C_f^*$.

Example 2.2: Let □:z□z be defined by f(*n*)= $\frac{1}{n^2}$ .Then fÎλ^1(*Z*). Therefore C_fÎ*B*(λ^2(*Z*))

$$C_f^* f)(n) \quad = \quad < C_f^* f, e_n >$$

$$= \quad < f, C_f e_n >$$

$$= \quad \sum_{m=-¥}^{¥} f(m)\ \overline{C_f}\ (m)$$

$$= \sum_{m=-\infty}^{\infty} f(m)\overline{(C_\varphi\ e_n)}\ (m)$$

$$= \quad \sum_{m=-¥}^{¥} f(m)\ \overline{e_n * f}\ (m)$$

$$= \quad \sum_{m=-\infty}^{\infty} f(m)\left(\sum_{p=-\infty}^{\infty} e_n(p)\phi(m-p)\right)$$

$$= \quad \sum_{m=-¥}^{¥} f(m) \frac{1}{(m-n)^2}$$

and

$$(C_f f)(n) \quad = \quad ác_f f, e_n ñ$$

$$= \quad \sum_{m=-¥}^{¥} (f*f)(m) e_n(m)$$

$$= \quad (f*f)(n)$$

$$= \quad \sum_{m=-¥}^{¥} f(m) f(n-m))$$

$$= \quad \sum_{m=-¥}^{¥} f(m) \frac{1}{(n-m)^2}$$

$$= \quad \sum_{m=-¥}^{¥} f(m) \frac{1}{(m-n)^2}$$

$$= \quad (C_f^* f)(n) \text{ for all } f Î \lambda^2(z) \text{ and } n Î Z.$$

Hence C_f is Hermition.

Theorem 2.3.: C_φ is Hamiltonian iff $\varphi = \varphi^*$

Proof.: if $\varphi = \varphi^*$, then clearly $C_\varphi = C_\varphi^* = C_{\varphi^*}$

Hence C_φ is Hamiltonian

Conversely, suppose $C_\varphi = C_{\varphi^*}$ or $C_\varphi = C_{\varphi^*}$

Now $(C_\varphi \; e_n)(m) = (e_n \times \varphi)(m)$

$$= \sum_{m=-\infty}^{\infty} e_n(p)\varphi(m-p)$$

$= \varphi(m-n)$,

and

$= (C_\varphi * e_n)(m) = \varphi^*(m-n)$

$= \varphi(n-m)$

$= \varphi(m-n)$

$= (C_\varphi \; e_n)(m)$

This is true for every m $\in$ X so that $(C_\varphi \; e_n) = C_{\varphi^*} \; e_n$ for every n $\in$ Z. Hence $C_\varphi = C_\varphi^*$, since $\{e_n\}_{n\in z}$ is a basis for l^2 (z).

3. Composite convolution operators

In this section we study the composite convolution operators on $l^2(z)$.

Theorem 3.1:- Let $C_{T,\varphi} \in B(l^2(z))$. Then $C_{T,\varphi}$ is Hamiltonian iff

$\bar{\Phi}(T(m)-n) = \Phi(T(n)-m)$

Proof: suppose the condition is true n $\in$ Z, we have

$$(C_{T,\varphi} \; e_n)(m) = (e_n * \varphi)(T(m))$$

$$= \sum_{p=-\infty}^{\infty} e_n(p)\varphi(T(m)-p)$$

$$= \varphi(T(m)-p) \text{-------------------------------------} \quad 1$$

and

$$(C_{T,\varphi}^* \; e_n)(m) = \sum_{p=-\infty}^{\infty} e_n(p)\bar{\varphi}(T(n)-m)$$

$$= \sum_{p=-\infty}^{\infty} \bar{\varphi}(T(n)-m) \text{ ------------------------------------ } 2$$

Since 1 and 2 are equal

so,

$(C_{T,\varphi}^* \; e_n) = C_{T,\varphi} \; e_n \forall \; n \in z$

Hence $C_{T,\varphi}^* = C_{T,\varphi}$

Since $C_{T,\varphi}$ is Hamiltonian

The proof of converse part is obvious.

Example 3.2 Let $\varphi: Z \to Z$ be defined by $\varphi = \frac{1}{(n-1)^2}$ and

$T: Z \to Z$ Be defined by T(m)= m+1 ∀ m ∈ z

Then

$\varphi \in l^{'}(z)$ and f_0 (m)=1 ∀ m∈ Z

Hence $C_{T,\varphi} \in B(l^2(z))$.

Now $(C_{\varphi}^{*} f)(n) = < C_{T,\varphi}^{*} f, e_n >$

$$= < f, (C_{T,\varphi}), e_n >$$

$$= \sum_{m=-\infty}^{\infty} f(m)\overline{(C_{T,\varphi}\, e_n)T}(m)$$

$$= \sum_{m=-\infty}^{\infty} f(m)\overline{(C_{\varphi} * e_n)T}(m)$$

$$= \sum_{m=-\infty}^{\infty} f(m) \left(\sum_{p=-\infty}^{\infty} e_n(p)\varphi(T(m) - p)\right)$$

$$= \sum_{m=-\infty}^{\infty} f(m)\varphi(\overline{T(m) - n})$$

$$= \sum_{m=-\infty}^{\infty} f(m)\varphi(\overline{(m+1) - n})$$

$$= \sum_{m=-\infty}^{\infty} f(m) \frac{1}{(m-n)^2} \text{ ------------------------------------1}$$

$$(C_{T,\varphi}, f)(m) = < C_{T,\varphi} f, e_n > = \sum_{p=-\infty}^{\infty} (f * \varphi)(T(p))\overline{e_n(p)}$$

$$= (f * \varphi)(T(n))$$

$$= \sum_{m=-\infty}^{\infty} f(m)\varphi(T(n) - m)$$

$$= \sum_{m=-\infty}^{\infty} f(m)\ \varphi((n+1)-m)$$

$$= \sum_{m=-\infty}^{\infty} f(m)\ \frac{1}{(m-n)^2} \quad \text{---2}$$

From 1 and 2, we have

$$(C_{T,\varphi} f)(n) = (C^*_{T,\varphi} f)(n)$$

Hence $C_{T,\varphi}$ is Hermition

Theorem 3.3 Let $C_{T,\varphi} \in B(l^2(z))$. Then $C_{T,\varphi}$ n. Compact if and only if i.e. $C_{T,\varphi} = 0$

Proof. If $C_{T,\varphi} \neq 0$ Then f_0 (p) $(T_n\ \varphi)$ (p) $\neq \varphi$ for some p $\in$ Z

For n $\in$ Z

$$\|C_{T,\varphi}\ e_n\| = \sum_{m=-\infty}^{\infty} |(e_n * \varphi)\ (T(m))\ |^2$$

$$= \sum_{m=-\infty}^{\infty} f_0(m) |(e_n * \varphi)\ (T(m))\ |^2$$

$$\leq \sum_{m=-\infty}^{\infty} f_0(m)\ [\ \sum_{p=-\infty}^{\infty} |(e_n(p))\ |^2\ |\varphi(m-p)|^2\ ]$$

$$= \sum_{m=-\infty}^{\infty} f_0(m) |\varphi(m-n)\ |^2$$

$$= \sum_{m=-\infty}^{\infty} f_0(m) |(T_n\ \varphi)(m)\ |^2$$

$$\geq f_0(p)\ |(T_n\ \varphi)\ (p)|^2$$

This is the sequence $\{e_n\}$ does not converge to zero. This proves that $C_{T,\varphi}$ does not compact.

Thus if $C_{T,\varphi}$ is compact, then $C_{T,\varphi} = 0$

Cor 3.4. If $\varphi(0) = 0$, then $C_{T,\varphi}$ is not compact.

Proof:- From the above theorem, we can conclude that

$\|C_{T,\varphi}\ e_n\| \geq \varphi(0)$ for infinitely many n $\in$ T(z).

This proves that $C_{T,\varphi}\ e_n \to 0$ strongly. But $e_n \to 0$ weakly hence $C_{T,\varphi}$ cannot be compact.

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