Distance-2 Chromatic Number for the Duplicate Graph of the Mycielskian Graphs

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Abstract- *Graph colouring is one of the most important area of research in graph theory. A distance-2 colouring of a graph G is a proper vertex colouring of G such that every two vertices at a distance-2 or less are assigned different colours. The least integer k for which there is a k-colouring satisfying this condition is the distance-2 chromatic number of G and is denoted by* $\chi^2(G)$. *In this paper, we present algorithms to determine the distance-2 chromatic number for the duplicate graph of the Mycielskian graph of paths, cycles and complete graphs.*

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Keywords: *Distance-2 colouring, Distance-2 chromatic number, Duplicate graph, Mycielskian graph.*

1. INTODUCTION

Let G = (V, E) be a simple, finite and undirected graphs. Let Δ(G) denote the maximum degree of vertices of a graph G. For vertices u and v in a graph G, the distance d(u,v) between u and v is the length of a shortest u-v path in G. In [3,4] an L-distance colouring of a graph G is defined as a proper vertex colouring of G such that every two vertices at distance L or less are assigned different colours. The least integer k for which there is a k-colouring satisfying this condition is the L-distance chromatic number of G. Borodin, Invova and Neustroeva [1] have studied sparse planar graphs and they proved more general results in distance-2 colouring. In 1955, Jan Mycielski [5] has given the construction of Mycielskian graph for the graphs. We consider Mycielskian graphs that are in spired by G.J.Chang, L. Huang and X. Zhu [2]. The concept of extended duplicate graph was introduced by P.P. Ulaganathan, K. Thirusangu and B. Selvam in [7]. In this paper, we present algorithms to determine the distance-2 chromatic number for the duplicate graph of the Mycielskian graph of paths, cycles and complete graphs.

2. PRELIMINARIES

*I*n this section, we present some basic definitions and results which are relevant to this paper.

Definition 2.1 (Colouring):

A (proper) colouring of a graph G is a function
c: V(G) → N having the property that c(u) ≠ c(v) for every pair u,v of adjacent vertices of G. A k-colouring of G uses k colours. The chromatic number $\chi(G)$ is the least positive integer k for which G admits a k-colouring.

Definition 2.2 (Distance-2 colouring)*:*

A distance-2 colouring of a graph G(V,E) is a proper colouring of the vertices such that any two vertices at a distance atmost 2, receive distinct colours and the distance-2 chromatic number χ^2 (G) is the least positive integer k for which G has distance-k colouring.

Definition 2.3 (Mycielskian Graph):

Let G be a graph with m vertices denoted by $v_1, v_2, \ldots, v_m$. The Mycielskian graph $\mu(G)$ is obtained by adding to each v_i, a new vertex u_i such that u_i is adjacent to the neighbors of v_i. Finally add a new vertex w such that w is adjacent to each and every vertex u_i.

Definition 2.4 (Mycielskian graph of Path):

Let $\mu(P_m)$ be the Mycielskian graph of path P_m, where m is the number of vertices in P_m. The vertex set and the edge set of $\mu(P_m)$ are given as follows

V={v_i, u_i,w for $1 \le i \le m$ }

$$E=\{v_i v_{i+1}; v_i u_{i+1}; u_i v_{i+1} / 1 \le i \le m-1\} \cup \{u_i w / 1 \le i \le m\}$$

Clearly $\mu(P_m)$ has 2m+1 vertices and 4m-3 edges, where 'm' is the number of vertices in path P_m.

Definition 2.5 (Mycielskian graph of Cycle):

Let $\mu(C_m)$ be the Mycielskian graph of cycle C_m, where 'm' is the number of vertices in C_m. The vertex set and the edge set of $\mu(C_m)$ are given as follows.

$V=\{v_i,u_i,w$ for $1 \le i \le m\}$

$$E=\{v_i v_{i+1}; v_i u_{i+1}; u_i v_{i+1} / 1 \le i \le m-1\} \cup \{u_i w / 1 \le i \le m\} \cup \{u_1 v_m; v_1 u_m; v_1 v_m\}$$

Clearly $\mu(C_m)$ has 2m+1 vertices and 4m edges, where m is the number of vertices in cycle C_m.

Definition 2.6 (Mycielskian graph of Complete graph):

Let $\mu(K_m)$ be the Mycielskian graph of complete graph K_m, where 'm' is the number of vertices in K_m. The vertex set and the edge set of $\mu(K_m)$ are given as follows.

$V=\{v_i,u_i,w$ for $1 \le i \le m\}$

$$E=\{wu_i / 1 \le i \le m\} \cup \{u_i v_{i+j}; u_{i+j} v_i; v_i v_{i+j} / 1 \le i \le m-1, 1 \le j \le m-i\}$$

Clearly $\mu(K_m)$ has 2m+1 vertices and $\frac{3m^2-m}{2}$ edges, where m is the number of vertices in complete graph K_m.

Definition 2.7 (Duplicate graph):

Let G(V, E) be a simple graph. A duplicate graph of G is DG = (V_1, E_1) where the vertex set V_1= V∪V$^/$ and V∩V$^/$=∅ and f : V→V$^/$ is bijective and the edge set E_1 of DG is defined as follows. The edge uv is in E if and only if both uv$^/$ and u$^/$ v are edges in E_1.

Definition 2.8 (Duplicate graph of Mycielskian graph of Path):

Let DGM(P_m) be the duplicate graph of the Mycielskian graph of path P_m (m ≥ 2). The vertex set and the edge set of DGM(P_m) are given as follows.

$$V=\{w, v_i, u_i, w^/, v_i^/, u_i^/ \ /1 \le i \le m\}$$

and

$$E=\{wu_i^/, w^/ u_i / 1 \le i \le m\} \cup \{u_i v^/_{i+1}; u_i^/ v_{i+1}; u_{i+1} v_i^/; u^/_{i+1} v_i; v_i v^/_{i+1}; v_i^/ v_{i+1} / 1 \le i \le m-1\}$$

.

Clearly DGM(P_m) has 4m+2 vertices and 8m-6 edges, where 'm' is the number of vertices in P_m.

Definition 2.9 (Duplicate graph of Mycielskian graph of Cycle):

Let DGM(C_m) be the duplicate graph of the Mycielsikian graph of cycle C_m ($m \ge 3$). The vertex set and the edge set of DGM(C_m) are given as follows.

$$V=\{w, v_i, u_i, w^/, v_i^/, u_i^/ \ /1 \le i \le m\}$$

$$E=\{wu_i^/, w^/ u_i / 1 \le i \le m\} \cup \{u_i v^/_{i+1}; u_i^/ v_{i+1}; u_{i+1} v_i^/; u^/_{i+1} v_i; v_i v^/_{i+1}; v_i^/ v_{i+1} / 1 \le i \le m-1\} \cup \{u_1 v_m^/, u_1^/ v_m, v_1 v_m^/, v_1^/ v_m, v_1 u_m^/, v_1^/ u_m\}$$

Clearly DGM(C_m) has 4m+2 vertices and 8m edges, where m is the number of vertices in cycle C_m, ($m \ge 3$).

Definition 2.10 (Duplicate graph of Mycielskian graph of Complete graph):

Let DGM(K_m) be the duplicate graph of the Mycielskian graph of the complete graph K_m, where 'm' is the number of vertices in K_m ($m \ge 4$). The vertex set and the edge set of DGM(K_m) are given as follows.

$$V=\{w, v_i, u_i, w^/, v_i^/, u_i^/ \ /1 \le i \le m\}$$

$$E=\{wu_i^/, w^/ u_i / 1 \le i \le m\} \cup \{u_i v^/_{i+j}; u_i^/ v_{i+j}; u_{i+j} v_i^/; u^/_{i+j} v_i; v_i v^/_{i+j}; v_i^/ v_{i+j} / 1 \le i \le m-1, 1 \le j \le m-i\}$$

Clearly DGM(K_m) has 4m+2 vertices and $3m^2-m$ edges, where m is the number of vertices in complete graph K_m ($m \ge 4$).

3. Main Results

In this paper, we present algorithms to determine the distance-2 chromatic number for the duplicate graph of the Mycielskian graph of paths, cycles and complete graphs.

Algorithm 3.1.

Procedure: Distance-2 colouring of DGM(P_m), ($m \geq 2$).

Input:

$V \leftarrow \{w, u_1, u_2 \dots u_m, v_1, v_2 \dots v_m, w', u_1', u_2' \dots u_m', v_1', v_2' \dots v_m'\}$ $E \leftarrow \{e_1, e_2 \dots e_{8m-6}\}$

\\assignment of colours to the vertices DGM(P_m), ($m \geq 2$).

$w, u_m \leftarrow c_1$, $w' \leftarrow c_2$
for $1 \leq i \leq m$
{
$u_i' \leftarrow c_{i+1}$
}
end for
for $1 \leq i \leq m-1$
{
$u_i \leftarrow c_{i+2}$
}
end for

if $2 \leq m \leq 4$
$v_m, v_m' \leftarrow c_{m+2}$
$v'_{m-1} \leftarrow c_{m+1}$
for $1 \leq i \leq m-1$
{
$v_i \leftarrow c_{i+1}$
}
end for

if $2 < m \leq 4$
for i= 1to m-2
{
$v_i' \leftarrow c_1$
}

else

if 4<m<7
$v_2', v_3' \leftarrow c_1$
$v_1, v'_{m-1} \leftarrow c_{m+1}$
$v_1', v_{m-1} \leftarrow c_m$
$v_m \leftarrow c_{m-1}$
for $1 \leq i \leq 2$
{
$v_{i+1} \leftarrow c_{4-i}$
}
end for
for i=1 to m-4
{
$v'_{m+2-2i} \leftarrow c_{i+2}$
}
end for

if m=6
$v_4 \rightarrow c_2$

else

if $m \geq 7$
$v_4 \leftarrow c_2, v_4' \leftarrow c_1$
for $3 \leq i \leq m-2$
{
$v_{i+2}, v'_{i+2} \leftarrow c_i$
}
end for
for $i = 1$ *to* 3 *do*
{
$v_i, v_i' \leftarrow c_{m-2+i}$
}

end if

end procedure.

Output: Distance-2 coloured of DGM(P_m), ($m \geq 2$).

Theorem 3.1:

If DGM(P_m) is the duplicate graph of the Mycielskian graph of path P_m ($m \geq 2$), where m is the number of vertices in path P_m, then $\chi^2(DGM(P_m)) \leq m+2$.

Proof.

Let DGM(P_m) be the duplicate graph of the Mycielskian graph of path P_m ($m \geq 2$), where m is the number of vertices in path P_m. Define a function f :V $\rightarrow$ {1,2......4m+2} such that $f(u) \neq f(v)$ if $uv \in E$, where V is the vertex set and E is edge set of DGM(P_m) as follows.

Case (i) if 1< m < 5

First we assign f(w)=c_1 and f(w$^/$)=c_2. Since

deg(w) = deg(w$^/$), $|N(w)| = |N(w^/)| = m$

Hence m+1 colours are needed to colour the vertices N[w]. Since $N(w) \cap N(w^/) = \phi$ and $d(N(w), N(w^/)) > 2$ N(w) and N(w$^/$) receives same colours whereas w and w$^/$ receives different colours among m+1 colours.Using algorithm 3.1, for $1 \leq i \leq m-1$,v_i and $v_i^/$ are coloured by m+1 colours.In this case, $N(v_{m-1}) = \Delta$ and $m \leq |N(v_{m-1})| \leq m+1$. If $|N(V_{m-1})| = m+1$ we need m+2 colours to N[v_{m-1}] and if m= $|N(v_{m-1})|$ we need m+1 colours to colour the vertices of N[v_{m-1}], but d(w$^/$,$v_i^/$)< 2, $f[w^/] \neq f[v_i^/]$ where $1 \leq i \leq m$ we need m+2 th colour to $v_m \in N(v_{m-1}^/)$ and $v_m^/ \in N(v_{m-1})$, f(v_m)= f($v^/_m$) Since d(v_m,$v^/_m$) >2. Hence $\chi^2(DGM(P_m)) \leq m+2$.

Case (ii) if $m \geq 5$

First we assign the maximum degree vertices f(w)=c_1 and f(w$^/$)=c_2. Since deg(w)=deg(w$^/$), $|N(w)| = |N(w^/)| = m$ and $|N(v_i)| = |N(v_i^/)| < m$ where $1 \leq i \leq m$ and $|N(u_i)| = |N(u_i^/)| < m$ Hence m+1 colours are needed to colour N[w]. Since $N(w) \cap N(w^/) = \phi$ and $d(N(w), N(w^/)) > 2$, N(w) and N(w$^/$) receives same colours whereas w and w$^/$ receives different colours among m+1 colours. Using algorithm 3.1 for 1$\leq$ i$\leq$ m-1, v_i and $v_i^/$ are coloured by m+1 colours. Hence $\chi^2(DGM(P_m)) \leq m+2$.

Algorithm 3.2.

Procedure: Distance-2 colouring of DGM(C_m), (m $\geq$ 3).

Input: V←{w, u_1,u_2….u_m, v_1,v_2….v_m , w$^/$, $u_1^/$,$u_2^/$…. $u_m^/$,$v_1^/$,$v_2^/$….$v_m^/$}

E ← {e_1,e_2……....e_{8m}}

\\ assignment of colours to the vertices of DGM(C_m), (m $\geq$ 3).

w, u_m ← c_1 , w$^/$ ← c_2
for i= 1 to m do
{
$u_i^/$ ← c_{i+1}
}
end for
for i= 1to m-1 do
{
u_i ← c_{i+2}
}
end for

if $2 < m < 5$

$v_m, v_m^/ \leftarrow c_{m+3}$

$v_{m-1}, v^/_{m-1} \leftarrow c_{m+2}$

$v^/_{m-2} \leftarrow c_m$

$v_{m-2} \leftarrow c_{m-1}$

else

if $m = 4$

$v_1 \leftarrow c_2, v_1^/ \leftarrow c_5$

else

if $m = 5$

$v_1, v_4^/ \leftarrow c_4, v_2^/ v_3^/ \leftarrow c_1, v_2 \leftarrow c_3$

$v_3, v_1^/ \leftarrow c_{m+1}, v_5, v_5^/ \leftarrow c_{m+2}, v_4 \leftarrow c_2$

else

if $m = 6$

$v_1, v_4^/ \leftarrow c_4,\ v_2^/ v_3^/ \leftarrow c_1,\ v_1^/, v_4 \leftarrow c_5,$
$v_6 \leftarrow c_3\ \ v_3, v_6^/ \leftarrow c_6,\ \ v_2, v_5^/ \leftarrow c_{m+1},$
$v_5 \leftarrow c_2$

else

if $m > 6$

$v_4 \leftarrow c_2, v_4^/ \leftarrow c_1$
for $i = 3$ *to* $m-2$ *do*
{
$v_{i+2}, v^/_{i+2} \leftarrow c_i$
}
end for

$$for\ i = 1\ to\ 3\ do$$
{
$v_i, v_i^{/} \leftarrow c_{m+i-2}$
}

end if
end procedure

Output: Distance-2 coloured of DGM(C_m), (m ≥ 3).

Theorem 3.2.

If DGM(C_m) is the duplicate graph of the Mycielskian graph of cycle C_m (m ≥ 3), where m is the number of vertices in cycle C_m, then $\chi^2(DGM(C_m)) \le m+3$.

Proof:

Let DGM(C_m) be the duplicate graph of the Mycielskian graph of cycle C_m, where m is the number of vertices in cycle C_m. Define a function f :V →{1,2......4m+2} such that $f(u) \ne f(v)$ if $uv \in E$, where V is the vertex set and E is edge set of DGM(C_m) as follows.

Case (i) if 2< m <5

Using algorithm 3.2, f(w)=f(u_m)=c_1,f($w^/$)=c_2, f(v_m)=f($v_m^/$)= c_{m+3}, f(v_{m-1})=f($v^/_{m-1}$)= $c_{m+2,}$ f($v^/_{m-2}$)=c_m, f(v_{m-2})=c_{m-1}, f($u^/_i$)=c_{i+1} for 1≤i≤m, f (u_i)=c_{i+2} for $1 \le i \le m-1$ and if m=4 f(v_1)=$c_{2,}$ f($v^/_1$)=$c_{m+1,}$ Since f(v_m)= f($v^/_m$)= c_{m+3}, we need atmost m+3 colours to colour the vertices v_m & $v^/_m$. Hence $\chi^2(DGM(C_m)) = m+3.$

Case (ii) if m=5

Using algorithm 3.2, f(w)=f(u_m)=c_1,f($w^/$)=c_2, f(v_1)=f($v_4^/$)= c_4, f(v_2)=c_3, f(v_3)= f($v_1^/$)= c_{m+1}, f(v_4)=c_2, f($v_2^/$)=f($v_3^/$)= c_1 and f($u_i^/$)=c_{i+1} for $1 \le i \le m$ and f(u_i)=c_{i+2} for $1 \le i \le m-1$. In this case

$$\left|N(v_{m-1}^{\ /})\right| = \{u_3, u_m, v_3, v_m\},$$

$$v_{m-1}^{\ /} \in c_4, u_3 \in c_m, v_3 \in c_{m+1}, u_m \in c_1$$

and f(v_m)≠c_2Uc_3 ,since $v_2 \in c_3$,$d(v_{4,}v_{m,}) < 2$ and $v_2 \in c_3$,$d(v_{2,}v_{m,}) < 2$. Therefore f(v_m)=$c_{m+2,}$ and f($v_m^/$)=$c_{m+2,}$ since $d(v_{m,}, v_m^/)$>2. Hence $\chi^2(DGM(C_m)) \le m+3.$

Case (iii) if $m \ge 5$

First we assign the maximum degree vertices f(w)=c_1 and f($w^/$)=c_2. Since deg(w)=deg($w^/$), $|N(w)| = |N(w^/)| = m$ and $|N(v_i)| = |N(v_i^/)| < m$ where $1 \le i \le m$ and $v_4 \in c_2$ .Hence m+1 colours are needed to colour N[w]. Since $N(w) \cap N(w^/) = \phi$ and $d(N(w), N(w^/)) > 2$, N(w) and N($w^/$) receives same colours whereas w and $w^/$ receives different colours among m+1 colours. Using algorithm 3.2, v_i and $v_i^/$ are coloured by m+1 colours for 1≤ i ≤ m . Hence $\chi^2(DGM(C_m)) \le m+3.$

Algorithm 3.3.

Procedure: Distance-2 colouring of DGM(K_m), m ≥ 4.

Input: $V \leftarrow \{w, u_1, u_2, ..., u_m, v_1, v_2, ..v_m, w', u_1', u_2', .., u_m', v_1', v_2', ..., v_m'\}$

$E \leftarrow \{e_1, e_2, ..., e_{3m^2-m}\}$

if m ≥ 4

$w, u_1, v_1' \leftarrow c_1$

$w', v_1 \leftarrow c_2$
for i = 1 to m do
{
$u_i' \leftarrow c_{i+1}$
}
end for
for i = 1 to m–1 do
{
$u_{i+1} \leftarrow c_{i+2}$
$v_{i+1}, v_{i+1}' \leftarrow c_{m+1+i}$
}

end if
end procedure.

Theorem 3.3.

Let DGM(K_m) be the duplicate graph of the Complete graph K_m, where m is the number of vertices in complete graph, then $\chi^2(DGM(K_m)) = \Delta + 2.$

Proof.

Let DGM(K_m) be the duplicate graph of the Complete graph K_m, where m is the number of vertices in complete graph. Define a function f : V → {1, 2, …} such that f(u) ≠ f(v) if uv ∈ E, where

V is the vertex set and E is the edge set of DGM(K_m) as follows. From the structure of DGM(K_m), $|N(v_i)| = |N(v'_i)| = \Delta$ and $|N(w)| = |N(w')| = |N(u_i)| = |N(u'_i)| < \Delta$.

Using algorithm 3.3, $f(w) = f(u_1) = f(v'_1) = c_1$, $f(w') = f(v_1) = c_2$ and $f(u'_i) = c_{i+1}$ $1 \le i \le m$ and $f(u_{i+1}) = c_{i+2}$ where $1 \le i \le m-1$ and $f(v_{i+1}) = f(v'_{i+1}) = c_{m+1+i}$ for $1 \le i \le m-3$. Since $|N(v_{m-1})| = \Delta$, hence $\Delta+1$ colours are neede to colour $N[v_{m-1}]$, $f(v_m) = f(N(v_{m-1}))$ and $d(v_m, v_{m-1}) \le 2$. Hence $\Delta+2$ colours are needed to colour $N[v_m]$ and $d(v_{m-1}, v'_{m-1}) > 2$, $f(v_{m-1}) = f(v'_{m-1})$ and $d(v_m, v'_m) > 2$, $f(v_m) = f(v'_m)$ .Hence $\chi^2(DGM(K_m)) = \Delta + 2.$

Conclusion

In this paper, we presented algorithms and determined the distance-2 chromatic number for the duplicate graph of the Mycielskian graph of paths, cycles and complete graphs.

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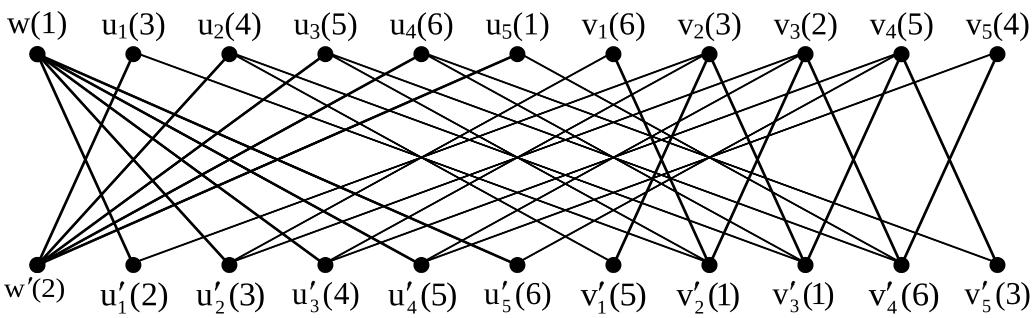
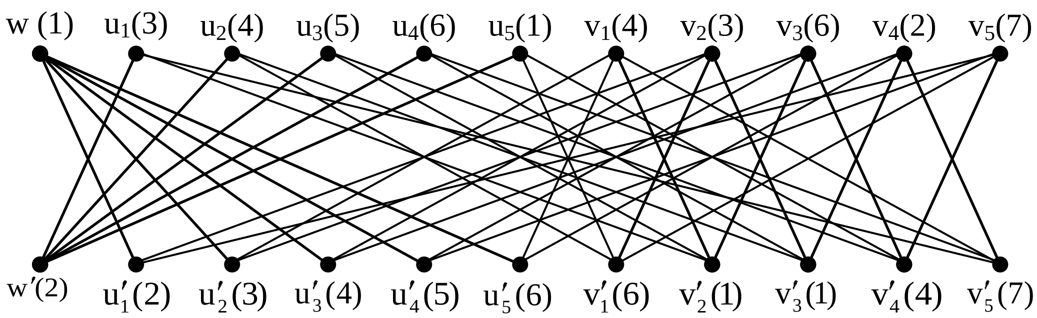
Example: Distance- 2 colored of DGM(P_5), ($m \ge 2$).



Fig(i) $\chi^2(\mu(P_5)) = 6$

Example: Distance- 2 colored of DGM(C_5),



Fig(ii) $\chi^2(\mu(C_5)) = 7$