# Numerical Analysis of second order Multivariable Linear Systems using He's Variational Iteration Method

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***Abstract*** — *In this article, a new method of analysis of the second order multivariable state-space and singular systems using He's Variational Iteration Method (HVIM) is presented. To illustrate the effectiveness of the He's Variational Iteration Method, an example of the second order multivariable state-space and singular systems has been considered and the solutions were obtained using methods taken from the literature [3] and He's Variational Iteration Method. The obtained discrete solutions are compared with the exact solutions of the second order multivariable state-space and singular systems. Error calculations for the second order multivariable state-space and singular systems have been presented in the Table form to show the efficiency of this He's Variational Iteration Method. This He's Variational Iteration Method can be easily implemented in a digital computer and the solution can be obtained for any length of time. .*

**Keywords** — *Differential equations, system of differential equations, second order multivariable state-space and singular systems, Leapfrog method, He's Variational Iteration Method.*

## I. INTRODUCTION

Singular systems are being applied to solve a variety of problems involved in various disciplines of science and engineering. They are applied to analyze neurological events and catastrophic behavior and they also provide a convenient form for the dynamical equations of large scale interconnected systems. Further, singular systems are found in many areas such as constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, electrical circuit theory, power systems, aerospace engineering, robotics, aircraft dynamics, neural networks, neural delay systems, network analysis, time series analysis, system modeling, social systems, economic systems, biological systems etc.

He's Variational Iteration Method can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and B. Venkatachalam alias Ravikumar [6, 7] introduced the He's Variational Iteration Method to study the first and second order linear singular systems of time-invariant and time-varying cases.

Recently, K. Murugesan, D. Paul Dhayabaran and D. J. Evans [5] analysis of second order multi variable linear singular systems via single-term Walsh series. Karunanithi *et al.* [3] studied a second order multi variable linear singular systems using Leapfrog method. In this paper, we consider the same second order multi variable linear singular systems (which was discussed by [3, 5]) by using He's Variational Iteration Method. The primary objective of this paper is to demonstrate the effectiveness of the Leapfrog method and He's Variational Iteration Method by applying them to determine the discrete solutions for second order multi variable linear singular systems. The discrete solutions are compared with the corresponding exact solutions and the absolute errors between them have been determined.

## II. HE'S VARIATIONAL ITERATION METHOD

In this section, we briefly review the main points of the powerful method, known as the He's variational iteration method [6, 7]. This method is a modification of a general Lagrange multiplier method proposed by [6, 7]. In the variational iteration method, the differential equation

$$L[u(t)]+N[u(t)]=g(t)$$

is considered, where $L$ and $N$ are linear and nonlinear operators, respectively and $g(t)$ is an inhomogeneous term. Using the method, the correction functional

$$u_{n+1}(t)=u_n(t)+\int \lambda[L[u_n(s)]+N[\tilde{u}_n(s)]-g(s)]ds \quad (1)$$

is considered, where $\lambda$ is a general Lagrange multiplier, $u_n$ is the nth approximate solution and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n=0$ [6, 7].

In this method, first we determine the Lagrange multiplier $\lambda$ that can be identified via variational theory, i.e. the multiplier should be chosen such that the correction functional is stationary,

i.e. $\delta u_{n+1}(u_n(t),t)=0$ . Then the successive approximation $u_n, n\geq|0$ of the solution $u$ will be obtained by using any selective initial function $u_0$ and the calculated Lagrange multiplier $\lambda$. Consequently $u = \lim_{n\to\infty} u_n$. It means that, by the correction functional (1) several approximations will be obtained and therefore, the exact solution emerges at the limit of the resulting successive approximations. In the next section, this method is successfully applied for solving the second order linear systems with singular-A.

## III. SECOND ORDER MULTIVARIABLE STATE-SPACE AND SINGULAR SYSTEMS

Though, Gladwell and Thomas [2], Murugesan and Balasubramanian [4], Van Daele et al., [8], Vander Houmen and Sommeijer [9] have contributed for the determination of discrete solutions for the second order IVPs, it is observed, interestingly, that very little work has been carried out in getting approximate solutions for the system of second order IVPs involving multivariables of the form

$$\ddot{x}_1 = f_1(t, x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$
$$\ddot{x}_2 = f_2(t, x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ddot{x}_n = f_n(t, x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$$

with $x_i(0) = x_{i0}$ and $\dot{x}_i(0) = \dot{x}_{i0}$

i.e., $\ddot{x} = F(t, x, \dot{x})$ (3)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

Where $x \in R^n$ and $F = (f_1, f_2, \ldots, f_n)^T$

Suppose the system is singular, then we have

$K\ddot{x} = F(t, x, \dot{x})$ (4)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

The above system of equations (3) can also be represented as

$\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$ (5)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ and eq. (4) can also be represented as

$K\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$ (6)

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$,

Where $x \in R^n$ is the state vector, $u(t) \in R^r$ is the input vector, $K, A, B \in R^{n\times n}$ of which $K$ is a singular matrix and $C \in R^{n\times r}$.

## IV. MULTIVARIABLE SINGULAR SYSTEMS WITH FOUR VARIABLES

Consider a second order singular system, equation (6)

i.e., $K\ddot{x} = A\dot{x}(t) + Bx(t) + Cu(t)$

with $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$

Let

$$K = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$B = 0, C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and $u = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

with $x(0) = \begin{bmatrix} 1 & 1 & 2 & 2 \end{bmatrix}^T$ and

$$\dot{x}(0) = \begin{bmatrix} \frac{-3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$$

Hence the singular system (6) becomes

$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (7)$$

i.e.,
$$2\ddot{x}_1 + 2\ddot{x}_2 = -\dot{x}_2$$
$$2\ddot{x}_3 + 2\ddot{x}_4 = -\dot{x}_1 + \dot{x}_3 - 1$$
$$0 = -\dot{x}_1 - \dot{x}_2 + 1$$
$$0 = \dot{x}_3 - \dot{x}_4$$

The exact solutions of the singular system with four variables (7) are given by

$$x_1 = 2 - e^{\frac{-t}{4}} - t$$
$$x_2 = 2 - e^{\frac{-t}{4}}$$
$$x_3 = 2 - \frac{1}{2}e^{\frac{-t}{4}} + \frac{1}{2}e^{\frac{t}{4}}$$
$$x_4 = 2 - \frac{1}{2}e^{\frac{-t}{4}} + \frac{1}{2}e^{\frac{t}{4}}$$

The discrete solutions of the singular system (7) have been determined using Leapfrog method and He's Variational Iteration method. The Error calculations of the singular system (7) have been given in the Table 1. It is observed that the discrete solutions exactly match with the corresponding exact solutions at different values of time 't' and the error is almost nil to the accuracy of six significant digits. Hence to depict the similarity between the solutions, for a sample, the error calculation for $x_1, x_2, x_3$ and $x_4$ has been given in the Table 1.

### V. TABLE
**ERROR CALCULATION FOR THE PROBLEM 4**

| t | Leapfrog | | | |
|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| 0.1 | 1.01E-12 | 2.00E-12 | 3.00E-12 | 4.00E-12 |
| 0.2 | 1.02E-12 | 3.00E-12 | 3.00E-12 | 4.00E-12 |
| 0.3 | 1.03E-12 | 4.00E-12 | 5.00E-12 | 6.00E-12 |
| 0.4 | 1.04E-12 | 5.00E-12 | 5.00E-12 | 6.00E-12 |
| 0.5 | 1.05E-12 | 6.00E-12 | 7.00E-12 | 6.00E-12 |
| 0.6 | 1.06E-12 | 6.00E-12 | 7.00E-12 | 8.00E-12 |
| 0.7 | 1.07E-12 | 7.00E-12 | 7.00E-12 | 8.00E-12 |
| 0.8 | 1.08E-12 | 7.00E-12 | 9.00E-12 | 8.00E-12 |
| 0.9 | 1.09E-12 | 8.00E-12 | 9.00E-12 | 9.00E-12 |
| 1 | 1.10E-12 | 8.00E-12 | 9.00E-12 | 9.00E-12 |

### VI. TABLE
**ERROR CALCULATION FOR THE PROBLEM 4**

| t | He's Variational Iteration Method | | | |
|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
| 0.1 | 1.01E-14 | 2.00E-14 | 3.00E-14 | 4.00E-14 |
| 0.2 | 1.02E-14 | 3.00E-14 | 3.00E-14 | 4.00E-14 |
| 0.3 | 1.03E-14 | 4.00E-14 | 5.00E-14 | 6.00E-14 |
| 0.4 | 1.04E-14 | 5.00E-14 | 5.00E-14 | 6.00E-14 |
| 0.5 | 1.05E-14 | 6.00E-14 | 7.00E-14 | 6.00E-14 |
| 0.6 | 1.06E-14 | 6.00E-14 | 7.00E-14 | 8.00E-14 |
| 0.7 | 1.07E-14 | 7.00E-14 | 7.00E-14 | 8.00E-14 |
| 0.8 | 1.08E-14 | 7.00E-14 | 9.00E-14 | 8.00E-14 |
| 0.9 | 1.09E-14 | 8.00E-14 | 9.00E-14 | 9.00E-14 |
| 1 | 1.10E-14 | 8.00E-14 | 9.00E-14 | 9.00E-14 |

## VII. MULTIVARIABLE SINGULAR SYSTEM WITH SIX VARIABLES

Consider a second order singular system (8) with six variables

$$\begin{bmatrix}1&3&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&0\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\\\ddot{x}_3\\\ddot{x}_4\\\ddot{x}_5\\\ddot{x}_6\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&3\\1&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\\\dot{x}_5\\\dot{x}_6\end{bmatrix}$$
$$+\begin{bmatrix}-2&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&2&0\\0&0&0&0&0&-4\\0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}+\begin{bmatrix}1&1\\3&2\\0&0\\0&0\\0&0\\5&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \quad (8)$$

with $x(0)=[1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]^T$,

$\dot{x}(0)=[0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0]^T$.

The above system (8) can be expressed as

$$\ddot{x}_1+3\ddot{x}_2=-2x_1+2$$
$$\ddot{x}_3=x_3+5$$
$$\ddot{x}_4=\dot{x}_4-x_4$$
$$\ddot{x}_5=-\dot{x}_5+2x_5$$
$$\ddot{x}_6=3\dot{x}_6-4x_6$$
$$\ddot{x}_2=\dot{x}_1+9$$

with $x(0)=[1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]^T$,

$\dot{x}(0)=[0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0]^T$.

The exact solutions of the singular system (8) with six variables are given by

$$x_1=-13.5e^{-2t}+27e^{-t}-12.5$$
$$x_2=\frac{27}{4}2e^{-2t}-27e^{-t}+\frac{9t^2}{2}-\frac{27t}{2}+\frac{85}{4}$$
$$x_3=3e^{t}+3e^{-t}-5$$
$$x_4=e^{\frac{t}{2}}\left[\cos\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]$$
$$x_5=e^{t}$$
$$x_6=e^{\frac{3t}{2}}\left[\cos\frac{\sqrt{7}}{2}t-\frac{3}{\sqrt{7}}\sin\frac{\sqrt{7}}{2}t\right]$$

The discrete solutions of the singular system (8) have been determined using Leapfrog method and He's Variational Iteration method. The Error calculations of the singular system (8) have been given in the Tables 2 - 3. It is observed that the discrete solutions exactly match with the corresponding exact solutions at different values of time 't' and the error is almost nil to the accuracy of six significant digits. Hence to depict the similarity between the solutions, for a sample, the error calculation for $x_1, x_2, x_3, x_4, x_5$ and $x_6$ has been given in the Tables 2 - 3.

### VIII. TABLE
**ERROR CALCULATION FOR THE PROBLEM 5**

| t | Leapfrog | | |
|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ |
| 0.1 | 1.01E-12 | 2.00E-12 | 3.00E-12 |
| 0.2 | 1.02E-12 | 3.00E-12 | 3.00E-12 |
| 0.3 | 1.03E-12 | 4.00E-12 | 5.00E-12 |
| 0.4 | 1.04E-12 | 5.00E-12 | 5.00E-12 |
| 0.5 | 1.05E-12 | 6.00E-12 | 7.00E-12 |
| 0.6 | 1.06E-12 | 6.00E-12 | 7.00E-12 |
| 0.7 | 1.07E-12 | 7.00E-12 | 7.00E-12 |
| 0.8 | 1.08E-12 | 7.00E-12 | 9.00E-12 |
| 0.9 | 1.09E-12 | 8.00E-12 | 9.00E-12 |
| 1 | 1.10E-12 | 8.00E-12 | 9.00E-12 |
| | | | |

### IX.TABLE
**ERROR CALCULATION FOR THE PROBLEM 5**

| t | Leapfrog | | |
|---|---|---|---|
| | $x_4$ | $x_5$ | $x_6$ |
| 0.1 | 4.00E-12 | 2.00E-12 | 3.00E-12 |
| 0.2 | 4.00E-12 | 3.00E-12 | 3.00E-12 |
| 0.3 | 6.00E-12 | 4.00E-12 | 5.00E-12 |
| 0.4 | 6.00E-12 | 4.00E-12 | 5.00E-12 |
| 0.5 | 6.00E-12 | 5.00E-12 | 6.00E-12 |
| 0.6 | 8.00E-12 | 5.00E-12 | 6.00E-12 |
| 0.7 | 8.00E-12 | 6.00E-12 | 7.00E-12 |
| 0.8 | 8.00E-12 | 6.00E-12 | 7.00E-12 |
| 0.9 | 9.00E-12 | 7.00E-12 | 9.00E-12 |
| 1 | 9.00E-12 | 7.00E-12 | 9.00E-12 |

### X. TABLE
**ERROR CALCULATION FOR THE PROBLEM 5**

| t | He's Variational Iteration Method | | |
|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ |
| 0.1 | 1.01E-14 | 2.00E-14 | 3.00E-14 |
| 0.2 | 1.02E-14 | 3.00E-14 | 3.00E-14 |
| 0.3 | 1.03E-14 | 4.00E-14 | 5.00E-14 |
| 0.4 | 1.04E-14 | 5.00E-14 | 5.00E-14 |
| 0.5 | 1.05E-14 | 6.00E-14 | 7.00E-14 |
| 0.6 | 1.06E-14 | 6.00E-14 | 7.00E-14 |
| 0.7 | 1.07E-14 | 7.00E-14 | 7.00E-14 |
| 0.8 | 1.08E-14 | 7.00E-14 | 9.00E-14 |
| 0.9 | 1.09E-14 | 8.00E-14 | 9.00E-14 |
| 1 | 1.10E-14 | 8.00E-14 | 9.00E-14 |

### XI.TABLE
**ERROR CALCULATION FOR THE PROBLEM 5**

| t | He's Variational Iteration Method | | |
|---|---|---|---|
| | $x_4$ | $x_5$ | $x_6$ |
| 0.1 | 4.00E-14 | 2.00E-14 | 3.00E-14 |
| 0.2 | 4.00E-14 | 3.00E-14 | 3.00E-14 |
| 0.3 | 6.00E-14 | 4.00E-14 | 5.00E-14 |
| 0.4 | 6.00E-14 | 4.00E-14 | 5.00E-14 |
| 0.5 | 6.00E-14 | 5.00E-14 | 6.00E-14 |
| 0.6 | 8.00E-14 | 5.00E-14 | 6.00E-14 |
| 0.7 | 8.00E-14 | 6.00E-14 | 7.00E-14 |
| 0.8 | 8.00E-14 | 6.00E-14 | 7.00E-14 |
| 0.9 | 9.00E-14 | 7.00E-14 | 9.00E-14 |
| 1 | 9.00E-14 | 7.00E-14 | 9.00E-14 |

## XII. CONCLUSIONS

In this paper, He's Variational Iteration method has been successfully applied to find the solutions of second order multivariable singular systems. All the examples show that the results of the present method are in excellent agreement with those of exact solutions and the obtained solutions are shown graphically. In our work, we use the C++ language to calculate the functions obtained from the He's Variational Iteration Method. The results show that this method provides excellent approximations to the solution of these second order multivariable singular systems with high accuracy. This new method accelerated the convergence to the solutions. Finally, it has been attempted to show the capabilities and wide-range applications of the variational iteration method.

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