Global Domination in Permutation

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Abstract

If i, j belong to a permutation π on n symbols $A = \{1, 2, ..., n\}$ and $i < j$ then the line of i crosses the line of j in the permutation if i appears after j in the image sequence $s(\pi)$ and if the no. of crossing lines of i is less than the no. of crossing lines of j then i global dominates j. A subset D of A, whose closed neighborhood is A in π is a dominating set of π. D is a global dominating set of π if every i in $A - D$ is global dominated by some j in D. In this paper the global domination number of a permutation is investigated by means of crossing lines.

Keywords

Perumutation - Permutation graph - Global domination.

1 Introduction

Sampathkumar introduced the Global Domination Number of a Graph. Adin and Roichman introduced the concept of permutation graphs and Peter Keevash, Po-Shen Loh and Benny Sudakov identified some permutation graphs with maximum number of edges. J.Chithra, S.P.Subbiah and V.Swaminathan introduced the concept of Domination in Permutation graphs. If i, j belongs to a permutation on n symbols $\{1, 2, ..., n\}$ and i is less than j then there is an edge between i and j in the permutation graph if i appears after j. (i. e) inverse of i is greater than the inverse of j. So the line of i crosses the line of j in the permutation. So there is a one to one correspondence between crossing of lines in the permutation and the edges of the corresponding permutation graph. In this paper we found the global domination number of a permutation and also derived the global domination number of permutation graph through the permutation.

2 Permutation Graphs

Definition 2.1.

Let π be a *permutation* on n symbols $\{a_1, a_2, ..., a_n\}$ where image of a_i is $a_i^{'}$.
Then the *permutation graph* G_π is given by (V_π, E_π) where $V_\pi = \{a_1, a_2, ..., a_n\}$ and $a_i, a_j \in E_\pi$ if $(a_i - a_j)(\pi^{-1}(a_i) - \pi^{-1}(a_j)) < 0$.

Definition 2.2.

Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, ..., a_n\}$ given by
$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4... & a_n \\ a_1^{'} & a_2^{'} & a_3^{'} & a_4^{'}... & a_n^{'} \end{pmatrix}$, where $|a_{i+1} - a_i| = c, c > 0, 0 < i \leq n-1$.

The sequence of π is given by $s(\pi) = \{a_1^{'}, a_2^{'}, a_3^{'}, ..., a_n^{'}\}$. When elements of A are ordered in L_1 and the sequence of π are represented in L_2 , then a line joining a_i inL_1 and a_i in L_2 is represented by l_i . This is known as line representation of a_i in π.

Definition 2.3.

The element a_i is said to *dominate* a_j if their lines cross each other in π. The set of collection of elements of π whose lines cross all the lines of the elements $a_1, a_2, ..., a_n$ in π is said to be a *dominating set* of π. $V = \{a_1, a_2, ..., a_n\}$ is always a dominating set.

Definition 2.4.

The subset D of $\{a_1, a_2, ..., a_n\}$ is said to be a *Minimal Dominating Set* (MDS) of π if $D - \{a_i\}$ is not a dominating set of π, for all $a_j \in D$.

Definition 2.5.

The *Neighbourhood* of a_i in π is a set of all elements of π whose lines cross the line of a_i and is denoted by $N_\pi(a_i)$.

Propositon 2.6.

The domination number of a permutation π is equal to the domination number of the corresponding permutation graph realized by π. (i.e) $\gamma(\pi) = \gamma(G_\pi)$, the minimum cardinality of a minimal dominating set of G_π.

Example 2.7.

Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$, Then $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = \{1, 2, 3, 4, 5\}$ and $E_\pi = \{(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5)\}$. The complement of $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$, Then $\bar{G}_\pi = (V_{\bar{\pi}}, E_{\bar{\pi}})$ where $V_{\bar{\pi}} = \{1, 2, 3, 4, 5\}$ and $E_{\bar{\pi}} = \{(1, 2), (1, 4), (3, 4), (3, 5)\}$.

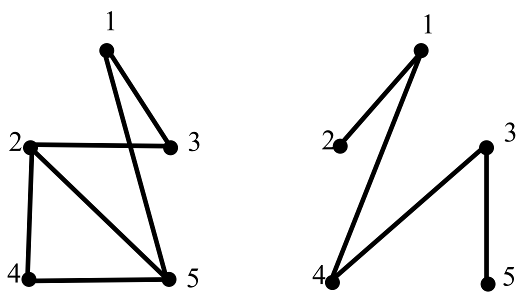

Figure 1: Perumutation graph of G_π and $\bar{G}_\pi$

3 Global Domination of a Permutation

Definition 3.1.

A graph $G_\pi = (V_\pi, E_\pi)$, $D \subseteq V$ is said to dominate G_π when every vertex in $V - D$ is adjacent to (a neighbor of) a vertex in D. A *global dominating set* (GDS) is a set of vertices that dominates both G_π and the complement graph $\bar{G}_\pi$.

Definition 3.2.

Let $a_i, a_j \in A$. Then the residue of a_i and a_j in π is denoted by $Res(a_i, a_j)$ and is given by $(a_i - a_j)(\pi^{-1}(a_i) - \pi^{-1}(a_j))$.

Definition 3.3.

The neighbourhood of a_i in π is a set of all elements of π whose lines cross the line of a_i and is denoted by $N_\pi(a_i)$, equal to $\{a_r \in \pi / l_i \text{ crosses } l_r \text{ in } \pi\}$ and $d(a_i) = |N_\pi(a_i)|$ is the number of lines that cross l_i in π.

Definition 3.4.

Let $Res(a_i, a_j) < 0$ and let $d(a_i) \geq d(a_j)$ then we say a_i dominates a_j and a_j weakly dominates a_i .

Definition 3.5.

A subset D of V_π is said to be a *(global) dominating set* of π if $N_\pi[D] = V_\pi$ and $d(a_i) \geq d(a_j)$ such that for atleast one $a_i \in D$, $a_j \in V_\pi - D$, $Res(a_i, a_j) < 0$

Definition 3.6.

The *dominating number* of a permutation π is the minimum cardinality of a set in MDS(π) and is denoted by $\gamma(\pi)$.
The *global dominating number* of a permutation π is the minimum cardinality of a set in MDS(π) and is denoted by $\gamma_g(\pi)$.

Theorem 3.7.

The global domination number of a permutation π is $\gamma_g(\pi) = \gamma_g(G_\pi)$, the minimum cardinality of the minimal (global) dominating sets ($MGDS$) of G_π .

Proof.

Let π be a permutation on a finite set $V = \{a_1, a_2, a_3, ..., a_n\}$ given by
$\pi = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 ... & a_n \\ a_1^{'} & a_2^{'} & a_3^{'} & a_4^{'} ... & a_n^{'} \end{pmatrix}$,
Let $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = V$ and $a_i a_j \in E_\pi$, if $Res(a_i, a_j) < 0$.
Let $a_i \in V$ such that $d(a_i) = max\{d(a_j)/a_j \in V\}$.
Then $D = \{a_i\}$ and let $T = N_\pi(a_i)$.
Let $V_1 = V - (D \cup T)$.
If there exists only one such a_i and if $V_1 = \emptyset$, then D is $MGDS(\pi)$.
If $V_1 \neq \emptyset$, and $< V_1 >= \emptyset$ then $D_1 = D \cup V_1$ is a $MGDS(\pi)$.
If $V_1 \neq \emptyset$, and $< V_1 > \neq \emptyset$ then choose $a_r \in V - D$ such that
$d(a_r) = max\{d(a_i)/a_i \in V_1\}$.
If $d(a_r) > d(a_i) \forall a_i \in N_\pi(a_r)$ then $D_1 = D \cup \{a_r\}$ and $T_1 = N_\pi(a_r)$ and $V_2 = V_1 - (D_1 \cup T_1)$
Otherwise choose $a_t \in N_\pi(a_r)$ such that $d(a_t) = max\{d(a_i)/a_i \in N_\pi(a_r)\}$.
Now $D_1 = D \cup \{a_t\}$ and $T_1 = N_\pi(a_t)$ and $V_2 = V_1 - (D_1 \cup T_1)$. If $V_2 = \emptyset$, then D_1 is $MGDS(\pi)$.
If $V_2 \neq \emptyset$, and $< V_2 >_\pi = \emptyset$ then $D_2 = D_1 \cup V_1$ is a $MGDS(\pi)$.

If $V_2 \neq \emptyset$, and $< V_2 >_\pi \neq \emptyset$, then proceed as before to obtain a MGDS.
If there are more than one a_i such $d(a_i)$ is max then by applying the same procedure to all $a_{r_1}, a_{r_2}, ..., a_{r_m}$ where $0 \leq r_1, r_2, ..., r_m \leq n$ all $MGDS(\pi)$ are obtained. V is finite and no. of subsets of E_π is finite. Hence within 2^n approaches all minimal global dominating sets including minimum global dominating set are produced. The minimum cardinality of the sets in all $MGDS(\pi)$ is the global domination number of π which is $\gamma_g(\pi)$. So calculation of $\gamma_g(\pi)$ is of polynomial time. Hence by Propositon 2.6, $\gamma_g(\pi) = \gamma_g(G_\pi)$.

Theorem 3.8.

A dominating set D of G_π is a global dominating set iff for each $a_j \in V_\pi - D$, there exists a $a_i \in D$ such that a_i is not adjacent to a_j.
Let $\bar{\gamma}(\pi) = \gamma(\bar{G}_\pi)$ and $\bar{\gamma}_g(\pi) = \gamma_g(\bar{G}_\pi)$. Then the permutation graph $\gamma_g(\pi) = \bar{\gamma}_g(\pi)$.

Proof

Let $f \in \gamma_g(G_\pi)$ and let $a_i, a_j \in V(G_\pi)$.
Then a_i, a_j are adjacent in $\bar{G}_\pi \Leftrightarrow a_i, a_j$ are not adjacent in G_π.
$\Leftrightarrow f(a_i), f(a_j)$ are not adjacent in G_π
since f is an automorphism of G_π
$\Leftrightarrow f(a_i), f(a_j)$ are adjacent in $\bar{G}_\pi$
Hence f is an automorphism of $\bar{G}_\pi$.
There four $f \in \gamma_g(\bar{G}_\pi)$ and hence $\gamma_g(G_\pi) \subseteq \gamma_g(\bar{G}_\pi)$.
Similarly $\gamma_g(\bar{G}_\pi) \subseteq \gamma_g(G_\pi)$ so that $\gamma_g(G_\pi) = \gamma_g(\bar{G}_\pi)$
Hence $\gamma_g(\pi) = \bar{\gamma}_g(\pi)$.

Propositon 3.9.

For any permutation graph G_π

$$\gamma(\pi) \leq \gamma_g(\pi)$$
$$\frac{\gamma(\pi)+\bar{\gamma}(\pi)}{2} \leq \gamma_g(\pi) \leq \gamma(\pi) + \bar{\gamma}(\pi)$$

Note

Any complete graph does not global domination.

Example 3.10.

let $G_\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 1 & 8 & 3 & 6 & 4 \end{pmatrix}$, Here $D = \{4, 5\}$ is minimal global dominating sets. $\gamma_g(\pi) = \gamma_g(G_\pi) = 2$.

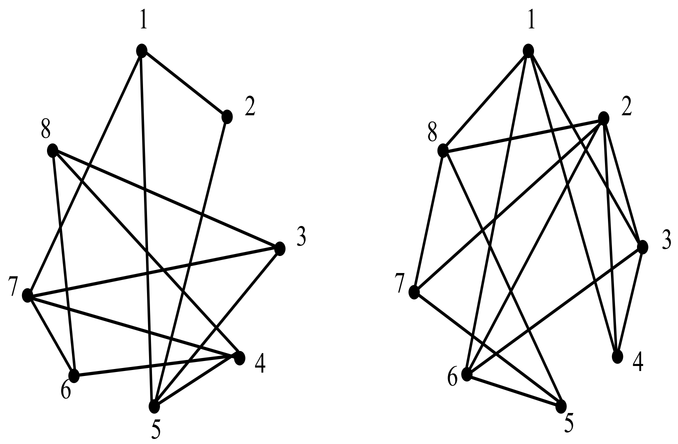

Figure 2: Global domination in permutation graph G_π and $\bar{G}_\pi$

4 Some Theorems of Global Domination

Theorem 4.1.

(i) For a graph G_π with p vertices, $\gamma_g(G_\pi) = p$ iff $G = K_p$ or $\bar{K}_p$.
(ii)$\gamma_g(K_{m,n}) = 2$ for all $m, n \geq 1$
(iii) $\gamma_g(C_4) = 2, \gamma_g(C_5) = 3$ and $\gamma_g(C_n) = \lceil \frac{n}{3} \rceil$for all $m, n \geq 6$
(iv) $\gamma_g(P_n) = 2$ for $n = 2, 3$ and $\gamma_g(P_n) = \lceil \frac{n}{3} \rceil$ for $n \geq 6$.

Proof

we prove only (i), and (ii)-(iv) are obvious. Clearly, $\gamma_g(K_p) = \gamma_g(\bar{K}_p) = p$. Suppose $\gamma_g(G_\pi) = p$ and $G_\pi \neq K_p$ or $\bar{K}_p$ Then G_π has at least one edge uv and a vertex w not adjacent to, say v. Then $V_\pi - \{v\}$ is a global domination set and $\gamma_g(G_\pi) = p - 1$.
For some graphs including trees, γ_g is almost equal to γ

Theorem 4.2.

Let D be a minimum dominating set of G_π. If there exists a vertex v in $V - D$ adjacent to only verticesin D, then

$$\gamma_g \leq \gamma + 1$$

Proof

This follows since $D \cup \{v\}$ is a global dominating set.

Corollary 4.2.1.

Let $G_\pi = (V_1 \cup V_2, E_\pi)$ be a bipartite graph without isolates, where$\mid V_1 \mid= m, \mid V_2 \mid= n$ and $m \leq n$. Then $\gamma_g \leq m + 1$.

Proof

This follows from $\gamma_g \leq \gamma + 1$ since $m \leq n$

Corollary 4.2.2.

For any graph with a pendant vertex, $\gamma_g \leq \gamma + 1$ holds. In particular, $\gamma_g \leq \gamma + 1$ holds for a tree.

Corollary 4.2.3.

If $V - D$ is independent, then $\gamma_g \leq \gamma + 1$ holds.
Let α_0 and β_0 respectively denote the covering and independence number of a graph.

Theorem 4.3.

For a (p, q) graph G_π without isolates.

$$\frac{2q-p(p-3)}{2} \leq \gamma_g \leq p - \beta_0 + 1$$

Proof

Let D be a minimum global dominating set. Then every vertex in $V_\pi - D$ is not adjacent to atleast one vertex in D. This imlies
$q \leq pC_2 - (p - \gamma_g)$ and the lower bound follows.
To establish the upper bound, let B be an independent set with β_0 vertices. Since G_π has no

isolates. $V - B$ is a dominating set of G_π.
Clearly, for any $V \in B$, $(V - B) \cup \{V\}$ is a global dominating set of G_π, and the upper bound follows.
Since $\alpha_0 + \beta_0 = p$ for eny graph of order p without isolates.

Corollary 4.3.1.

$$\gamma_g \leq \alpha_0 + 1$$

The *independent domination number* $i(G)$ of G_π is the minimum cardinality of a dominating set which is also independent. It is well-known that

$$\gamma \leq i \leq \beta_0$$

Corollary 4.3.2.

For any graph G_π of order p without isolates.
(i) $\gamma + \gamma_g \leq p + 1$, (ii) $i + \gamma_g \leq p + 1$.

Theorem 4.4.

For any graph $G_\pi = (V_\pi, E_\pi)$

$$\gamma_g \leq max\{\chi(G_\pi).\chi(\bar{G_\pi})\}$$

where $\chi(G_\pi)$ is the chromatic number of G_π.

Proof

Let $\chi(G_\pi) = m,\ \chi(\bar{G_\pi}) = n$ and $m \leq n$. Consider a $\chi(G_\pi)$ partition $a_1, a_2, ...a_m$ and a $\chi(\bar{G_\pi})$ partition $a_1^{'}, a_2^{'}, ...a_n^{'}$ of v.
Cleary, no two vertices of any a_i can belong to any $a_j^{'}$ and conversely.We can select m vertices $a_1, a_2, ...a_m$ such that
(i) $a_i \in V_\pi, 1 \leq i \leq m$, and (ii)$a_1, a_2, ...a_m$ belong to different sets in $a_1^{'}, a_2^{'}, ...a_n^{'}$, say $a_j \in V_\pi^{'}, 1 \leq j \leq m$. Choose $a_j \in V_\pi^{'}, m + 1 \leq j \leq n$. Clearly,$a_1, a_2, ...a_m$ is a dominating set of $\bar{G}$, and $a_1, a_2, ...a_m, a_{m+1}, ...a_n$ is a dominating set of G_π and $\bar{G_\pi}$.
Let Δ and δ respectively be the maximum and minimum degrees of a graph G_π, and $\bar{\Delta} = \Delta(\bar{G_\pi}, \bar{\delta} = \delta(\bar{G_\pi})$.
It is well known that $\chi(G_\pi) \leq \Delta + 1$ and if G_π is neither complete nor an odd cycle,then $\chi(G_\pi) \leq \Delta$.

Corollary 4.4.1.

For any graph G_π of order p

$$\gamma_g \leq max\{\Delta + 1, \bar{\Delta} + 1\} = max\{p - \bar{\delta}, p - \delta\}$$

and If G_π is neither complete nor an odd cycle

$$\gamma_g \leq max\{\Delta, \bar{\Delta}\} = max\{p - 1 - \bar{\delta}, p - 1 - \delta\}$$

since $\gamma \leq \gamma_g$ and $\bar{\gamma} \leq \gamma_g$

Corollary 4.4.2.

Let $t = \gamma$ or $\bar{\gamma}$. For any graph G_π

$$t \leq max\{\Delta + 1, \bar{\Delta} + 1\}$$

if g_π is neither complete nor an odd cycle

$$t \leq max\{\Delta, \bar{\Delta}\}$$

Let k and $\bar{k}$ respectively denote the connectivity of G_π and $\bar{G}_\pi$. it is well know that $k \leq \delta$.

Corollary 4.4.3.

For any graph G_π of order p

$$\gamma_g \leq max\{p - k - 1, p - \bar{k} - 1\}$$

For $v \in V_\pi$, let $N(v) = \{u \in V_\pi : uv \in E_\pi\}$ and $N[v] = (v) \cup \{v\}$.
A set $D \subset V_\pi$ is *full* if $N(v) \cap V_\pi - D \neq \emptyset$ for all $v \in D$. Also D is g-full if $N(v) \cap V_\pi - D \neq \emptyset$ both in G_π and $\bar{G}_\pi$.
The *full number*$f = f(G_\pi)$ of G_π is the maximum cardinality of a full set of G_π and the g- *full number*$f_g = f_g(G_\pi)$ of G_π is the maximum cardinality of a g-full set of G_π.
Clearly $f_g(G_\pi) = f_g(\bar{G}_\pi)$

Proposition 4.5.

If G_π is of order $\gamma + f = p$
Analogously we have

Theorem 4.6.

If G_π is of order $\gamma_g + f_g = p$

Proof

Let D be a minimum global dominating set and $v \in V_\pi - D$. Then $N(v) \cap D \neq \emptyset$ both in G_π and $\bar{G}_\pi$.
Hence $V_\pi - D$ is g- full and $p - \gamma_g = |V_\pi - D| \leq f_g$.
On the otherhand,
Suppose D V_π is g-full with $|D| = f_g$. Then, for all $v \in D$, $N(v) \cap V_\pi - D \neq \emptyset$ both in G_π and $\bar{G}_\pi$.
This implise that $V_\pi - D$ is aglobal dominating set.
Hence $\gamma_g \leq |V_\pi - D| = p - f_g$.

5 The Global Domination Number

A partition $\{a_1, a_2, ..., a_n\}$ of V is a domination (global domination) partition of G_π if each V_i is a dominating set(global dominating set). The domination number $d = d(G_\pi)$ (global domination number $d = d(G_\pi)$) of G_π is the maximum order of a domination (global domination) partition of G_π.
Clearly, for any graph G_π, $d_g(G_\pi) = d_g(\bar{G}_\pi)$

Propositon 5.1.

(i) $d_g(K_n) = d_g(\bar{K}_n)$=1
(ii) For any $n \geq 1$, $d_g(C_{3n}) = 3$, and $d_g(C_{3n+1}) = d_g(C_{3n+2})$=2
(iii) For any $2 \leq m \leq n$, $d_g(K_{m,n}) = n$. when $\bar{d} = d(\bar{G}_\pi)$ and $\bar{d}_g = d_g(\bar{G}_\pi)$

Propositon 5.2.

If G_π is of order p, then $\gamma + d \leq p + 1$ and $\gamma_g + d_g \leq p + 1$ if and only if $G_\pi = K_p$ or $\bar{K}_p$.

References

[1] **M.Murugan** *"Topics in Graph Theory and Algorithms"*, Muthali Publishing House, Chennai, India,2003.

[2] **E.Sampathkumar** *"The Global Domination Number of a Graph"*, Journal of math. Phy. Science, Volume:23, no:5.(1989).

[3] **J.Chithra, S.P.Subbiah and V.Swaminathan** *"Domination in Permutation Graphs"*, International Journal of Computing Algorithm, Volume:03, Pages:549-553.(2014).