On Embedding of Every Finite Group into a Group of Automorphism

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***Abstract*— In this Article, We have proved that every group of finite order can be embedded into a group of automorphism. We have used the famous classic result of cayley, which states that every group can be embedded into a group of permutations.**

Keywords— *Embedding, Finite Group, Group of Automorphism,* $Aut\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\cdots n\ copies\right)$

Notation:

$Aut(\mathbb{Z}^n) = Aut\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\cdots n\ copies\right)$ is group of automorphisms with respect to composition of function.

$$GL(n,\mathbb{Z}) = \left\{ \ [a_{ij}]_{n\times n} \ : a_{ij} \in \mathbb{Z} \ \& \ \det([a_{ij}]_{n\times n}) = \pm 1 \ \right\}$$

Theorem 1: (***Cayley's***) Every finite group can be embedded in S_n for some $n \in N$.

Theorem 2: $GL(n,\mathbb{Z})$ is isomorphic to $Aut\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\cdots n\ copies\right)$.

Theorem 3: S_n can be embedded in $GL(n,\mathbb{Z})$ for all $n \in N$.

Main Theorem: Using theorem 1, 2 &3 we can say that every finite group can be embedded into group $Aut\left(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\cdots n\ copies\right)$ for some $n \in N$.

Proof of Theorem 2:

Let $f, g \in Aut(\mathbb{Z}^n)$

Then

$$f(1,0,0,0....) = (a_{11}, a_{12}, ...a_{1n})$$

$$f(0,1,0,0....) = (a_{21}, a_{21}, ...a_{2n})$$

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$$f(0,0,0,....,1) = (a_{n1}, a_{n2}, ...,a_{nn})$$

Now define $\varphi : Aut(\mathbb{Z}^n) \longrightarrow GL(n,\mathbb{Z})$

Such that

$$\varphi(f) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$\Rightarrow f$ is associated with a $n \times n$ matrix with determinant ± 1.

If f is an automorphism then f has an inverse automorphism which is multiplication by

$$N = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

Such that $MN = I$.

The determinants are integers satisfying $\det(M).\det(M^{-1}) = 1$

$\Rightarrow \det(M) = \pm 1$

$\Rightarrow$ There is a bijection between $GL(n,\mathbb{Z})$ and $Aut(\mathbb{Z}^n)$.

Composition of automorphisms corresponds to multiplication of matrices.

So, it is an isomorphism.

$\Rightarrow Aut(\mathbb{Z}^n)$ is isomorphic to $GL(n,\mathbb{Z})$.

Proof of Theorem 3:

Let S_n be the permutation group on n symbols.

Define $\varphi : S_n \longrightarrow GL\left(n,\mathbb{Z}\right)$ such that:

$$\varphi(\sigma) = \left[\sigma\right]_{n\times n} \ \forall \sigma \in S_n$$

Where $\left[\sigma\right]_{n\times n}$ is permutation matrix obtained by σ .

i.e. if $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix}$ then

$$\left[\sigma\right]_{n\times n} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Where R_i is $R^{th}_{\beta_i}$ row of identity matrix.

Clearly φ is a homomorphism.

Now consider the kernel of this homomorphism.

$$\ker\varphi = \{\sigma : \varphi(\sigma) = I_{n\times n}\}$$

$$\Rightarrow i = \beta_i \ \ \forall i$$

$\Rightarrow \ker\varphi$ is trivial.

Hence the homomorphism is injective.

$\Rightarrow$: S_n can be embedded in $GL(n,\mathbb{Z})$ for all $n \in N$.

Proof of Main Theorem:

Since every finite group can be embedded in S_n as stated in theorem 1 & by theorem 3 we can say that S_n can be embedded in $GL(n,\mathbb{Z})$.

So, every finite group can be embedded in $GL(n,\mathbb{Z})$.

And in theorem 2, we have proved that $GL(n,\mathbb{Z})$ is isomorphic to $Aut(\mathbb{Z}^n)$.

Now, combining all these statements one can easily conclude that every finite group is isomorphic to a subgroup of a group of automorphisms.

CONCLUSION

Every finite group is isomorphic to a subgroup of group of permutations by the famous cayley's theorem.

But we have proved that every finite group is isomorphic to the subgroup of a group of automorphisms i.e. $Aut(\mathbb{Z}^n)$.

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