Inclusion Attributes with Applications for Certain Subclasses of Analytic Subroutines

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1. Introduction

Let A be the class of functions f analytical in the open unit disc E = {z: |z|< 1} and are normalized with the condition f(0) = 0, f'(0) = 1. Also let S.S * (γ). C(γ) announce the subclasses of A comprising of function that are severally monovalent star like of order γ and convex of order γ, $0 \leq \gamma < 1$. In E.

Let f and g be analytic in E then f is said to be subordinate to g, written as $f \propto g$ and $f(z) \propto g(z)$, $z \in E$. If there exists a Schwarz function w analytic in E with w(0) = 0and w(z)<1 fot $z \in E$ such that

$$f(z) = g(w(z)).$$

If g is univalent in E then $f \propto g$ if only if f(0) = g(0) and $f(E) \subset g(E)$.

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ the convolution (Hadamard product) of t and g is defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). z \in E.$

Let h be analytic , convex and univalent in E with h(0) = 1. $\Re h(z) > 0$. We denote the class of all analytic functions p with p(0) = 1 as p(h) if $p \propto h$ in E. Let For f, $\phi \in A$, $(f * g)(z) \neq 0$ and let $f(z) * \phi(z) = f_1(z)$. Also we define $F(z) = (1-\lambda)f_1(z) + \lambda z f_1'(z)$: $0 \leq \lambda \leq 1$.

We now define the following.

Definition 1. Let f, $\phi \in A$ and let F be Defined by (1). Then $f \in S(h,\phi,\lambda)$ if and only if

$$\frac{zF'(z)}{F(z)} \propto h(z).$$

Where h is analytic convex and univalent in E with h(0) = 1.

In this Case, we say $F \in S(h)$.

The corresponding class C(h,ϕ,λ) is defined as follows.

Let $f \in A$. Then

$f \in C(h,\phi,\lambda)$ if and only if $zf' \in S(h,\phi,\lambda)$.

In other words.

$f \in C(h,\phi,\lambda)$ if and only if $\frac{zF'(z)}{F(z)} \propto h(z)$, $z \in E$.

Definition 2. Let f,$\phi \in A$ and let F be defined by (1). Then $f \in K(h,\phi,\lambda)$ if there exists $g \in S(h,\phi,\lambda)$ with

$$G = (1-\lambda)(g * \phi) + \lambda(g * \phi)'$$

Such that $\frac{zF'(z)}{G(z)} \propto h(z)$, $z \in E$. Where $\lambda \in (0,1)$ and h is analytic convex univalent in E. h(0)=1.

Definition 3. Let $f, \phi \in A$ $(f * g)(z) \neq 0$ and let h be analytic and convex univalent in E with h(0) = 1. Then $f \in R(h,\phi,\lambda)$, for $\lambda \geq 0$. If and only if

$$(f * g)' + \lambda z(f * \phi)'' \propto h, z \in E.$$

The corresponding class $T(h,\phi,\lambda)$ can be defined as follows. Let $f \in A$. Then $f \in T(h,\phi,\lambda)$ if and only if $zf' \in R(h,\phi,\lambda)$.

Let S_σ be the class of prestarlike functions of order $\sigma \le 1$. We recall that $f \epsilon S_\sigma$ whenever $f \epsilon A$ and f satisfies

$$\Re\left\{ f(z) * \frac{z}{(1-z)^{2-2\sigma}}\right\} > \sigma . if\ \sigma < 1.$$

while

$$\Re \frac{f(z)}{z} > \frac{1}{2}, \text{ if } \sigma = 1.$$

For special cases , we have:

(i). $S_0 = C$

(ii).$S_{1/2}$= S(1/2), the class of starlike functions of order ½.

(iii). $S_1 = \bar{C}oC$, where $\bar{C}oC$ is the closed convex hull of C.

A prestart like functions of order σ is univalent.

2 Preliminaries

Lemma 1([25]). For $\sigma \le 1$, let $f \in S_\sigma$, g be star like of order σ. H be analytic in E. Then

$$\frac{f*gH}{f*g}(E) \subset \bar{C}o(H(E)).$$

Also, for $\sigma < 1$

$$S_\sigma * K(\sigma) \subset K(\sigma).$$

Where K(σ) is the class of close-to-convex functions of order σ.

Lemma 2([12]). Let be analytic univalent convex in E with h(0)=1 and $\Re[\beta h(z) + \delta] > 0. \beta . \delta \in C. z \in E$ If p is analytic in E with p(0) = h(0), then

$$\left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \delta}\right\} \propto h(z).$$

Implies

$$P(z) \propto q(z) \propto h(z).$$

Where q(z) is the best dominant and is given as

$$q(z) = [\{\left(\int_0^1 exp \int_t^{tz} \frac{h(u)-1}{u} du\right)dt\}^{-1} - \frac{\delta}{\beta}].$$

Lemma 3([4]). Let β,γ be complex numbers. Let h(z) be convex univalent in E with h(0) = =1. And $\Re[\beta h(z) + \gamma] > 0. z \epsilon E \text{ and } q \epsilon A$ with $q(z) \propto h(z), z \epsilon E.$

If p is analytic in E with p(0) = 1. $\Re p(z) >$ 0.then

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \propto h(z).$$

Implies p(z) ∝ h(z) in E.

Lemma 4. Let p(z) and q(z) be analytic in E p(0) = q(0) = 1 and $\Re q(z) > \frac{1}{2}$ for $|z| < p$ $(0<p\le 1)$. Then the image of $E_p = \{z: |z|< p\}$ under p*q is a subset of the closed convex hull of p(E).

The above Lemma is a simple consequence of a result due to Nehari and Netanyahu |13|.

Lemma 5. Let g(z)be analytic in E and h(z) be analytic and convex univalent in E with h(0) = g(0). If

$$\left\{ g(z) + \frac{1}{\delta} zg'(z)\right\} \propto h(z), \ (\Re(\delta) \ge 0, \delta \ne 0),$$
(2)

then

$$g(z) \propto h(z) = \delta z^{-\delta} \int_0^z t^{\delta-1} h(t)dt \propto h(z).$$

And h(z) is the best dominant of (2).

3 Main Results

Theorem 1. $C(h, \phi, \lambda) \subset S(h, \phi, \lambda)$.

Proof. Let $f \in C(h, \phi, \lambda)$. Set

$$\frac{zF^{'}(z)}{F(z)} = p(z) \qquad (3)$$

where F is defined by (1), and p(z)is analytic in E with p(0) = 1.

With simple computation, we get from (3)

$$p(z) + \frac{zp^{'}(z)}{p(z)} = \frac{(zF^{'}(z))'}{F^{'}(z)} \propto h(z).$$

And using Lemma 2, it follows that

P(z) ∝ h(z), z ∈ E.

This implies that f ∈ S(h, ϕ, λ)in E.

Theorem 2. Let class S(h, ϕ, λ) is invariant under convex convolution.

This result also hold for the classes

C(h, ϕ, λ), K(h, ϕ, λ), R(h, ϕ, λ) and T(h, ϕ, λ).

Proof. Let ψ ∈ C and f ∈ S(h, ϕ, λ). WE want to show that (ψ*f) ∈ S(h, ϕ, λ). Consider

$$\frac{z[(1-\lambda)\{\phi*(\psi*f)\}^{'} + \lambda\{z(\phi*(\psi*f))^{'}\}^{'}]}{z[(1-\lambda)\{\phi*(\psi*f)\} + \lambda z\{(\phi*(\psi*f))\}^{'}]} = \frac{\psi*[\{(1-\lambda)z(\phi*f)^{'}\} + \lambda\{z(z(\phi*f)^{'})^{'}\}]}{\psi*\{(1-\lambda)(\phi*f) + \lambda z(\phi*f)^{'}\}}$$

For ψ ∈ C,[(1-λ)(ϕ*f)+λz(ϕ*f)’] = F ϵ S(h) and S(h) ϵ S, p = $\frac{zF'}{F}$ ∝ h. we have

$$\frac{z[(1-\lambda)\{\phi*(\psi*f)\}^{'} + \lambda\{z(\phi*(\psi*f))^{'}\}^{'}]}{(1-\lambda)\{\phi*(\psi*f)\} + \lambda z\{(\phi*(\psi*f))\}'} = \frac{\psi*p\{(1-\lambda)(\phi*f) + \lambda z(\phi*f)^{'}\}}{\psi*\{(1-\lambda)(\phi*f) + \lambda z(\phi*f)^{'}\}} \qquad (4)$$

WE now apply Lemma 1 with σ = 0 to(4) and have

(ψ*f) ⊂ S(h, ϕ, λ) in E.

The proof of this result for other classes follows on similar lines.

AS an application of Theorem 2. We have the following.

Remark 1. Since the classes S(h, ϕ, λ), K(h, ϕ, λ), R(h, ϕ, λ) and T(h, ϕ, λ). are preserved under convolution with convex functions, it follows that these classes are invariant under the following integral operators.

$$f_1(z) = \int_0^z \frac{f(t)}{t} dt = [\log(1-z)] * f(z) = (\psi_1 * f)(z).$$

$$f_2(z) = \frac{2}{z}\int_0^z f(t)dt = \left[\frac{-2}{z}\{z + \log(1-z)\}\right] * f(z) = (\psi_2 * f)(z).$$

$$f_3(z) = \frac{b_1+1}{z^{b_1}}\int_0^z t^{b_1-1} f(t)dt. \Re b_1 > -1$$

$$= (\sum_{n=1}^{\infty} \frac{b_1+1}{b_1+n} z^n)^{f(z)} = (\psi_3 * f)(z).$$

It can easily be verified that $\psi_1, \psi_2 \epsilon\ C\ and\ we\ refer\ to$ [26,27] for ψ_3 to be convex. We apply Theorem 2 to obtain the required result.

Theorem 3 For λ ≥ 0, S (h,ϕ,λ) ⊂ S(h,ϕ,0)

Proof. The case, When λ = 0 , is trivial, so we suppose λ > 0. Let f ϵ S (h,ϕ,λ) and let $f_1 = f * \phi$. Define

$$F(z) = (1-\lambda)f_1(z) + \lambda z f_1'(z).$$

Then F ∈ S(h), that is $\frac{zF^{'}(z)}{F(z)} \propto h(z)$ in E. We want to show that $\frac{zf_1^{'}(z)}{f_{1}(z)} \propto h(z)$ in E.

Let

$$\frac{zf_1'(z)}{f_{1(z)}} = p(z).$$

Then p(z) is analytic in E with p(0) = 1.

Now

$$\frac{zF'(z)}{F(z)} = \frac{zf'_1(z) + z^2f''_1(z)}{(1-\lambda)f_1(z) + \lambda zf'_1(z)}$$

$$= \frac{zf'_1(z) + \lambda z\left(zf'_1(z)\right)' - \lambda zf'_1(z)}{(1-\lambda)f_1(z) + \lambda zf'_1(z)}$$

$$= \frac{(1-\lambda)\frac{zf'_1(z)}{f_1(z)} + \lambda z\frac{(zf'_1(z))'}{f_1(z)}}{(1-\lambda) + \lambda\frac{zf'_1(z)}{f_1(z)}}$$

$$= \frac{(1-\lambda)p(z) + \lambda(p^2(z) + zp'(z))}{(1-\lambda) + \lambda p(z)}$$

$$= \left[p(z) + \frac{zp'(z)}{p(z) + \left(\frac{1}{\lambda} - 1\right)}\right] \propto h(z), z \in E$$

We know use Lemma 2 to have p(z) ∝ h(z) in E.

Theorem 4. For $\lambda \geq 0$, K(h,ϕ,λ) ⊂ K(h,ϕ,0).

Proof. The case $\lambda = 0$ is trivial. We assume $\lambda > 0$, Let

$$F(z) = (1-\lambda)(f*\phi) + \lambda z(f*\phi)' \text{ and } G(z) = (1-\lambda)(g*\phi) + \lambda z(g*\phi)' \quad (5)$$

Let f ϵ K(h,ϕ,λ). Then there exists g ϵ S(h,ϕ,λ) such that

$$\frac{zF'(z)}{G(z)} \propto h(z), z \in E.$$

Where F and G are defined by (5) . Set

$$\frac{z(f*\phi)'(z)}{(f*\phi)(z)} = p(z) \quad (6)$$

We note that p is analytic in E with p(0) = 1.

Then, from (6) and with $\frac{z(g*\phi)'}{(g*\phi)} = p_0 \propto h$, we have, after some simple computation.

$$\frac{zF'(z)}{G(z)} = p(z) + \frac{\lambda zp'(z)}{(1-\lambda) + \lambda p_0(z)} \propto h(z) \text{in E.}$$

Using Lemma 3, we obtain the required result, that is

$$\frac{z(f(z)*\phi(z))'}{(f(z)*\phi(z))} = p(z) \propto h(z), z \in E.$$

Theorem 5. R(h,ϕ, λ) ⊂ T(h,ϕ, λ).

Proof. Let f ∈ R(h,ϕ λ) and let $\left\{(1-\lambda)\frac{(f(z)*\phi(z))}{z} + \lambda\left(f(z)*\phi(z)\right)'\right\} = p(z),$

Then $(f(z)*\phi(z))'+\lambda(f(z)*\phi(z))'' = p(z) + zp'(z)$.

Since f ∈ R(h,ϕ, λ,), We have p + zp' ∝ h and, applying Lemma 3, it follows that p ∝ h in E. This proves that

$$f \in T(h,\phi,\lambda)$$

And the inclusion relation is established.

Theorem 6. The class R(h,ϕ, λ) is a convex set.

Proof. Let $f_1, f_2 \in R(h,\phi,\lambda)$ and let

$$F_1 = (1-\lambda)(f_1*\phi)' + \lambda(z(f_1*\phi)')'$$

$$F_2 = (1-\lambda)(f_2*\phi)' + \lambda(z(f_2*\phi)')'$$

Let

$$F(z) = \alpha[F_1(z) + (1-\alpha)F_2(z), 0 \leq \alpha \leq 1.$$

Then

$F'(z) + \lambda F''(z) = \alpha[F_1(z) + (1-\alpha)F_2]' + \lambda z[\alpha F_1''(z) + (1-\alpha)F_2''] = \alpha p_1(z) + (1-\alpha)p_2(z) = p(z).$

Where $p_i(z) = F_i'(z) + \lambda z F_i''(z), i = 1,2, p_i \propto h$.

Since P(h)is a convex set, $P \propto h$ and hence $F \in R(h,\phi,\lambda)$ in E.

Remark 2. Functions in R(h,ϕ,λ) can be obtained by taking convolution (Hadamard product) of the function

$$k(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z \frac{t^{\frac{1}{\lambda}-1}}{1-t} dt, \lambda > 0 \quad (7)$$

With function

$$j(z) = \int_0^z p(t)dt, \qquad p \propto h \qquad (8)$$

The following facts about the classes R(h,ϕ, λ) and T(h,ϕ, λ) can easily be established.

(i) $R(h, \frac{z}{1-z}, 0)$ and $T(h, \frac{z}{1-z}, 1)$ consist entirely of univalent functions.

(ii) T(h,ϕ, λ) is a convex set.

(iii) $T(h,\phi,\lambda_1) \subset T(h,\phi,\lambda_2)$, $0 \leq \lambda_2 \leq \lambda_1$.

(iv) $T(h,\phi,\lambda) \subset T(h,\phi,1)$, $\lambda \geq 1$.

(v) $R(h,\phi,\lambda_1) \subset R(h,\phi,\lambda_2)$, $0 \leq \lambda_2 \leq \lambda_1$.

We prove the result (v) as follows, For $A_2 = 0$ we need the Lemma given below.

Lemma 6. Let $\lambda \geq 0$ and $D(z) \in S(h)$. Let N(z) be analytic in E and $N(0) = D(0) = 0$, $N'(0) = D'(0) = 1$.

Let, for $z \in E$, h convex univalent , $\Re h(z) > 0$.

$$\left\{(1-\lambda)\frac{N(z)}{D(z)} + \lambda\frac{N'(z)}{D'(z)}\right\} \propto h(z),$$

Then

$$\frac{N(z)}{D(z)} \propto h(z) \;\; for\ z \in E.$$

Proof. The proof of this Lemma is quite straightforward when we put $\frac{N(z)}{D(z)} = p(z)$,and obtain

$$(1-\lambda)\frac{N(z)}{D(z)} + \lambda\frac{N'(z)}{D'(z)} = \left\{p(z) + p_0(z)\left(zp'(z)\right)\right\} \propto h(z),$$

where $\Re p_0(z) = \Re\frac{D(z)}{D'(z)} > 0$ in E. Now using Lemma 3 we have the required result that $\frac{N(z)}{D(z)} \propto h(z)$ in E.

We now proceed to prove the inclusion result (v). We assume $\lambda_2 > 0$ and $f \in R(h,\phi,\lambda_1)$. Then

$$(1-\lambda_2)(f*\phi)' + \lambda_2(z(f*\phi)')'$$

$$= \frac{\lambda_2}{\lambda_1}\left\{(1-\lambda_1)(f*\phi)' + \lambda_1\left(z(f*\phi)'\right)'\right\} + \left(1-\frac{\lambda_2}{\lambda_1}\right)(f*\phi)'$$

$$= \frac{\lambda_2}{\lambda_1}p_1(z) + \left(1-\frac{\lambda_2}{\lambda_1}\right)p_2(z).$$

Since $f \in R(h,\phi,\lambda_1)$, $p_1 \propto h$ and from Lemma 6, $p_2 \propto h$. Now $\frac{\lambda_2}{\lambda_1} < 1$ *and* $h(E)$ *is convex, it follows that*

$$\left\{(1-\lambda_2)(f*\phi)' + \lambda\left(z(f*\phi)'\right)'\right\} \propto h$$

Thus $f \in R(h,\phi,\lambda_2)$.

Theorem 7. Let $f \in T(h,\phi,\lambda)$, $0 < \lambda < 1$.Then $f \in T(h,\phi,1)$ and hence univalent for $|z| < r_0$.where r_0 is the radius of the largest disc centered at the origin for which $\Re k'(z) > \frac{1}{2}$, k(z) is defined by (7) and r_0 is given by the smallest positive root of the equation.

$$\frac{\frac{2}{\lambda}-1-r}{1+r}-\frac{2}{\lambda}(\frac{1}{\lambda}-1)\int_0^1\frac{\xi^{\frac{1}{\lambda}-1}}{1+\xi r}d\xi=0. \quad (9)$$

This result is sharp.

Proof. Since f ∈ T(h, ϕ, λ),we may write , by using Remark 2.

$$f(z)=zF'(z), \qquad F \in \mathrm{R}(h,\phi,\lambda)$$
$$=z(k(z)*J(z))'$$
$$=zp(z)*k(z).$$

Where p ∝ hand k(z)is given by (7).

Hence

$$f'(z)=\frac{zp(z)*zk'(z)}{z} = \frac{zp(z)*zk'(z)}{z*zk'(z)} \quad (10)$$

Let zk'(z)=H(z)

Then H'(z)=k'(z) + zk''(z).

It is easy to see that k'(0)=1.Therefore, for $\Re k'(z) > ½$ for $|z| < r_0$,we have

$$\Re\frac{H(z)}{zH'(z)}>\frac{1}{2}.$$

In $|z|= r_0$.

Hence H is a prestarlike function of order σ =1. Also, since g(z)=z ∈ S_1, we can apply Lemma 1 and it follows f ∈ T(h, ϕ, λ) for $|z|< r_0$.

The function $f_0 \in T(h,\phi,\lambda)$, defined as

$$f_0(z)=zh(z)*k(z).$$

Shows that the above radius given by (9) is sharp. To find the radius r_0. We proceed as follows.

From (7), we have

$$k'(z)=\frac{1}{\lambda(1-z)}-\frac{1}{\lambda}(\frac{1}{\lambda}-1)z^{-\frac{1}{\lambda}}\int_0^z\frac{t^{\frac{1}{\lambda}-1}}{1-t}dt. \quad (11)$$

Power in (11) are meant as principal values. The function k'(z) is analytic in E, k'(0)=1 and

$$2k'(z)-1=\frac{2-\lambda+\lambda_z}{\lambda(1-z)}-\frac{2}{\lambda}\left(\frac{1}{\lambda}-1\right)z^{-\frac{1}{\lambda}}\int_0^z\frac{t^{\frac{1}{\lambda}-1}}{1-t}dt.$$

So

$$2\Re k'(z)-1\geq\frac{\frac{2}{\lambda}-1-r}{1+r}-\frac{2}{\lambda}\left(\frac{1}{\lambda}-1\right)\int_0^1\frac{\xi^{\frac{1}{\lambda}-1}}{1+\xi r}d\xi.$$

Therefore $\Re k'(z) > \frac{1}{2}$ for $|z| < r_0$, where r_0 is the smallest positive root of (9).

For the function $f_0(z)=h(z)*k(z),\ f'(r_0)=0$.

This shows that the above result is sharp and proof is complete.

Theorem 8 Let f ∈ R(h, ϕ, 0). Then f ∈ R(h, ϕ, λ) for $|z| < r_\lambda$, where

$$r_\lambda=(1+\lambda^2)^{\frac{1}{2}}-\lambda.$$

Proof. Let $F_1(z) = (f*\phi)(z)$. Then $F_1 \propto h$. Now

$F'_1(z)+(z)+\lambda zF''_1(z) = \psi_\lambda(z)* F'_1(z)$. (12)

Where

$\psi_\lambda(z) = \frac{z(1-(1-\lambda)z)}{(1-z)^2} = z + \sum_{n=2}^{\infty}(1 + (n-1)\lambda)z^n.$

It is known (11) that $\Re\frac{\psi_\lambda(z)}{z} > \frac{1}{2}$ in $|z| < r_\lambda$. Now, from (12) and Lemma 4,it follows that

$$(F'_1 + \lambda z F''_1) \propto h \text{ in } |z| < r_\lambda.$$

This gives us $F_1 \in R(h, \phi, \lambda)$ in $|z| < r_\lambda$, and the proof is complete.

Using Lemma 6,the following result can be easily proved.

Theorem 9 Let

$$F = (1-\lambda)(f*\phi) + \lambda z(f*\phi)', \; f,\phi \in A, \lambda \geq 0.$$

And

$$G = (1-\lambda)(g*\phi) + \lambda z(g*\phi)', \; g \in S(h,\lambda,\phi),$$

Then

$$\frac{(zF'(z))'}{G'(z)} \propto h(z) \text{ implies } \frac{zF'(z)}{G(z)} \propto h(z) \text{ in } E.$$

We prove the following.

Theorem10 Let $f \in R(h, \phi, \lambda)$, $\Re h > 0$. Then $(f*\phi) \in C(h) for \; |z| < (\sqrt{2}-1)$.

This result is sharp.

Proof. Since $f \in R(h, \phi, \lambda)$, $h \in P$, we have

$$(f*\phi)'(z) = k(z) * \int_0^z h(t)dt, \quad h(z) \propto \frac{1+z}{1-z}$$

And k(z) given by (7) is convex function in E. If show that

$$J(z) = \int_0^z h(t)dt$$

.is convex for $|z| < (\sqrt{2}-1)$, then $(f*\phi) = k*J$ is also convex for $(\sqrt{2}-1)$ due to a well known result, see [27]. Now $J'(z) = h(z)$, and

$$1 + \frac{zJ''(z)}{J'(z)} = 1 + \frac{zh'(z)}{h(z)}, \qquad h \propto \frac{1+z}{1-z}$$

Then

$$\Re[1 + \frac{zJ''(z)}{J'(z)} \geq 1 - \frac{2r}{1-r^2}$$

$$= \frac{1-2r-r^2}{1-r^2}$$

Since $|\frac{zh'(z)}{h(z)}| \leq \frac{2r}{1-r^2}$, see (6). Thus $J \in C$ for $|z| < (\sqrt{2}-1)$ and consequently $f \in C(\phi)$in $|z| < (\sqrt{2}-1)$. The sharpness follows from the function $f_1 \in R(h, \phi, \lambda)$ given as

$$(f_1 * \phi)(z) = k(z) * \int_0^z \frac{1+t}{1-t}dt$$

4 Applications

We shall have different choice of analytic functions ϕ and h to illustrate the application of the main results.

I. Choices for h(z)

Let

$$h(z) = \frac{1+Az}{1+Bz}, \qquad A \in C \text{ and } B \in [-1,0], \; A \neq B.$$

For $-1 \leq B < A \leq 1$. These functions are called Janowski functions (6). By taking $A = 1 - 2\alpha$, $B = -1$, $0 \leq \alpha \leq 1$. We have

$$h(z) = h_\alpha(z) = \frac{1+(1-2\alpha)z}{1-z}.$$

This gives us $\Re h_\alpha(z) > \alpha$ and $with$ $\alpha = 0$, we have

$h_0(z) = \frac{1+z}{1-z}$, $\Re h(z) > 0.$ see [6]. (13)

II. For $K \geq 0$, let $h(z) = p_k(z)$, where

$$p_k(z) = \frac{1+z}{1-z}, k = 0$$

$$p_k(z) = 1 + \frac{2}{\pi^2}(\log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2, \quad k = 1$$

$$p_k(z) = 1 + \frac{2}{1-k^2}\sin h^2\left[\left(\frac{2}{\pi}\arccos k\right) arctan\sqrt{(z)}\right], \quad 0 < k < 1,$$

$$p_k(z) = 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{(1-tx)^2}}dx\right) + \frac{1}{k^2-1}, \quad k > 1$$

Here $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, R(t) is Legender's complete elliptic integral of the first kind and R'(t) is the complementary integral of R(t).

The function $p_k(z)$ play the role of extremal functions mapping E onto the conic domain Ω_k given below

$$\Omega_k = \{u + iv : u > k\sqrt{(u-1)^2 + v^2}, k > 0\} \quad (15)$$

For fixed k, Ω_k represents the conic region bounded, successively,by the imaginary axis (k=0),the right branch of hyperbola (0<k<1),a parabola (k=1) and (k>1). It is noted that the functions $p_k(z)$ are univalent in E and belong to the class Pof Caratheodry functions of positive real part. For detail,we refer to [9,10,15,17,18,19,20,21].

Now, by choosing $h(z) = p_k(z)$ in Theorem 3, we can easily prove the following.

Corollary 1. $S(p_k, \phi, \lambda) \subset S(q_k, \phi, 0)$, where

$$q_k(z) = [\int_0^1 exp\int_t^{tz}\frac{p_k(u)-1}{u}du]^{-1}. \quad (16)$$

Some of the special cases are given below.

(i) Let k = 0. Then $f \in S(\frac{1+z}{1-z}, \phi, \lambda)$ implies that $f \in S(\frac{1}{1-z}, \phi, 0)$. That is $\Re\left[\frac{z(f*\phi)'}{f*\phi}\right] > \frac{1}{2}$, for $z \in E$.

(ii) For k>1 and $f \in S(p_k, \phi, \lambda)$, we obtain fom Theorem3 and [18] that $f \in S\left(\frac{z}{(z-k)\log\left(1-\frac{z}{k}\right)}, \phi, 0\right)$. That is

$$\left[\frac{z\{f(z)*\phi(z)\}'}{f(z)*\phi(z)}\right] \propto \frac{z}{(z-k)\log(1-\frac{z}{k}}, z \in E.$$

Since, in this case $q_k(-1) = \frac{1}{(k+1)\log(1+\frac{1}{k}}$, we have

$$\Re\left[\frac{z(f(z)*\phi(z))'}{f(z)*\phi(z)}\right] > \frac{1}{(k+1)\log(1+\frac{1}{k})}.$$

(iii) For the case k=2. We note that

$$S(p_2, \phi, \lambda) \subset S(q_2, \phi, \lambda)$$

This gives us

$$\Re\left\{\frac{z(f(z)*\phi(z))'}{f(z)*\phi(z)}\right\} > q_2(-1) = \frac{1}{3\log\frac{3}{2}} \approx 0.813.$$

(iv) Let k=1,Then

$$S\left(\left[1+\frac{2}{\pi^2}(log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2\right],\phi,\lambda\right)\subset S(q_1,\phi,0).$$

And

$$\Re\frac{z(f(z)*\phi(z))'}{f(z)*\phi(z)}> q_2(-1)=\frac{1}{2}.$$

Corollary 2. Let

$$h(z)=h_\alpha(z)=\frac{1-(1-2\alpha)z}{1-z}$$

Then , from Theorem 1 and result given in [12.p 115], it follows that

$$C(h_\alpha,\phi,\lambda)\subset S(q_\alpha,\phi,\lambda).$$

Where

$$q_\alpha(z)=\frac{(1-2\alpha)z}{(1-z)[1-(1-z)]^{1-2\alpha}},\qquad \text{if } \alpha\neq\frac{1}{2}$$

$$q_\alpha(z)=\frac{z}{(z-1)\log(1-z)},\qquad if\ \alpha=\frac{1}{2}$$

Corollary 3. Let $f\ \in S(h_\alpha,\phi,\lambda$ and $f_2(z)=\frac{2}{z}\int_0^z f(t)dt.$

Then it follows from Remark 2 and a result in [12.p116] that $\in S(H_\pi,\phi,\lambda)$, where

$$H_\alpha(z)=\frac{2\alpha(2\alpha-1)z^2}{(1-z)[(1-z)^{1-2\alpha}+(2\alpha-1)z-1]}-1,\quad \alpha\neq\frac{1}{2},\alpha\neq 0,$$

$$H_\alpha(z)=\frac{z^2}{(z-1)[\log(1-z)+z]}-1,\alpha=\frac{1}{2},$$

$$H_\alpha(z)=\frac{z^2}{(1-z)[(1-z)\log(1-z)+z]}-1,\alpha=0.$$

(2) Choice for $\phi(z)$

[2(a)] Consider the operator $D^n(n\ \in\ N_0=\{0,1,2,\dots\})$ which is called the Salagcan derivative operator defined as

$$D^n f(z)= D\left(D^{n-1}f(z)\right)= Z\left(D^{n-1}f(z)\right),$$

With $D^0 f(z)= f(z)$. see [28].

Also one-parameter Jung-Kim-Srivastava integral operator[8,29] is defined as

$$I^\sigma f(z)=\frac{2^\sigma}{z\tau(\sigma)}\int_0^z(log\frac{z}{t})^{\sigma-1}f(t),(\sigma\ real)$$

$$=z+\sum_{m=2}^{\infty}(\frac{z}{m+1})^\sigma a_m z^m, \qquad (17)$$

The operator I^σ is closely related to the multiplier transformation studied by Flett|5|.

We can express

$$D^n f(z)=z+\sum_{m=2}^{\infty}m^n a_m z^m\ , \qquad (18)$$

Where f(z)= $z+\sum_{m=2}^{\infty}m^n a_m z^m$

From (17), the following identity can easily be deduced.

$$z[I^{\sigma+1}f(z)]'=2I^\sigma f(z)-I^{\sigma+1}f(z). \qquad (19)$$

Combining the operators D^n and I^σ , operator

$$I_n^\sigma : A\to A$$

Is defined by taking

$$F_{n,\sigma}(z)=z+\sum_{m=2}^{\infty}m^n(\frac{2}{m+1})^\sigma z^m.$$

As follows

$$I_n^\sigma f(z) = D^n\big(I^\sigma f(z)\big) = I^\sigma(D^n f(z))$$

$$= F_{n,\sigma}(z) * f(z)$$

$$= Z$$

$$+ \sum_{m=2}^{\infty} m^n(\frac{2}{m+1})^\sigma z^m. \qquad (20)$$

We note that $\mathrm{I}_n^0 f(z) = \mathrm{D}^n f(z), = \mathrm{D}^n f(z),$

$$\mathrm{I}_0^\sigma f(z) = \mathrm{I}^\sigma f(z) \quad .$$

From (20) we easily derive

$$\mathrm{I}_{n+1}^{\sigma+1} f(z) = 2\mathrm{I}_n^\sigma f(z) - \mathrm{I}_n^{\sigma+1} f(z) \qquad (21)$$

Now, taking $\phi(z)=F_{n,\sigma}(z)$, we have:

Theorem11.

$$S(h, F_{n+1}^\sigma, \lambda) \subset S(h, F_n^\sigma, \lambda) \subset S(h, F_n^{\sigma+1}, \lambda)$$

[2(b)]. $\Phi(z) = f_{a,b}(z)$.

In [3],the operator $J_{a,b}$ is defined as $J_{a,b} : A \to A$ by

$$J_{a,b}\, f(z) = f_{a,b}(z) * f(z)\,, \qquad (a>0,b>0),$$

Where $\frac{z}{(1-z)^a} * f_{a,b}(z) = \frac{z}{(1-z)^b}$ see also [14].

We note that, by taking a= n+1, n $\in$ N and b=2. We obtain the operator considered by Noor [16,19]. Also $J_{a,b} = L(b,a)$ is Carlson-Shaffer operator introduced in [2] as follows.

$$L(b,a) f(z) = \Psi(b,a,z) * f(z),$$

Where

$$\Psi(b,a,z) = \sum_{m=0}^{\infty} \frac{(b)_m}{(a)_m} z^{m+1} \quad , \qquad a \neq 0, -1 \ldots$$

Is an incomplete beta function related to the Gauss hypergeometric function by

$$\Psi(b,a;z) = z\,{}_2F_1(1,b;a;z) \text{ and } (b)_m = b(b+1)\ldots(b+m-1), (b)_0 = 1.$$

We note that , by taking a = n+1 , n $\in$ N and b = 2, we obtain the operator considered in [15,16],and

$$J_{1,n+1} f(z) = L(n+1,1) f(z)$$

$$= D^n f(z)$$

$$= \frac{z(z^{n-1} f(z))^{(n)}}{n!}$$

The Rusheweyh derivative of order n.

The following identities hold for a>0,b>0

$$z\big(J_{a,b} f\big)' = aJ_{a,b} f - (a-1) J_{a+1,b} f,$$

$$z\big(J_{a,b} f\big)' = bJ_{a,b+1} - (b-1) J_{a,b}$$

Using [22] and some computations,we have:

Corollary 4. (i). For $b \geq 1$.

$$S\big(h, f_{a,b+1}, \lambda\big) \subset S\big(h, f_{a,b}, \lambda\big)\,, z \in E.$$

(ii). For $0 \leq \delta \geq 1$, $b \geq 1$.

$$S\left(\frac{1+(1-2\delta)z}{1-z}, f_{a,b+1}, \lambda\right) \subset S\left(\frac{1+(1-2\beta)z}{1-z}, f_{a,b}, \lambda\right)$$

Where

$$\beta = \frac{1}{4}\{-(2b-2\delta-1) + \sqrt{(2b-2\delta-1)^2 + 8(2\delta b - 2\delta + 1)}\}$$

The result (ii) has been established in [15].

As a special case of Corollary 4, we deduce that ,for λ = 0, a = b = 1,

$$S\left(\frac{1+(1-2\delta)z}{1-z}, f_{1,2}, 0\right) \subset S\left(\frac{1+(1-2\beta_1)z}{1-z}, f_{1,1}, 0\right)$$

Where β_1 is given by (22) with b= 1. That is C(δ) ⊂ S(β$_1$).

$$\beta_1 = \frac{1}{4}\{(2\delta-1) + \sqrt{4\delta^2-4\delta+9}\}. \quad (22)$$

For δ = 0 ,we obtain a well known result that every convex function is starlike of order $\frac{1}{2}$. For more results related to Corollary 4.we refer to [14].

[2(c)] $\phi(z) = f^s_{\gamma,\mu}(z), f(z) = z + \sum_{m=2}^{\infty} a_m z^m$.

Define $f^s_{\gamma,\mu}(z)$ as $f^s_{\gamma}(z) * f^s_{\gamma,\mu}(z) = \frac{z}{(1-z)^{\mu}}$, $(\mu > 0, z \in E)$. (23)

Where $f^s_{\gamma}(z) = z + \sum_{m=2}^{\infty}(\frac{m+\gamma}{1-\gamma})^s z^m$, (γ>-1).

Then, using (24), the operator $L^s_{\gamma,\mu} : A \to A$ is introduced as

$$L^s_{\gamma,\mu} f(z) = f^s_{\gamma,\mu}(z) * f(z), (f \in A; s \in \mathbb{R}; \lambda > -1, \mu > 0), (24)$$

We note that

$$L^0_{0,z} f(z) = zf'(z), L^1_{0,2} f(z) = f(z),$$

$$L^{-1}_{1,1} f(z) = z + \sum_{m=2}^{\infty} \frac{1}{m} a_m z^m = \int_0^z \frac{f(t)}{t} dt ,$$

And obtain the following relation

$$z\left(L^s_{\gamma,\mu} f(z)\right)' = \mu L^s_{\gamma,\mu+1} f(z) - (\mu - 1) L^s_{\gamma,\mu} f(z) \quad (25)$$

$$z\left(L^{s+1}_{\gamma,\mu} f(z)\right)' = (\gamma+1) L^s_{\gamma,\mu} f(z) - (\gamma) L^{s+1}_{\gamma,\mu} f(z) \quad (26)$$

We can now derive the following results easily

Corollary 5.

$S(p_k, f^s_{\gamma,\mu}, \lambda) \subset S(\frac{1-(1-2p)z}{1-z} , f^{s+1}_{\gamma,\mu}, \lambda)$

Where

$$p = \frac{2(1+2p_0\gamma)}{[1-2(\gamma-p_0)]+\sqrt{([1+2(\gamma-p_0)]^2+8(1+2p_0\gamma)}}. \quad and \ p_0 = \frac{k}{k+1}$$

For this result we refer to [17,18].

Corollary 6. Let f$_3$(z) be defined as in Remark 1, with f ∈ R(p$_k$, $f^s_{\gamma,m\mu}, \lambda$). Then f$_3$ ∈ R(q$_k$, $f^s_{\gamma,\mu}, \lambda$) in E.

Proof. The operator defined by f$_3$ is known as Bernardi integral operator for b$_1$=1,2,3…..see[1]. We have

$$f_3(z) = \frac{b_1+1}{z^{b_1}} \int_0^z t^{b_1-1} f(t)dt , \quad b_1 > -1, f \in \mathrm{R}(q_k, f^s_{\gamma,\mu}, \lambda).$$

Then

$$(b_1+1)f(z) = zf'_3(z) + b_1 f_3(z). \quad (27)$$

Now writing

$$h(z) = (1-\lambda)\left(L^s_{\gamma,\mu} f_3(z)\right)' + \lambda\left[z\left(L^s_{\gamma,\mu} f_3(z)\right)'\right]'.$$

We obtain from (27).

$$(1-\lambda)\left(L^s_{\gamma,\mu} f_3(z)\right)' + \lambda(z\left(L^s_{\gamma,\mu} f_3(z)\right)')' = h(z) + z\frac{1}{b_1+1}h'(z).$$

Since f ∈ R(p_k, $f^{s}_{\gamma,\mu} f(z), \lambda$), it follows that

$$\left(h + \frac{1}{b_1 + 1} zh'\right) \propto p_k, \qquad z \in E.$$

Applying Lemma 5,we have, for z ∈ E

$$h(z) \propto \widetilde{q_k}(z) \propto p_k(z).$$

Where

$$\widetilde{q_k}(z) = \frac{b_1 + 1}{z^{b_1}} \int_0^z t^{b_1} p_k(t) dt.$$

There h $\propto \widetilde{q_k}$ and consequently $f_3 \in$ R$\left(\widetilde{q_k}, f^{s}_{\gamma,\mu}, \lambda\right)$.

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