A note on Bicomplex Linear operators on bicomplex Hilbert spaces

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Abstract: *In this paper we define the isomorphism between the bicomplex Hilbert spaces. We also give some simple and basic results on bicomplex isomorphism with respect to hyperbolic-valued norm on the bicomplex Hilbert spaces.*

Keywords: Bicomplex numbers, hyperbolic norm, bicomplex isomerty.

1 Introduction and preliminaries

Firstly, we give some basic definitions and important properties of bicomplex numbers and bicomplex holomorphic functions. Let i, j be two imaginary units such that $ij = ji, i^2 = j^2 = -1$. The set of bicomplex numbers $\mathbf{BC}$ is defined as

$$\mathbf{BC} = \{Z = z + jw : z, w \in \mathbf{C}(i)\}.$$

Let $Z_1 = z + jw$ and $Z_2 = s + jt$, be any two bicomplex numbers. Then the addition and multiplication of bicomplex numbers is defined as $Z_1 + Z_2 = (z + jw) + (s + jt) = (z + s) + j(w + t)$, $Z_1.Z_2 = (z+jw)(s+jt) = (zs-wt)+j(ws+zt)$. With respect to above binary operations a bicomplex number set turn out to be a ring and is a module over itself. A bicomplex number is a combination of two complex numbers. The set of complex numbers is a subring of bicomplex numbers. If we denote the real number z by x and imaginary number w by iy such that $ij = k$, then the set of hyperbolic numbers is defined as

$$\mathbf{D} = \{x + ky : k^2 = 1 \text{ with } k \notin \mathbf{R}\}.$$

The bicomplex numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ are zero divisions and are linearly independent in the complex plane and have the following properties: $e_1 + e_2 = 1, e_1 - e_2 = ij, e_1.e_2 = 0, {e_1}^2 = e_1, {e_2}^2 =$

e_2. Thus, the bicomplex number is not a division algebra. The numbers e_1 and e_2 are also known as hyperbolic numbers and form the basis for bicomplex numbers. For any bicomplex number $Z = z + jw$, the three kind of congugations can be defined as follows: (i) $Z^\sharp = \bar{z}+j\bar{w}$, (ii) $Z^\dagger = z-jw$, (iii) $Z^\ddagger = \bar{z} - j\bar{w}$, where $\bar{z}, \bar{w}$ denote the complex congugates to $z, w \in \mathbf{C}(i)$. For each congugation, a bicomplex modulus is defined by

$$|Z|_j^2 = Z.Z^\sharp, |Z|_i^2 = Z.Z^\dagger, |Z|_k^2 = Z.Z^\ddagger.$$

If $Z = z + jw$, then the idempotent representation of a bicomplex number can be written as

$$Z = e_1 z_1 + e_2 z_2,$$

where $\{z_1 = z - iw$ and $z_2 = z + iw\}$ are in $\mathbf{C}(i)$. By the above representation of bicomplex number, we can write

$$\mathbf{BC} = e_1\mathbf{BC} + e_2\mathbf{BC}.$$

There is a vast literature on bicomplex analysis, see [1], [7] and [15] for details. Now, we begin with the definition of norms, Inner products on BC-modules. We refer to [1], [6], [7] for following definitions.

Definition 1.1 *Let X be a $\mathbf{BC}$-module and X_1 and X_2 be the complex linear spaces, (see [17]) then we can write*

$$X = e_1 X_1 + e_2 X_2,$$

and is called the idempotent decomposition of X.

If X_1 and X_2 are normed linear spaces with norms $\|.\|_1$ and $\|.\|_2$ respectively. Then for each $x = e_1x_1 + e_2x_2 \in X$, for all $x_1 \in X_1$ and $x_2 \in X_2$, we have

$$\begin{aligned} \|x\|^2 &= \|e_1x_1 + e_2x_2\|^2 \\ &= \|e_1x_1\|_1^2 + \|e_2x_2\|_2^2 \\ &= |e_1|^2\|x_1\|_1^2 + |e_2|^2\|x_2\|_2^2 \\ &= \tfrac{1}{2}(\|x_1\|_1^2 + \|x_2\|_2^2). \end{aligned}$$

It is well known that $\|.\|$ defines a real-valued norm on X and $\|\alpha x\| \leq \sqrt{2}|\alpha|\|x\|$ for any $x \in X, \alpha \in \mathbf{BC}$. This norm shows that for any bicomplex numbers Z_1 and Z_2,

$$|Z_1.Z_2| < \sqrt{2}|Z_1||Z_2|.$$

Since, the inner product square is positive hyperbolic number so it gives an idea of hyperbolic-valued ($\mathbf{D}$-valued) norm of a $\mathbf{BC}$-module and is defined as:

$$\begin{aligned} \|x\|_{\mathbf{D}}^2 &= \|e_1x_1 + e_2x_2\|_{\mathbf{D}}^2 \\ &= e_1\|x_1\|_1^2 + e_2\|x_2\|_2^2 \\ &= e_1 < x_1, x_1 >_1 + e_2 < x_2, x_2 >_2 \\ &= < e_1x_1 + e_2x_2, e_1x_1 + e_2x_2 >_{\mathbf{D}} \\ &= < x, y >_{\mathbf{D}}. \end{aligned}$$

A $\mathbf{BC}$-module X endowed with a bicomplex inner product $< .,. >$ is called $\mathbf{BC}$-inner product module. Let X_1 and X_2 be two linear spaces. Assume that X_1 and X_2 are inner product spaces with inner product $< .,. >_1$ and $< .,. >_2$, respectively and corresponding norms $\|.\|_1$ and $\|.\|_2$. Then for any $x, y \in X$, $< x, y >_X = < e_1x_1 + e_2x_2, e_1y_1 + e_2y_2 >_X = e_1 < x_1, y_1 >_1 + e_2 < x_2, y_2 >_2$ defines a bicomplex inner product on the bicomplex module X. Moreover, the above bicomplex inner product norm introduces a hyperbolic-valued norm on inner product $\mathbf{BC}$-module X defined as

$$\|x\|_{\mathbf{D}} = e_1\|x_1\|+e_2\|x_2\|_{\mathbf{D}} =< x, x >^{\frac{1}{2}} . \quad (1)$$

Theorem 1.2 *[9, Theorem 3.5] Let X be a $\mathbf{BC}$-module. Then $X = e_1X_1 + e_2X_2$ is a bicomplex Banach-module if and only if X_1, X_2 are complex Banach spaces.*

Definition 1.3 *[1, P. 53] A $\mathbf{BC}$-module X with inner product $< ., . >_X$ is said to be a bicomplex Hilbert space if it is complete with respect to the $\mathbf{D}$-valued norm generated by the bicomplex inner product. Thus $X = e_1X_1 + e_2X_2$ is bicomplex Hilbert space if and only if $(X_1, < ., . >_1)$ and $(X_2, < ., . >_2)$ are complex Hilbert spaces.*

Definition 1.4 *Let X and Y be two $\mathbf{BC}$-modules with hyperbolic norms. A mapping $T : X \to Y$ is said to be bicomplex linear operator on X if $T(\alpha x + \gamma y) = \alpha T(x) + \gamma T(y),\ \ for\ all$ $x, y \in X$ and $\alpha, \gamma \in \mathbf{BC}$. The idempotent representation of T is given by*

$$T = e_1T_1 + e_2T_2, \quad (2)$$

where T_1, T_2 are complex linear operators and e_1, e_2 form basis of bicomplex numbers and so called idempotent basis of bicomplex numbers.The set B(X,Y) denotes the space of all $\mathbf{D}$-bounded $\mathbf{BC}$- linear operators and the norm for each $T \in B(X, Y)$ (see [1]) is defined as

$$\|T\|_{\mathbf{D}} = \sup\{\|T(x)\|_{\mathbf{D}}, x \in X, \|x\|_{\mathbf{D}} <' 1\}$$

and is called hyperbolic norm on T and so we can write

$$\|T\|_{\mathbf{D}} = \|e_1T_1+e_1T_2\|_{\mathbf{D}} = e_1\|T\|_1+e_2\|T\|_2,$$

where $\|.\|_1$ and $\|.\|_2$ are usual norms defined on complex linear operators T_1 and T_2 respectively.

Definition 1.5 *Let X and Y be two bicomplex Hilbert spaces with inner product $< ., . >_X$ and $< ., . >_Y$ respectively. Then the bicomplex adjoint operator $T^* : Y \to X$ for a bounded linear operator $T : X \to Y$ is defined by*

$$< Tx, y >_Y =< x, T^*y >_X .$$

The bicomplex adjoint operator T^* can be written as

$$T^* = e_1T_1^* + e_2T_2^*,$$

where T_1^* and T_2^* are the complex adjoint operators of T_1 and T_2 respectively.

Definition 1.6 *Let X be a bicomplex Hilbert space and $T \in B(X)$. Then T is said to be a*

(a) bicomplex self adjoint if $T = T^*$,

(b) bicomplex normal operator if $TT^* = T^*T$,

(c) bicomplex unitary if $TT^* = T^*T = I$, where I is an identity operator on X.

For systematic study of bicomplex functional analysis, we refer to [1], [2], [6], [7], [8], [11], [12] and references therein.

2 Bicomplex isomorphism

In this section, we investigate bicomplex isometry between the bicomplex Hilbert spaces..

Definition 2.1 *A bicomplex isomorphism T of a bicomplex Hilbert space X onto $\tilde{X}$ is a bicomplex bijective linear operator $T : X \to \tilde{X}$ which preserves the bicomplex inner product, i.e., for all $x, y \in X$*

$$< Tx, Ty >=< x, y >$$

$\tilde{X}$ is said to be the bicomplex isomorphic to X. X and $\tilde{X}$ are called bicomplex isomorphic inner product spaces.

We can easily prove the following proposition.

Proposition 2.2 *Let X be a bicomplex Hilbert space and $T \in B(X)$, the space of all bicomplex $\mathbf{D}$-bounded linear operators from X to itself. Then T is bicomplex isometric on X if and only if T_1 and T_2 are isometric on X.*

Proposition 2.3 *Let X be a bicomplex Hilbert space with hyperbolic norm $\|.\|_{\mathbf{D}}$ and $T \in B(X)$. Then T is a bicomplex isometry if and only if $< Tx, Ty >_X=< x, y >_X$; for all $x, y \in X$.*

Proof. Since $T \in B(X)$, so we can write $T = e_1T_1 + e_2T_2$, where T_1 and T_2 are complex linear operators. Also for any $x, y \in X$, we have $x = e_1x_1 + e_2x_2$ and $y = e_1y_1 + e_2y_2$.
Suppose $< Tx, Ty >_X=< x, y >_X$. Then

$$\|Tx\|^2_{\mathbf{D}} =< Tx, Tx >_X$$
$$= < x, x >_X$$
$$= < e_1x_1 + e_2x_2, e_1x_1 + e_2x_2 >_X$$
$$= e_1 < x_1, x_1 >_X + e_2 < x_2, x_2 >_X$$
$$= e_1\|x_1\|^2_{\mathbf{D}} + e_2\|x_2\|^2_{\mathbf{D}}$$
$$= \|x\|^2_{\mathbf{D}}.$$

Hence T is isometry.
Conversely, if $x, y \in X$ and $\alpha \in \mathbf{BC}$, then
$\|x + \alpha y\|^2_{\mathbf{D}} = \|Tx + \alpha Ty\|^2_{\mathbf{D}}$
$\Rightarrow < x + \alpha y, x + \alpha y >_X=< Tx + \alpha Ty, Tx + \alpha Ty >_X$
$\Rightarrow< e_1(x_1 + \alpha_1y_1) + e_2(x_2 + \alpha_2y_2), e_1(x_1 + \alpha_1y_1) + e_2(x_2 + \alpha_2y_2) >_X$
$=< e_1(T_1x_1 + \alpha_1T_1y_1) + e_2(T_2x_2 + \alpha_2T_2y_2), e_1(T_1x_1 + \alpha_1Ty_1) + e_2(Tx_2 + \alpha_2Ty_2) >_X$

$\Rightarrow e_1 < x_1 + \alpha_1y_1, x_1 + \alpha_1y_1 >_1 +e_2 < x_2 + \alpha_2y_2, x_2 + \alpha_2y_2 >_2$
$= e_1 < T_1x_1+\alpha_1T_1y_1, T_1x_1+\alpha_1T_1y_1 >_1 +e_2 < T_2x_2+\alpha_2T_2y_2, T_2x_2+\alpha_2T_2y_2 >_2$

$\Rightarrow e_1 < x_1, x_1 >_1 +e_1 < x_1, \alpha_1y_1 >_1 +e_1 < \alpha_1y_1, x_1 >_1 +e_1 < \alpha_1y_1, \alpha_1y_1 >_1 +e_2 < x_2, x_2 >_2 +e_2 < x_2, \alpha_2y_2 >_2 +e_2 < \alpha_2y_2, x_2 >_2 +e_2 < \alpha_2y_2, \alpha_2y_2 >_2$
$= e_1 < T_1x_1, T_1x_1 >_1 +e_1 < T_1x_1, \alpha_1T_1y_1 >_1 +e_1 < \alpha_1T_1y_1, T_1x_1 >_1 +e_1 < \alpha_1T_1y_1, \alpha_1T_1y_1 >_1 +e_2 < T_2x_2, T_2x_2 >_2 + e_2 < T_2x_2, \alpha_2T_2y_2 >_2 +e_2 < \alpha_2T_2y_2, T_2x_2 >_2 +e_2 < \alpha_2T_2y_2, \alpha_2T_2y_2 >_2$

$\Rightarrow (e_1\|x_1\|^2_{\mathbf{D}} + e_2\|x_2\|^2_{\mathbf{D}}) + (e_1|\alpha_1|^2\|y_1\|^2_{\mathbf{D}} + e_2|\alpha_2|^2\|y_2\|^2_{\mathbf{D}}) + e_1$

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$< x_1, \alpha_1 y_1 >_1 + e_2 < x_2, \alpha_2 y_2 >_2$
$+ e_1 < \alpha_1 y_1, x_1 >_1 + e_2 < \alpha_2 y_2, x_2 >_2$
$= (e_1 \|T_1 x_1\|^2{}_{\mathbf{D}} + e_2 \|T_2 x_2\|^2{}_{\mathbf{D}}) + (e_1 |\alpha_1|^2 \|T_1 y_1\|^2{}_{\mathbf{D}} + e_2 |\alpha_2|^2 \|T_2 y_2\|^2{}_{\mathbf{D}})$
$+ e_1 < T_1 x_1, \alpha_1 T_1 y_1 >_1 + e_2 < T_2 x_2, \alpha_2 T_2 y_2 >_2 + e_1 < \alpha_1 T_1 y_1, T_1 x_1 >_1$
$+ e_2 < \alpha_2 T_2 y_2, T_2 x_2 >_2$

$\Rightarrow \|x\|^2{}_{\mathbf{D}} + |\alpha|^2 \|y\|^2{}_{\mathbf{D}} + < e_1 x_1 + e_2 x_2, e_1 \alpha_1 y_1 + e_2 \alpha_2 y_2 >_X + < e_1 \alpha_1 y_1 + e_2 \alpha_2 y_2, e_1 x_1 + e_2 x_2 >_X$
$= \|Tx\|^2{}_{\mathbf{D}} + |\alpha|^2 \|Ty\|^2{}_{\mathbf{D}} + < e_1 T_1 x_1 + e_2 T_2 x_2, e_1 \alpha_1 T_1 y_1 + e_2 \alpha_2 T_2 y_2 >_X$
$+ < e_1 \alpha_1 T_1 y_1 + e_2 \alpha_2 T_2 y_2, e_1 T_1 x_1 + e_2 T_2 x_2 >_X$

$\Rightarrow \|x\|^2{}_{\mathbf{D}} + |\alpha|^2 \|y\|^2{}_{\mathbf{D}} + < x, \alpha y >_X + < \alpha y, x >_X$
$= \|Tx\|^2{}_{\mathbf{D}} + |\alpha|^2 \|Ty\|^2{}_{\mathbf{D}} + < Tx, \alpha Ty >_X + < \alpha Ty, Tx >_X$

$\Rightarrow \|x\|^2{}_{\mathbf{D}} + |\alpha|^2 \|y\|^2{}_{\mathbf{D}} + 2Re\ \alpha < x, y >_X$
$= \|Tx\|^2{}_{\mathbf{D}} + |\alpha|^2 \|Ty\|^2{}_{\mathbf{D}} + 2Re\ \alpha < Tx, Ty >_X$

$\Rightarrow < x, y >_X = < Tx, Ty >_X$. ■

Proposition 2.4 *Let $T \in B(X)$ such that $T = e_1 T_1 + e_2 T_2$ be its idempotent decomposition with $I = e_1 I_1 + e_2 I_2$. Then following conditions are equivalent:*

(i) T is a bicomplex isometry;

(ii) $T^* T = I$;

(iii) $< Tx, Ty >_X = < x, y >_X$; for all $x, y \in X$.

Proof. By Proposition 2.3, (i) and (iii) are equivalent. Further, for any $x, y \in X$, we have
$< Tx, Ty >_X = < x, y >_X$

$$\begin{aligned}
&\Leftrightarrow && < e_1 T_1 x_1 + e_2 T_2 x_2, e_1 T_1 y_1 + e_2 T_2 y_2 >_X \\
&= && < e_1 x_1 + e_2 x_2, e_1 y_1 + e_2 y_2 >_X \\
&\Leftrightarrow && e_1 < T_1 x_1, T_1 y_1 >_1 + e_2 < T_2 x_2, T_2 y_2 >_2 \\
&= && e_1 < x_1, y_1 >_1 + e_2 < x_2, y_2 >_2 \\
&\Leftrightarrow && e_1 < T_1^* T_1 x_1, y_1 >_1 + e_2 < T_2^* T_2 x_2, y_2 >_2 \\
&= && e_1 < I_1 x_1, y_1 >_1 + e_2 < I_2 x_2, y_2 >_2 \\
&\Leftrightarrow && < (T_1^* T_1 - I_1) x_1, y_1 >_1 = 0 \text{ and } < (T_2^* T_2 - I_2) x_2, y_2 >_2 = 0 \\
&\Leftrightarrow && T_1^* T_1 = I_1 \text{ and } T_2^* T_2 = I_2 \\
&\Leftrightarrow && T^* T = I.
\end{aligned}$$

Hence $(ii) \Leftrightarrow (iii)$ condition holds. ■
The next result is the immediate consequence of Proposition 2.4.

Corollary 2.5 *Let X be a bicomplex Hilbert space and $T \in B(X)$. Then following conditions are equivalent:*
(i) $T^* T = T T^* = I$;
(ii) T is bicomplex normal isometry;
(iii) T is bicomplex isometry.

Proposition 2.6 *If X is a bicomplex Hilbert space and $T \in B(X)$ such that $< Tx, x >_X = 0$; for all $x \in X$, then $T = 0$.*

Proof. Since $T \in B(X)$, using (2) we write $T = e_1T_1 + e_2T_2$. Further, for any complex operators T_l for $l = 1, 2$ such that $< T_lx_l, x_l >_X = 0$, we have $T_l = 0$ for $l = 1, 2$.
Now $< Tx, x >_X = 0$
$\Rightarrow < e_1T_1x_1 + e_2T_2x_2, e_1x_1 + e_2x_2 >_X = 0$
$\Rightarrow e_1 < T_1x_1, x_1 >_1 + e_2 < T_2x_2, x_2 >_2 = 0$
$\Rightarrow e_1 < T_1x_1, x_1 >_1 = 0$ and $e_2 < T_2x_2, x_2 >_2 = 0$
$\Rightarrow T_1 = 0$ and $T_2 = 0$ which gives $T = e_1T_1 + e_2T_2 = 0$ and hence $T = 0$. ■

The following proposition follows easily.

Proposition 2.7 *Let X be a bicomplex Hilbert space with real valued norm $\|.\|_X$ and $T \in B(X)$. Then T is bicomplex isometry on X if and only if*

$$< Tx, Ty >_X = < x, y >_X; \quad \textit{for all} \quad x, y \in X.$$

Lemma 2.1 *Let X be a bicomplex Hibert space with $\mathbf{D}$-valued norm and $T \in B(X)$. If T is self-adjoint, i.e., $T^* = T$, then $\|T\|_{\mathbf{D}} = \sup\{| < Tx, x > |; \|x\|_{\mathbf{D}} = 1\}$.*

Proof. We can write $T = e_1T_1 + e_2T_2$, where $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ are complex linear operators. Also, any bicomplex Hilbert space can be written as $X = e_1X_1 + e_2X_2$.

Now $\sup\{| < Tx, x > |; \|x\|_{\mathbf{D}} = 1\}$

$$\begin{aligned}
&= \sup\{|e_1 < T_1x_1, x_1 > + e_2 < T_2x_2, x_2 > |; \\
&\quad \|x_1\|_{\mathbf{D}} = 1, \|x_2\|_{\mathbf{D}} = 1\} \\
&= e_1 \sup\{| < T_1x_1, x_1 >_1 |; \|x_1\|_{\mathbf{D}} = 1\} \\
&+ e_2 \sup\{< T_2x_2, x_2 >_2 |; \|x_2\|_{\mathbf{D}} = 1\} \\
&= e_1\|T_1\|_{\mathbf{D}} + e_2\|T_2\|_{\mathbf{D}} \\
&= \|T\|_{\mathbf{D}}.
\end{aligned}$$

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