Classification of Solutions of Second Order Nonlinear Neutral Delay Dynamic Equations with Positive and Negative Coefficients

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Abstract — *In this paper, the authors classified all solutions of the second-order nonlinear neutral delay dynamic equations with positive and negative coefficients into four classes and obtained conditions for the existence / non-existence of solutions in these classes. Examples are included to illustrate the validation of the main results.*

Keywords — *Second-order, neutral dynamic equations, oscillatory solution, positive and negative coefficients, weakly oscillatory solution, asymptotic behaviour.*

I. INTRODUCTION

In this paper, we are concerned to obtain the conditions for the existence / non-existence solutions of a class M^+, M^-, OS, WOS and asymptotic behaviour of solutions of a class of M^+ and M^- second order nonlinear neutral delay dynamic equations with positive and negative coefficients of the form

$$\begin{aligned}(r(t)(x(t)+p(t)x(\alpha(t)))^{\Delta})^{\Delta}+q(t)f(x(\beta(t)))\\ -h(t)g(x(\gamma(t)))=0\end{aligned} \tag{1.1}$$

for $t\in[t_0,\infty)_T$ where T is a time scale with $\sup T=\infty$. in what follows, it is always assume that

(i) $r\in C^1_{rd}([t_0,\infty)_T,(0,\infty))$, $p\in C^2_{rd}([t_0,\infty)_T,R)$ and $q,h\in C_{rd}([t_0,\infty)_T,R)$;

(ii) $f,g\in C(R,R)$ such that $uf(u)>0$ and $ug(u)>0$ for $u\neq 0$

(iii) $\alpha,\beta,\gamma\in C_{rd}(T,T)$ are strictly increasing functions such that $\alpha(t)\le t,\beta(t)\le t,\gamma(t)\le t$ and $\lim_{t\to\infty}\alpha(t)=\lim_{t\to\infty}\beta(t)=\lim_{t\to\infty}\gamma(t)=\infty.$

Let $t_x\in[t_0,\infty)_T$ such that $\alpha(t)\ge t_0,\beta(t)\ge t_0$ and $\gamma(t)\ge t_0$ for all $t\in[t_x,\infty)_T$. By a solution of equation (1.1), we shall mean a function $x\in C_{rd}([t_x,\infty)_T,R)$ which has the properties $x(t)+p(t)x(\alpha(t))\in C^1_{rd}([t_x,\infty)_T,R)$ and $r(t)(x(t)+p(t)x(\alpha(t)))\in C^1_{rd}([t_x,\infty)_T,R)$, satisfies equation (1.1) on $[t_x,\infty)_T$. As is customary, a solution of equation (1.1) is oscillatory solution (OS) if it is neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. A non-oscillatory solution x(t) of equation (1.1) is said to be weakly oscillatory solution (WOS) if x(t) is non-oscillatory and $x^{\Delta}(t)$ is oscillatory for large value of $t\in[t_0,\infty)_T$.

In recent years, there has been much research activity concerning the oscillation, non-oscillation and asymptotic behaviour of solutions of various differential equations, difference equations and dynamic equations. For instance in [7,8,9] etc., authors have been studied by classifying all solutions into four classes such as M^+, M^-, OS, WOS and obtained criteria for the existence / non-existence of solutions. In order to extend and generalize the papers [7,8,9], Rami Reddy et al. [17] were concerned the solutions of existence / non-existence of a class M^+, M^-, OS, WOS of second order nonlinear neutral delay dynamic equation of the form

$$(r(t)(x(t)+p(t)x(\alpha(t)))^{\Delta})^{\Delta}+q(t)f(x(\beta(t)))=0, \tag{1.2}$$

For $t\in[t_0,\infty)_T$, subject to the conditions:

(i) $r\in C^1_{rd}([t_0,\infty)_T,(0,\infty))$, $p\in C^2_{rd}([t_0,\infty)_T,R)$

(ii) $q\in C_{rd}([t_0,\infty)_T,R)$ and q does not vanish eventually;

(iii) $f\in C^1(R,R)$ such that f satisfies $uf(u)>0$ for $u\neq 0$ and $f'(u)\ge 0$ for $u\in R$;

(iv) $\alpha,\beta\in C_{rd}([t_0,\infty)_T,T)$ are strictly increasing functions such that $\alpha(t)\le t,\beta(t)\le t$ and $\lim_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\beta(t)$

The study of the oscillation and other asymptotic properties of solutions of neutral delay difference / differential / dynamic equations with positive and negative coefficients attracted a good bit of attention in the last several years. Da-Xue et al. [11] have been studied the oscillation criteria of second order nonlinear dynamic equations with positive and negative coefficients of the form

$$(r(t)x^{\Delta}(t))^{\Delta}+p(t)f(x(\zeta(t)))-q(t)h(x(\delta(t)))=0.$$

Thandapani et al. [18] have been studied the classification of solutions of second order nonlinear neutral delay differential equations with positive and negative coefficients of the form

$$(r(t)(x(t)+c(t)x(t-\tau))')'+p(t)f(x(t-\delta))$$
$$-q(t)g(x(t-\sigma))=0$$

Where $c,r,p,q\in C([t_0,\infty),R)$, $f,g\in C(R,R)$ and $\tau,\delta,\sigma\in(0,\infty)$. for results on the second order nonlinear equations, we refer the reader to the recent papers [12, 13, 14] and the references cited therein.

Motivated and inspired by the papers mentioned above, in this paper the authors are interested to study the solutions of existence / non-existence of a class M^{+}, M^{-}, OS, WOS and asymptotic behaviour of M^{+} and M^{-} of equation (1.1). Here we consider two cases, namely $q\geq 0$ and q changes sign for all large $t\in[t_0,\infty)_T$ to give sufficient conditions in order that every solution of equation (1.1) oscillates and to study the asymptotic nature of non-oscillatory solutions of equation (1.1). With respect to their asymptotic behaviour all the solutions of equation (1.1) may be priori divided in to the following classes:

$M^{+}=\{x\in S:$ there exists some $t_{x_1}\in[t_0,\infty)_T$ such that $x(t)x^{\Delta}(t)\geq 0$ for $t\in[t_{x_1},\infty)_T\}$;

$M^{-}=\{x\in S:$ there exists some $t_{x_1}\in[t_0,\infty)_T$ such that $x(t)x^{\Delta}(t)\leq 0$ for $t\in[t_{x_1},\infty)_T\}$;

$OS=\{x\in S:$ there exists a sequence $t_n\in[t_0,\infty)_T, t_n\to\infty$ s.t. $x(t_n)x(t_{n+1})\leq 0\}$

$WOS=\{x\in S: x(t)$ is non-oscillatory for large $t\in[t_0,\infty)_T$, but $x^{\Delta}(t)$ oscillates$\}$.

With a very simple argument we can prove that M^{+}, M^{-}, OS and WOS are mutually disjoint. By the above definitions, it turns out that solutions in the class M^{+} are eventually either positive non-decreasing or negative non increasing, solutions in the class M^{-} are eventually either positive non-increasing or negative non-decreasing solutions in the class OS are oscillatory, and finally solutions in the class WOS are weakly oscillatory.

In Section 2, we mentioned important lemma's which are existing in the literature. In Section 3, we obtain sufficient conditions for the existence / non-existence in the above said classes. In Section 4, we discuss the asymptotic behavior of solutions in the class of M^{+} and M^{-}. Finally section 5, follows conclusion.

2. Important Lemma's

Theorem2.1. (Chain rule) ([1, Theorem 1.90]) Let f: R→R is continuously differentiable and suppose g : T→ R is delta differentiable. Then fog: T → R is delta differentiable and the formula

$$(fog)^{\Delta}(t)=\left\{\int_0^1 f'(g(t)+h\mu(t)g^{\Delta}(t))dh\right\}g^{\Delta}(t)$$

Theorem2.2.([1, Theorem 1.117]) Let $a\in T^{k}$, $b\in T$ and assume $f:T*T^{k}\to R$ continuous at (t, t), where $t\in T^{k}$ with t > a. Also assume that $f^{\Delta}(t,.)$ is rd-continuous on $[a,\sigma(t)]$ Suppose that for each $\epsilon>0$ there exists a neighbourhood U of t, independent of $\tau\in[a,\sigma(t)]$, such that

$$|f(\sigma(t),\tau)-f(s,\tau)-f^{\Delta}(t,\tau)(\sigma(t)-s)|\leq\epsilon|\sigma(t)-s|$$

for all $s\in U$.

Where f^{Δ} denotes the derivative of f with respect to the first variable. Then

(i) $g(t):=\int_a^t f(t,\tau)\Delta\tau$ implies

$$g^{\Delta}(t)=\int_a^t f^{\Delta}(t,\tau)\Delta\tau+f(\sigma(t),t)$$

(ii) $h(t):=\int_t^b f(t,\tau)\Delta\tau$ implies

$$h^{\Delta}(t)=\int_t^b f^{\Delta}(t,\tau)\Delta\tau-f(\sigma(t),t)$$

For more basic concepts in the time scale theory the readers are referred to the books(see[1, 2]).

3. Existence and Non-Existence Of Solutions In M^{+}, M^{-}, OS and WOS

First, The existence of solutions of equation (1.1) in the class M^{+}

Theorem 3.1. Assume that

(H_1) $p(t)\geq 0$ non decreasing for all $t\in[t_0,\infty)_T$;

(H_2) $h(t)\geq 0$ for all $t\in[t_0,\infty)_T$;

(H_3) $\beta(t)\geq\gamma(t)$;

(H_4) there exists $M>0$ such that

$$\frac{g(u)}{f(u)}\leq M \quad \text{for} \quad u\neq 0;$$

(H_5) f is non-decreasing;

(H_6) $\limsup_{t\to\infty}\int_{t_0}^{t}(q(s)-Mh(s))\Delta s=\infty$ hold.

Then for equation (1.1) we have $M^{+}=\phi$.

Proof. Suppose that the equation (1.1) has a solution $x\in M^{+}$. Without loss of generality we may assume that $x(t)>0$ and $x^{\Delta}(t)\geq 0$ for large (the proof is similar if $x(t)<0$ and $x^{\Delta}(t)\leq 0$ for large $t\in[t_0,\infty)_T$. Then there exists $t_1\in[t_0,\infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$, $x(\gamma(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\beta(t))$, $x^{\Delta}(\gamma(t))$ all are non-negative for all $t\in[t_1,\infty)_T$. Define

$$z(t) = x(t) + p(t)x(\alpha(t)) \tag{3.1}$$

For $t \in [t_0, \infty)_T$. Then by condition (H_1) we have $z(t) > 0$ and $z^\Delta(t) \ge 0$ for all $t \in [t_1, \infty)_T$ Using (3.1), equation (1.1) becomes

$$(r(t)z^\Delta(t))^\Delta + q(t)f(x(\beta(t))) - h(t)g(x(\gamma(t))) = 0$$

Or

$$\frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} = -q(t) + h(t)\frac{g(x(\gamma(t)))}{f(x(\beta(t)))} \tag{3.2}$$

for $t \in [t_1, \infty)_T$. Now for $t \in [t_1, \infty)_T$,

$$\left(\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} +$$

$$(r(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{1}{f(x(\beta(t)))}\right)^\Delta$$

$$= \frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} -$$

$$(r(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{(f(x(\beta(t))))^\Delta}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right)$$

Implies that,

$$\left(\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} - (r(\sigma(t))z^\Delta(\sigma(t)))$$

$$\left(\frac{\{\int_0^1 f'(x(\beta(t)) + h\mu(t)(x(\beta(t)))^\Delta)dh\}(x(\beta(t)))^\Delta}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right)$$

Or

$$\left(\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} - (r(\sigma(t))z^\Delta(\sigma(t)))$$

$$\left(\frac{\{\int_0^1 f'(x(\beta(t)) + h\mu(t)(x(\beta(t)))^\Delta)dh\}x^\Delta(\beta(t))\beta^\Delta(t)}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right) \tag{3.3}$$

Therefore,

$$\left(\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta \le \frac{(r(t)z^\Delta(t))^\Delta}{f(x(\beta(t)))} \tag{3.4}$$

for all $t \in [t_1, \infty)_T$, due to (H_5) $z^\Delta(t) \ge 0$ and $x^\Delta(\beta(t)) \ge 0$ for all $t \in [t_1, \infty)_T$. From equation (3.2) and (3.4), we have

$$\left(\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta \le -q(t) + h(t)\frac{g(x(\gamma(t)))}{f(x(\beta(t)))}$$

$$\le -q(t) + h(t)\frac{g(x(\gamma(t)))}{f(x(\gamma(t)))}$$

$$\le -(q(t) - Mh(t))$$

for $t \in [t_1, \infty)_T$, due to (H_2) - (H_5) Integrating the last inequality from t_1 to t, we obtain

$$\frac{r(t)z^\Delta(t)}{f(x(\beta(t)))} - \frac{r(t_1)z^\Delta(t_1)}{f(x(\beta(t_1)))} \le -\int_{t_1}^t (q(s) - Mh(s))\Delta s$$

From (H_6), we obtain

$$\liminf_{t\to\infty} \frac{r(t)z^\Delta(t)}{f(x(\beta(t)))} = -\infty,$$

Which contradicts the assumption $z^\Delta(t) \ge 0$ for large t. Thus, the theorem is proved.

Example3.2. In Theorem3.1, the assumption (H_6) can not be dropped. For this suppose $T = ¢$, and consider the following difference equation

$$\Delta\left(\frac{1}{n}\Delta\big(x(n) + 2x(n-1)\big)\right) +$$

$$\frac{8(n-1)^2 + 1}{(n^3 + n^2)(n-1)^2(n^2+1)} x(n)(x^2(n)+1) -$$

$$\frac{5n^2 - 10n + 6}{(n^2+n)(n-1)^5} x^3(n-1) = 0, \tag{3.5}$$

$n \ge 2$. For this difference equation, all assumptions of Theorem (3.1) holds, but (H_6) The equation (3.5) has a solution $x(n) = n \in M^+$.

Theorem3.3. Assume that $-1 < p(t) \le 0$ and (H_2) - (H_5) hold. If

(H_7) $q(t) \ge Mh(t)$ for all $t \in [t_0, \infty)_T$;

(H_8) $\int_{t_0}^\infty \frac{1}{r(s)} \Delta s = \infty$; and

(H_9) $\int_{t_0}^\infty (q(s) - Mh(s))\Delta s = \infty$ then for equation (1.1), we have $M^+ = \phi$.

Proof. Suppose that the equation (1.1) has a solution $x \in M^+$. Proceeding as in the proof of Theorem3.1, we have (3.1) and (3.2). Since $x \in M^+$ and (iii), we have

$$z(t) = x(t) + p(t)x(\alpha(t)) \ge x(\alpha(t)) + p(t)x(\alpha(t))$$

$$= (1 + p(t))x(\alpha(t)) > 0$$

For $t \in [t_1, \infty)_T$. From (3.1), equation(1.1) becomes

$$(r(t)z^\Delta(t))^\Delta = -q(t)f(x(\beta(t))) + h(t)g(x(\gamma(t))) \tag{3.6}$$

From (3.6), (H_2) - (H_5) and (H_7), we obtain

$$(r(t)z^\Delta(t))^\Delta = f(x(\beta(t)))\left(-q(t) + h(t)\frac{g(x(\gamma(t)))}{f(x(\beta(t)))}\right)$$

$$\le -f(x(\beta(t)))\big(q(t) - Mh(t)\big) \le 0$$

For $t \in [t_1, \infty)_T$. This implies that $(r(t)z^\Delta(t))$ is non increasing on $t \in [t_1, \infty)_T$. Now suppose that $r(t)z^\Delta(t) < 0$ for large $t \in [t_1, \infty)_T$. Then there exists $t_2 \in [t_1, \infty)_T$ such that

$$r(t)z^\Delta(t) \le r(t_2)z^\Delta(t_2) < 0$$

$$z^{\Delta}(t) \le \frac{r(t_2)z^{\Delta}(t_2)}{r(t)}$$

Integrating from t_2 to t, we obtain

$$z(t) - z(t_2) \le \int_{t_2}^{t} \frac{r(t_2)z^{\Delta}(t_2)}{r(s)}\Delta s$$

This implies that $z(t) \to -\infty$ as $t \to \infty$ due to (H_8) which is a contradiction. Thus, $r(t)z^{\Delta}(t) \ge 0$ for large $t \in [t_1,\infty)_T$.

Hence, $z^{\Delta}(t) \ge 0$

Now proceeding as in the proof of Theorem 3.1, we obtain

$$\lim_{t\to\infty} \frac{r(t)z^{\Delta}(t)}{f(x(\beta(t)))} = -\infty$$

due to (H_9), which contradicts the assumption $z^{\Delta}(t) \ge 0$ for large $t \in [t_1,\infty)_T$.

This completes the proof of the theorem.

Example3.4. In theorem3.3, some of the assumptions cannot be dropped. For this, suppose $T = R$ and consider the differential equation

$$\Big(\frac{4}{(t^2+2t)}(x(t) - \frac{1}{2}x(t-1))'\Big)' + \frac{1}{t^2(t-1/2)^3}x^3(t-1/2) - \frac{1}{(t+2)^2(t-1)^3}x^3(t-1) = 0, \qquad (3.7)$$

$t(\ge 2) \in T$ For this difference equation all the conditions of Theorem3.3 are satisfied except (H_9). The equation (3.7) has a solution $x(t) = t \in M^{+}$.

Next, the existence of solutions of equation (1.1) in the class M^{-}.

Theorem3.5. Assume that (H_2), (H_4) and (H_5) hold. If (H_{10}) $\beta(t) \le \gamma(t) \le \alpha(t)$

(H_{11}) the function $\frac{1}{f(u)}$ is locally integrable on $(0,c)$ and $(-c,0)$ for some $c > 0$,

i.e $\int_0^c \frac{du}{f(u)} < \infty$, $\int_{-c}^0 \frac{du}{f(u)} > -\infty$; for some $c>0$

(H_{12}) f is sub multiplicative i.e $f(uv) \le f(u)f(v)$ for $u,v \in R$;

(H_{13}) $p(t) \ge 0$ and non-increasing and

(H_{14})

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{r(s)f(1+p(s))}\Big(\int_{t_0}^{s}(q(\theta) - Mh(\theta))\Delta\theta\Big)\Delta s = \infty$$

Then for equation (1.1), we have $M^{-} = \phi$

Proof. Suppose that equation (1.1) has a solution $x \in M^{-}$. Without loss of generality, we may assume that $x(t) > 0$ and $x^{\Delta}(t) \le 0$ for large $t \in [t_0,\infty)_T$ (the proof is similar if $x(t) < 0$ and $x^{\Delta}(t) \ge 0$ for large $t \in [t_0,\infty)_T$). Then there exists $t \in [t_1,\infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$, $x(\gamma(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\beta(t))$, $x^{\Delta}(\gamma(t))$ all are non-positive for all $t \in [t_1,\infty)_T$. Defining z(t) as in (3.1). Then by using (H_{13}), we see that $z(t) > 0$ and $z^{\Delta}(t) \le 0$ for all $t \in [t_1,\infty)_T$. Then equation (1.1) becomes

$$(r(t)z^{\Delta}(t))^{\Delta} + q(t)f(x(\beta(t))) - h(t)g(x(\gamma(t))) = 0$$

For $t \in [t_1,\infty)_T$, proceeding as in the proof of theorem (3.1), by using (H_2), (H_4) and (H_5) we obtain

$$\frac{r(t)z^{\Delta}(t)}{f(x(\beta(t)))} - \frac{r(t_1)z^{\Delta}(t_1)}{f(x(\beta(t_1)))} \le -\int_{t_1}^{t}\big(q(s) - Mh(s)\big)\Delta s$$

$$\frac{z^{\Delta}(t)}{f(x(\beta(t)))} \le -\frac{1}{r(t)}\int_{t_1}^{t}\big(q(s) - Mh(s)\big)\Delta s$$

Implies that,

$$\frac{z^{\Delta}(t)}{f(x(\beta(t)))} \le -\frac{1}{r(t)}\int_{t_1}^{t}\big(q(s) - Mh(s)\big)\Delta s \qquad (3.8)$$

Since x is non-increasing and $\beta(t) \le \alpha(t)$, we have

$$z(t) = x(t) + p(t)x(\alpha(t)) \le (1+p(t))x(\alpha(t)) \le (1+p(t))x(\beta(t))$$

For $t \in [t_1,\infty)_T$. Using (H_{12}), we have

$$f(z(t)) \le f(1+p(t))f(x(\beta(t))) \qquad (3.9)$$

For $t \in [t_1,\infty)_T$. From equations (3.8) and (3.9)

$$\frac{z^{\Delta}(t)f(1+p(t))}{f(z(t))} \le -\frac{1}{r(t)}\int_{t_1}^{t}\big(q(s) - Mh(s)\big)\Delta s$$

Implies that

$$\frac{z^{\Delta}(t)}{f(z(t))} \le -\frac{1}{f(1+p(t))r(t)}\int_{t_1}^{t}\big(q(s) - Mh(s)\big)\Delta s.$$

Integrating the last inequality from t_1 to t, we obtain

$$\int_{t_1}^{t}\frac{z^{\Delta}(t)}{f(z(t))}\Delta t \le -\int_{t_1}^{t}\frac{1}{f(1+p(s))r(s)}\int_{t_1}^{s}\big(q(\theta) - Mh(\theta)\big)\Delta\theta\Delta s$$

implies that

$$\int_{z(t_1)}^{z(t)} -\frac{1}{f(u)}\Delta u \ge \int_{t_1}^{t}\frac{1}{f(1+p(s))r(s)}\int_{t_1}^{s}\big(q(\theta) - Mh(\theta)\big)\Delta\theta\Delta s$$

or

$$\int_{z(t)}^{z(t_1)}\frac{du}{f(u)} \ge \int_{z(t_1)}^{z(t)} -\frac{1}{f(u)}\Delta u \ge \int_{t_1}^{t}\frac{1}{f(1+p(s))r(s)}\int_{t_1}^{s}(q(\theta) - Mh(\theta))\Delta\theta\Delta s$$

Using (H_{14}), we have

$$\limsup_{t\to\infty}\int_{z(t)}^{z(t_1)}\frac{du}{f(u)} = \infty$$

which contradicts to the condition (H_{11}). This completes the proof of the theorem.

Example3.6. In theorem 3.5, some of the assumptions cannot be dropped. For this, suppose $T = ¡$ and the differential equation

$$\left(t^2(2t-1)^2\left(x(t)+\frac{1}{2t+1}x(t-1/2)\right)'\right)' +$$
$$2t\left(\frac{t-2}{t-1}\right)^{1/3}x^{1/3}(t-2)-4\left(\frac{t-1}{t}\right)^{1/3}x^{1/3}(t-1)=0, \tag{3.10}$$

$t\in[t_0,\infty)_T, t_0\geq 3$. For the equation (3.10), all the assumptions of Theorem3.5 are hold, except (H_{14}).

The equation (3.10) has a solution $x(t)=\frac{t+1}{t}\in M^-$.

Theorem3.7. Assume that $(H_2),(H_4),(H_5)$ (H_8) and (H_{13}) hold. If

(H_{15}) $\beta(t)\leq\gamma(t)$ and

(H_{16}) $q(t)>Mh(t)$ for large $t\in[t_0,\infty)_T$ hold.

Then for equation (1.1), we have $M^-=\phi$.

Proof. Proceeding as in the proof of Theorem 3.5, we have

$$(r(t)z^\Delta(t))^\Delta = -q(t)f(x(\beta(t)))+h(t)g(x(\gamma(t)))$$

From $(H_2),(H_4),(H_5),(H_{15})$ and (H_{16}), we obtain

$$(r(t)z^\Delta(t))^\Delta = f(x(\beta(t)))\left(-q(t)+\frac{g(x(\gamma(t)))}{f(x(\beta(t)))}h(t)\right)$$
$$\leq -f(x(\beta(t)))\left(q(t)-Mh(t)\right)$$
$$<0 \quad \text{for} \quad t\in[t_1,\infty)_T$$

Then $r(t)z^\Delta(t)$ is decreasing. For $t>t_1$, we have

$$r(t)z^\Delta(t)<r(t_1)z^\Delta(t_1)<0$$

Or

$$z^\Delta(t)<r(t_1)z^\Delta(t_1)\frac{1}{r(t)}$$

Integrating from t_1 to t and using (vi), we get

$$z(t)\leq z(t_1)+(r(t_1)z^\Delta(t_1))\int_{t_1}^{t}\frac{1}{r(s)}\Delta s.$$

This implies that, $z(t)\to-\infty$ as $t\to\infty$.

which contradicts to our assumption that $z(t)>0$ for all $t\in[t_0,\infty)_T$. This complete the proof of the theorem.

Example3.8. $\left(t^2(t-1)^2\left(x(t)+x(t-1)\right)'\right)' +$

$$4t(t-1)^3x^3(t-1)-2\left(t-1/2\right)^3x^3(t-1/2)=0, \tag{3.11}$$

$t\in[t_0,\infty)_T$, $t_0\geq 1$.

For the equation (3.11), all assumptions of Theorem3.7 are hold, except (H_8). The equation (3.11) has a solution $x(t)=\frac{1}{t}\in M^-$, so $M^-\neq\phi$.

Theorem3.9. Assume that (H_2), (H_4), (H_5) (H_8), (H_9), (H_{15}) and (H_{16}) hold. Then for equation (1.1), we have $M^-=\phi$

Proof. Suppose that equation (1.1) has a solution $x\in M^-$. Proceeding as in the proof of Theorem 3.5, and defining $z(t)$ as in (3.1). we have,

$$z(t)=x(t)+p(t)x(\alpha(t))\geq x(\alpha(t))+p(t)x(\alpha(t))$$
$$=(1+p(t))x(\alpha(t))>0$$

and

$$(r(t)z^\Delta(t))^\Delta = f(x(\beta(t)))\left(-q(t)+\frac{g(x(\gamma(t)))}{f(x(\beta(t)))}h(t)\right)$$
$$\leq -f(x(\beta(t)))\left(q(t)-Mh(t)\right)<0$$

For $t\in[t_1,\infty)_T$, due to $(H_2),(H_4),(H_5)$, (H_{15}) and (H_{16}). It follows that, $r(t)z^\Delta(t)$ is decreasing for $t\in[t_1,\infty)_T$. Now proceeding as in the proof of Theorem 3.3. In view of (H_8), we find $r(t)z^\Delta(t)\geq 0$ for $t\in[t_1,\infty)_T$. Then We define,

$$w(t)=\frac{r(t)z^\Delta(t)}{f(z(\beta(t)))} \quad \text{for} \quad t\in[t_1,\infty)_T$$

$$w^\Delta(t)=\frac{(r(t)z^\Delta(t))^\Delta}{f(z(\beta(t)))}+(r(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{1}{f(z(\beta(t)))}\right)^\Delta$$

$$=\frac{(r(t)z^\Delta(t))^\Delta}{f(z(\beta(t)))} -$$

$$(r(\sigma(t))z^\Delta(\sigma(t)))\left(\frac{(f(z(\beta(t))))^\Delta}{f^\sigma(z(\beta(t)))f(z(\beta(t)))}\right)$$

Which implies that,

$$w^\Delta(t)=\frac{(r(t)z^\Delta(t))^\Delta}{f(z(\beta(t)))}-(r(\sigma(t))z^\Delta(\sigma(t)))$$
$$\left(\frac{\{\int_0^1 f'(z(\beta(t))+h\mu(t)(z(\beta(t)))^\Delta)dh\}(z(\beta(t)))^\Delta}{f^\sigma(z(\beta(t)))f(z(\beta(t)))}\right)$$

Implies that,

$$w^\Delta(t)\leq\frac{(r(t)z^\Delta(t))^\Delta}{f(z(\beta(t)))}$$
$$\leq -(q(t)-Mh(t))\frac{f(x(\beta(t)))}{f(z(\beta(t)))}$$

for $t\in[t_1,\infty)_T$. Clearly $z(t)\leq x(t)$, the last inequality will becomes

$$w^\Delta(t)\leq -(q(t)-Mh(t)) \tag{3.12}$$

for $t\in[t_1,\infty)_T$. Integrating the inequality (3.12), we get

$$w(t)\leq w(t_1)-\int_{t_1}^{t}(q(s)-Mh(s))\Delta s$$

which implies $w(t)\to-\infty$ as $t\to\infty$, due to (H_9), which is a contradiction. This completes the proof of the theorem.

Next we establish sufficient conditions under which equation (1.1) has no weakly oscillatory solutions.

Theorem3.10. Assume that $p(t)\equiv p(>-1)$, $(H_2),(H_4)$ and (H_7) hold. If

(H_{17}) $\alpha(t)=t$ and $\beta(t)=\gamma(t)$

Then for equation (1.1), we have $WOS=\phi$.

Proof. Let x be a weakly oscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)>0$ for large $t\in[t_0,\infty)_T$ (the proof is similar if x(t) < 0 for large $t\in[t_0,\infty)_T$). Then there exists $t\in[t_1,\infty)_T$ such that x(t), $x(\alpha(t))$, $x(\beta(t))$ all are positive for all $t\in[t_1,\infty)_T$. Define z(t) as in (3.1), then we see that $z(t)>0$ for $t\in[t_1,\infty)_T$ and $z^{\Delta}(t)$ oscillates for large t. Proceeding as in the proof of Theorem3.9, we have

$$(r(t)z^{\Delta}(t))^{\Delta}\le -f(x(\beta(t)))\big(q(t)-Mh(t)\big) \quad (3.13)$$

For $t\in[t_0,\infty)_T$, due to $(H_2),(H_4)$, and (H_7)

By taking $F(t)=r(t)z^{\Delta}(t)$, then equation (3.13)reduces to

$$F^{\Delta}(t)=-f(x(\beta(t)))\big(q(t)-Mh(t)\big)\le 0$$

for $t\in[t_1,\infty)_T$. This implies that F is non-increasing, which gives a contradiction, because F is an oscillatory function. This completes the proof of the theorem.

Example 3.11 In Theorem 3.10 the assumption can not be dropped. For this, suppose $T=q^{¥_0}$, where q>1 is a fixed real number and consider the following q-difference equation

$$\Delta_q\big(t\Delta_q(x(t)+2x(t))\big)+$$

$$\frac{(26+(-1)^{\log_q t})}{(q-1)^2t(2-(-1)^{\log_q^t})^3(3-(-1)^{\log_q t})}((1+x)x^3)(t/q)$$

$$-\frac{13(2+(-1)^{\log_q t})}{(q-1)^2t(2-(-1)^{\log_q t})^3}x^3(t/q)=0, \quad (3.14)$$

$t\in T$. For this q-difference equation, all assumptions of Theorem (3.10) hold, but (H_7) is violated. The equation (3.14) has a solution $2+(-1)^{\log_q t}\in$ WOS.

Theorem3.12. Assume that $p(t)\equiv p(>0)$, $(H_2),(H_4),(H_5)$, (H_7) (H_8) and (H_9) hold. if $\beta(t)=\gamma(t)$, then every solution of equation (1.1) is either oscillatory or weakly oscillatory.

Proof. From Theorem 3.1, it follows that for equation (1.1) the class $M^{+}=\phi$. In order to complete the proof of it suffices to show that $M^{-}=\phi$. Suppose that equation (1.1) has a solution $x\in M^{-}$. Without loss of generality, we may assume that $x(t)>0$ and $x^{\Delta}(t)\le 0$ for large $t\in[t_0,\infty)_T$ (the proof is similar if x(t) < 0 and $x^{\Delta}(t)\ge 0$ for large $t\in[t_0,\infty)_T$). Then there exists $t\in[t_1,\infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$ all are positive and $x^{\Delta}(t)$, $x^{\Delta}(\alpha(t))$, $x^{\Delta}(\beta(t))$ all are non-positive for all $t\in[t_1,\infty)_T$. Defining z(t) as in (3.1). Then we see z(t)>0 and $z^{\Delta}(t)\le 0$ for all $t\in[t_1,\infty)_T$. Then from equation (3.1), (1.1) reduces to

$$(r(t)z^{\Delta}(t))^{\Delta}(t)+q(t)f(x(\beta(t)))-h(t)g(x(\gamma(t)))=0$$

for $t\in[t_1,\infty)_T$. Now for $t>t_2\ge t_1(t_2\in[t_1,\infty)_T)$, we have

$$\Big(\frac{r(t)z^{\Delta}(t)}{f(x(\beta(t))}\Big)^{\Delta}=\frac{(r(t)z^{\Delta}(t))^{\Delta}}{f(x(\beta(t)))}+ (r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}$$

$$=-q(t)+h(t)\frac{g(x(\gamma(t)))}{f(x(\beta(t)))}+(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}$$

$$\le -q(t)+Mh(t)+(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}$$

$$\le (r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}$$

Integrating the last inequality from t_2 to t, we obtain

$$\frac{r(t)z^{\Delta}(t)}{f(x(\beta(t)))}-\frac{r(t_2)z^{\Delta}(t_2)}{f(x(\beta(t_2)))}\le\int_{t_2}^{t}(r(\sigma(s))z^{\Delta}(\sigma(s)))\Big(\frac{1}{f(x(\beta(s)))}\Big)^{\Delta}\Delta s$$

$$\le (r(\sigma(t_2))z^{\Delta}(\sigma(t_2)))\int_{t_2}^{t}\Big(\frac{1}{f(x(\beta(s)))}\Big)^{\Delta}\Delta s$$

This implies that,

$$\frac{r(t)z^{\Delta}(t)}{f(x(\beta(t)))}\le (r(\sigma(t_2))z^{\Delta}(\sigma(t_2)))\Big(\frac{1}{f(x(\beta(t)))}-\frac{1}{f(x(\beta(t_2)))}\Big)$$

$$(r(t)z^{\Delta}(t))\le (r(\sigma(t_2))z^{\Delta}(\sigma(t_2)))\Big(1-\frac{f(x(\beta(t)))}{f(x(\beta(t_2)))}\Big)$$

for large $t\in[t_2,\infty)_T$ since, x is non increasing, then we can find $k(<0)\in R$ such that

$$(r(\sigma(t_2))z^{\Delta}(\sigma(t_2)))\Big(1-\frac{f(x(\beta(t)))}{f(x(\beta(t_2)))}\Big)\le k$$

for all $t\in[t_2,\infty)_T$.

Therefore,

$$r(t)z^{\Delta}(t)\le k.$$

Or $$z^{\Delta}(t)\le k\frac{1}{r(t)}$$

Thus, for large $t \in [t_2, \infty)_T$, we have

$$z(t) - z(t_2) \le k\int_{t_2}^{t} \frac{1}{r(s)}\Delta s$$

which implies that $z(t) \to -\infty$ as $t \to \infty$ a contradiction to $z(t) > 0$. This completes the proof of the theorem.

Example3.13. Let T = hZ, where h is a ratio of odd positive integers and consider the following h-difference equation

$$\Delta_h\left(3e^{-t}\Delta_h(x(t)+2x(t-h))\right)+ \frac{5(2e^h-1)(1+e^{-h})(1+e^{-2h})e^{-6h}}{h^2(1+(-1)^t e^{2h-t})}e^t x^3(x+1)(t-2h) - \frac{2(2e^h-1)(1+e^h)(1+e^{-2h})e^{-6h}e^t}{h^2}x^3(t-2h)=0 \quad (3.15)$$

for $t \ge 3h$. It is easy to see that equation (3.15) satisfies all the conditions of Theorem3.12. Hence, every solution of equation(3.15) oscillatory or weakly oscillatory. In particular, $x(t) = (-1)^t e^{-t}$ is a solution of equation (3.15).

Remark 3.14. We can also find the existence / non-existence solutions of a class M^+, M^-, OS and WOS by reducing the equation (1.1), it into the equation of the form (1.2) with the help of the following assumptions,

(H_{19}) g is bounded;

(H_{20}) $\int_{t_0}^{\infty}\frac{1}{r(s)}\int_s^{\infty} h(\theta)\Delta\theta\Delta s < \infty.$

The following theorem shows that under which conditions the solutions of a class M^+ is empty of equation (1.1)

Theorem 3.15. Assume that (H_1), (H_2), (H_5), (H_{19}) and (H_{20}) hold. If

(H_{21}) $\limsup_{t\to\infty}\int_{t_0}^{t} q(s)\Delta s = \infty$. Then for equation (1.1), we have $M^+ = \phi$.

Proof. Suppose that the equation (1.1) has a solution $x \in M^+$. Without loss of generality we may assume that $x(t) > 0$ and $x^\Delta(t) \ge 0$ for large (the proof is similar if $x(t) < 0$ and $x^\Delta(t) \le 0$ for large $t \in [t_0, \infty)_T$. Then there exists $t_1 \in [t_0, \infty)_T$ such that $x(t)$, $x(\alpha(t))$, $x(\beta(t))$, $x(\gamma(t))$ all are positive and $x^\Delta(t)$,

$x^\Delta(\alpha(t))$, $x^\Delta(\beta(t))$, $x^\Delta(\gamma(t))$ all are non negative for all $t \in [t_1, \infty)_T$. Define z(t) as in (3.1), we obtain

$$(r(t)z^\Delta(t))^\Delta + q(t)f(x(\beta(t))) - h(t)g(x(\gamma(t))) = 0. \quad (3.16)$$

Define,

$$k(t) = \int_t^{\infty}\frac{1}{r(s)}\int_s^{\infty} h(\theta)g(x(\gamma(\theta)))\Delta\theta\Delta s. \quad (3.17)$$

Then in view of (H_1), we have $z(t) > 0$ and $z^\Delta(t) \ge 0$ for all $t \in [t_1, \infty)_T$. Notice that from conditions (H_{19}) and (H_{20}) imply that k(t) exists for all $t \in [t_0, \infty)_T$. Define

$$y(t) = z(t) - k(t) = x(t) + p(t)x(\alpha(t)) - k(t) \quad (3.18)$$

Then $y^\Delta(t) = z^\Delta(t) - k^\Delta(t) \ge 0$, because $k^\Delta(t) < 0$. The equation (3.16) becomes

$$(r(t)y^\Delta(t))^\Delta + q(t)f(x(\beta(t))) = 0$$

or

$$(r(t)y^\Delta(t))^\Delta = -q(t)f(x(\beta(t))) \quad (3.19)$$

for $t \in [t_1, \infty)_T$. Now for $t \in [t_1, \infty)_T$

$$\left(\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)y^\Delta(t))^\Delta}{f(x(\beta(t)))} + (r(\sigma(t))y^\Delta(\sigma(t)))\left(\frac{1}{f(x(\beta(t)))}\right)^\Delta$$

$$= \frac{(r(t)y^\Delta(t))^\Delta}{f(x(\beta(t)))} - (r(\sigma(t))y^\Delta(\sigma(t)))\left(\frac{(f(x(\beta(t))))^\Delta}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right)$$

Implies that,

$$\left(\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)y^\Delta(t))^\Delta}{f(x(\beta(t)))} - (r(\sigma(t))y^\Delta(\sigma(t)))\left(\frac{\{\int_0^1 f'(x(\beta(t))+h\mu(t)(x(\beta(t)))^\Delta)dh\}(x(\beta(t)))^\Delta}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right)$$

Or

$$\left(\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta = \frac{(r(t)y^\Delta(t))^\Delta}{f(x(\beta(t)))} - (r(\sigma(t))y^\Delta(\sigma(t)))\left(\frac{\{\int_0^1 f'(x(\beta(t))+h\mu(t)(x(\beta(t)))^\Delta)dh\}(x^\Delta(\beta(t)))\beta^\Delta(t)}{f^\sigma(x(\beta(t)))f(x(\beta(t)))}\right)$$

therefore,

$$\left(\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta \le \frac{(r(t)y^\Delta(t))^\Delta}{f(x(\beta(t)))} \quad (3.20)$$

for all $t \in [t_1, \infty)_T$, because $f'(u) \ge 0$ for $u \ne 0$ and $y^\Delta(t) \ge 0$, $x^\Delta(\beta(t)) \ge 0$, for all $t \in [t_1, \infty)_T$. From (3.19) and (3.20), we have

$$\left(\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))}\right)^\Delta \le -q(t)$$

for all $t \in [t_1, \infty)_T$. Integrating the last inequality from t_1 to t, we obtain

$$\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))} - \frac{r(t_1)y^\Delta(t_1)}{f(x(\beta(t_1)))} \le -\int_{t_1}^{t} q(s)\Delta s$$

From (H_{21}), we obtain

$$\liminf_{t\to\infty}\frac{r(t)y^\Delta(t)}{f(x(\beta(t)))} = -\infty,$$

which contradicts the assumption $y^{\Delta}(t) \ge 0$ for large t. Thus, the theorem is proved.

Example 3.16. In Theorem 3.15, some of the assumptions cannot be dropped. For this, suppose $T = R$ and consider the differential equation

$$\Big(\frac{2}{3}\Big(\frac{t^3+1}{t^2-1}\Big)\Big(x(t)+\frac{1}{2}x(t-1)\Big)'\Big)' + \Big(\frac{t^2(t-3)}{(t^2-1)^2(t-2)^3}\Big)$$

$$x^3(t-2) - \Big(\frac{2t(t^2-2t+2)}{(t-1)(t^2-1)^2}\Big)\frac{x}{x^2+1}(t-1) = 0, \quad (3.21)$$

$t \in [t_0, \infty)_T$, $t_0 \ge 3$. For the equation (3.21), all the assumptions of Theorem hold, except (H_{21}). The equation (3.21) has a solution $x(t) = t \in M^+$.

Remark3.17. We can also obtained the sufficient conditions for the remaining classes by taking the assumptions (H_{19}) and (H_{20}).

4. Behavior of Solutions in M^+ and M^- :

Theorem4.1. Assume that (H_1), (H_2), (H_5), (H_{10}) and (H_{12}) - (H_{14}) hold. Then every solution of equation in the class $x \in M^-$, we have $\lim_{t\to\infty} x(t) = 0.$

Proof. Proceeding as in the proof of Theorem 3.5, we have

$$\int_{z(t)}^{z(t_1)} \frac{du}{f(u)} \ge \int_{z(t_1)}^{z(t)} -\frac{1}{f(u)}\Delta u$$

$$\ge \int_{t_1}^{t} \frac{1}{f(1+p(s))r(s)} \int_{t_1}^{s} (q(\tau)-Mh(\tau))\Delta\tau\Delta s$$

Using (H_{14}), we obtain

$$\limsup_{t\to\infty} \int_{z(t)}^{z(t_1)} \frac{du}{f(u)} = \infty.$$

This implies that $\lim_{t\to\infty} z(t) = 0.$

So $\lim_{t\to\infty} x(t) = 0$, because $z(t) \ge x(t)$ and x is monotonic. This completes the proof of the theorem.

Theorem4.2. Assume that (H_1), (H_2), (H_7) and p(t) is bounded hold. If

$$(H_{22}) \quad \limsup_{t\to\infty} \int_{t_0}^{t} (q(s)-Mh(s)) \int_{t_0}^{s} \frac{1}{r(\tau)}\Delta\tau\Delta s = \infty$$

Then every solution of (1.1) in the class M^+ is unbounded.

Proof. Let x be a solution of (1.1) such that $x \in M^+$. Proceeding as in the proof of Theorem3.1, by defining z(t) as in equation (3.1), we see that $z(t) > 0$ and $z^{\Delta}(t) \ge 0$ for all $t \in [t_1, \infty)_T$ due to (i). Then from equation (1.1), we obtain equation (3.2). For the function

$$w(t) = -\frac{r(t)z^{\Delta}(t)}{f(x(\beta(t)))} \int_{t_1}^{t} \frac{1}{r(s)}\Delta s,$$

we have,

$$w^{\Delta}(t) = -(r(t)z^{\Delta}(t))^{\Delta}\Big(\frac{1}{f(x(\beta(t)))}\int_{t_1}^{t}\frac{1}{r(s)}\Delta s\Big) -$$

$$(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\int_{t_1}^{t}\frac{1}{r(s)}\Delta s\Big)^{\Delta}$$

$$\ge (q(t)-Mh(t))\int_{t_1}^{t}\frac{1}{r(s)}\Delta s -$$

$$(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\frac{1}{f(x(\beta(t)))}\Big(\int_{t_1}^{t}\frac{1}{r(s)}\Delta s\Big)^{\Delta}\Big) -$$

$$(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\int_{t_1}^{\sigma(t)}\frac{1}{r(s)}\Delta s\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}\Big)$$

Implies that

$$w^{\Delta}(t) \ge (q(t)-Mh(t))\int_{t_1}^{t}\frac{1}{r(s)}\Delta s -$$

$$(r(t)z^{\Delta}(t))\Big(\frac{1}{r(t)f(x(\beta(t)))}\Big) -$$

$$(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\int_{t_1}^{\sigma(t)}\frac{1}{r(s)}\Delta s\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}\Big)$$

$$= (q(t)-Mh(t))\int_{t_1}^{t}\frac{1}{r(s)}\Delta s - \frac{z^{\Delta}(t)}{f(x(\beta(t)))} -$$

$$(r(\sigma(t))z^{\Delta}(\sigma(t)))\Big(\int_{t_1}^{\sigma(t)}\frac{1}{r(s)}\Delta s\Big)\Big(\frac{1}{f(x(\beta(t)))}\Big)^{\Delta}$$

$$\ge (q(t)-Mh(t))\int_{t_1}^{t}\frac{1}{r(s)}\Delta s - \frac{z^{\Delta}(t)}{f(x(\beta(t)))}$$

Integrating the last inequality from t_1 to t, we obtain

$$w(t) \ge \int_{t_1}^{t}(q(s)-Mh(s))\int_{t_1}^{s}\frac{1}{r(\tau)}\Delta\tau\Delta s$$

$$-\int_{t_1}^{t}\frac{z^{\Delta}(s)}{f(x(\beta(s)))}\Delta s \qquad (4.22)$$

As the function $\frac{z^{\Delta}(t)}{f(x(\beta(t)))}$ is positive for $t \in [t_1, \infty)_T$ then

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{f(x(\beta(s)))}\Delta s \quad \text{exists.}$$

Assume that $\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{f(x(\beta(s)))}\Delta s = k < \infty$. Taking into account (H_{22}) and from (4.22) we get

$$\limsup_{t\to\infty} w(t) = \infty$$

Which gives a contradiction, because w is negative for all $t \in [t_1, \infty)_T$. Thus

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{f(x(\beta(s)))}\Delta s = \infty. \qquad (4.23)$$

Consequently,

$$\lim_{t\to\infty}\int_{t_1}^{t}\frac{z^{\Delta}(s)}{f(x(\beta(s)))}\Delta s \le \frac{1}{f(x(\beta(t_1)))}\int_{t_1}^{t} z^{\Delta}(s)\Delta s$$
$$= \frac{1}{f(x(\beta(t_1)))}(z(t)-z(t_1)).$$

From (4.23), we get

$$\lim_{t\to\infty} z(t) = \infty. \qquad (4.24)$$

Since $z(t) = x(t) + p(t)x(\alpha(t))$ and x is non negative, we have

$$z(t) \le (1+p(t))x(t).$$

From (4.24), we have

$$\lim_{t\to\infty} x(t) = \infty.$$

This completes the proof of the theorem

5. Conclusion

The present paper has introduced the concept of positive and negative coefficients in neutral delay dynamic equations and relatively we studied the non-existence solutions of class M^{+} and M^{-} for the ranges $p(t) \ge 0$ and $-1 < p(t) \le 0$, and the non-existence solutions of a class WOS is studied for $p(t) \equiv p(>-1)$, by taking some restriction in delays. The existence solutions of a class OS and WOS has been studied by the way, so that the class M^{+} and M^{-} is empty. In section 4, we have been studied the asymptotic behavior of M^{+} and M^{-}. It would be interesting to study the existence / non-existence of a class M^{+}, M^{-}, OS and WOS for different ranges of p(t). In addition, extending such results to higher order equation would also be of interest.

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