Homotopy Analysis for laminar thermal boundary layer over a flat plate with a convective surface boundary condition

M.S. Abdelmeguid*
*Mechanical Engineering Department
Akhbar El-Yom Academy, 6 October City, Giza 12573, Egypt

Abstract — *In this paper, we will study the resulting thermal similarity equation for laminar flow and heat transfer between two separated fluids for various Prandtl numbers and a range of values characterizing the hot fluid convection process. The problem of hydrodynamic and thermal boundary layers over a flat plate in a uniform stream of fluid has been solved analytically using homotopy analysis method (HAM) and numerically using Matlab bvp4c numerical routine. Velocity and temperature distributions were numerically discussed and presented in the graphs.Convergence of the HAM solution is checked. The effects of various Prandtl numbers and a range of values of the parameter characterizing the hot fluid convection process for similarity energy equation are considered. The temperature and heat transfer characteristics of the Blasius flow have been investigated if the convective heat transfer of the fluid heating the plate on its lower surface is proportional to* $x^{-\frac{1}{2}}$. *The comparison between analytical and numerical results has an excellent agreement with previously published works.*

Keywords — *Thermal; HAM; Convective; Boundary layer*

I. Introduction

The problem of a similarity solution for the laminar flow and heat transfer between two separated fluids has been attracted a lot of attention recently. Aziz [1] has demonstrated that a similarity solution is possible for a convective boundary condition at the plate, where the convective heat transfer of the fluid heating the plate on its lower surface is proportional to $x^{-\frac{1}{2}}$. Shokouhmand H. et al. [2] has described the development of local and non-local similarity solutions for laminar flow and heat transfer between two separated fluids. Magyari E. [3] has been investigated the exact solution for the temperature boundary layer in terms of the solution of the flow problem in a compact integral form.

The homotopy analysis method (HAM) is one of the well-known methods to solve non-linear equations that does not need to any small parameter. This method has been introduced by Liao in 1992 [4-9]. The method has been used by many authors [10–24] in a wide variety of scientific and engineering applications to solve different types of governing differential equations: linear and non-linear, homogeneous and non-homogeneous, and coupled and decoupled as well. This method offers highly accurate successive approximations of the solution. In this paper, we will study the resulting thermal similarity equation for laminar flow and heat transfer between two separated fluids for various Prandtl numbers and a range of values characterizing the hot fluid convection process. The system of nonlinear coupled ordinary differential equations is solved analytically using homotopy analysis method (HAM) and numerically using Matlab bvp4c numerical routine.

2. Problem formulation

Consider the problem of hydrodynamic and thermal boundary layer flows over a flat plate in a stream of cold fluid at temperature T_∞ moving over the top surface of the plate with a uniform velocity U_∞. Assuming steady, incompressible, laminar flow with constant fluid properties and negligible viscous dissipation, and recognizing that $\frac{dp}{dx} = 0$, the boundary layer equations can be written as:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \quad (1)$$

Momentum:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} , \quad (2)$$

Energy:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \propto \frac{\partial^2 T}{\partial y^2} \quad (3)$$

Where u and v are the x (along the plate) and the y (normal to the plate) components of the velocities, respectively, T is the temperature, ν is the kinematic

viscosity of the fluid, and α is the thermal diffusivity of the fluid.
The velocity boundary conditions can be expressed as

$$u(x\,,0) = v(x\,,0) = 0, \tag{4}$$

$$u(x\,,\infty) = \mathrm{U}_\infty. \tag{5}$$

As mentioned before, the bottom surface of the plate is heated by convection from a hot fluid at temperature T_f which provides a heat transfer coefficient h_f. The boundary conditions at the plate surface and far into the cold fluid may be written as

$$-k\,\frac{\partial \mathrm{T}}{\partial y}(x\,,0) = \mathrm{h}_f\left[\,\mathrm{T}_f - \mathrm{T}(x\,,0)\,\right], \tag{6}$$

$$\mathrm{T}(x\,,\infty) = \mathrm{T}_\infty. \tag{7}$$

A similarity solution of Eqs. (1) – (5) is obtained by defining an independent variable η and a dependent variable f in terms of the stream function ψ as

$$\eta = y\left(\frac{\mathrm{U}_\infty}{\nu\,x}\right)^{1/2}, \tag{8}$$

$$\mathrm{f}(\eta) = \frac{\psi}{\mathrm{U}_\infty\sqrt{\nu\,x/\mathrm{U}_\infty}}. \tag{9}$$

Similarly defining a dimensionless temperature θ as

$$\theta(\eta) = \frac{\mathrm{T}-\mathrm{T}_\infty}{\mathrm{T}_f-\mathrm{T}_\infty}. \tag{10}$$

Eqs. (1) – (3) reduce to:

$$\mathrm{f}''' + \frac{1}{2}\,\mathrm{f}\,\mathrm{f}'' = 0\,, \tag{11}$$

$$\theta'' + \frac{1}{2}\,\mathrm{Pr}\,\mathrm{f}\,\theta' = 0\,. \tag{12}$$

Here the primes denote differentiation of f with respect to η .
The boundary conditions in terms of the similarity variables are:

$$\mathrm{f}(0) = \mathrm{f}'(0) = 0\,, \tag{13}$$

$$\mathrm{f}'(\infty) = 1\,, \tag{14}$$

$$\theta'(0) = -\,a\left(1-\theta(0)\right), \tag{15}$$

$$\theta(\infty) = 0\,, \tag{16}$$

where

$$a = \frac{\mathrm{h}_f}{k}\sqrt{\nu\,x/\mathrm{U}_\infty}\,. \tag{17}$$

For the energy equation to have a similarity solution, the quantity a must be a constant and not a function of x as in Eq. (17). This condition can be met if the heat transfer coefficient h_f is proportional to $x^{-1/2}$. We therefore assume

$$\mathrm{h}_f = \mathrm{c}\,x^{-1/2}, \tag{18}$$

Where c is a constant. With the introduction of Eq. (18) into Eq. (17), we have

$$a = \frac{\mathrm{c}}{k}\sqrt{\nu/\mathrm{U}_\infty}\,. \tag{19}$$

With a defined by Eq. (19), the solutions of Eqs. (11) – (15) yield the similarity solutions. With a defined by Eq. (17), the solutions generated are the local similarity solutions.

3. Homotopy analysis solution

3.1. Zero-order deformation equations

Solving Eqs. (11)–(16) using HAM [25,26–29]. From the boundary conditions (13) – (16), it is obvious to choose:

$$\mathrm{f}_o(\eta) = \eta - 1 + e^{-\eta}, \tag{20}$$

$$\theta_o(\eta) = \frac{a}{a+\mathrm{Pr}}\,e^{-\mathrm{Pr}\,\eta}, \tag{21}$$

as the initial approximations of f(η) and θ(η), respectively, and to choose:

$$\mathrm{L}_f[\,\mathrm{f}(\eta\,;\mathrm{q})\,] = \frac{\partial^3\Phi(\eta\,;\mathrm{q})}{\partial\eta^3} + \frac{\partial^2\Phi(\eta\,;\mathrm{q})}{\partial\eta^2}, \tag{22}$$

$$\mathrm{L}_\theta[\,\square(\eta\,;\mathrm{q})\,] = \frac{\partial^2\Theta(\eta\,;\mathrm{q})}{\partial\eta^2} + \frac{\partial\Theta(\eta\,;\mathrm{q})}{\partial\eta}, \tag{23}$$

as the auxiliary linear operators, which have the following properties:

$$\mathrm{L}_f\,[\,c_1 + c_2\,\eta + c_3\,e^{-\eta}\,] = 0\,,$$

$$\mathrm{L}_\theta\,[\,c_4 + c_5\,e^{-\eta}\,] = 0. \tag{24}$$

where c_i $(i = 1-5)$ are arbitrary constants.
Based on (11) and (12), This paper is led to define the non-linear operators:

$$N_f[\Phi(\eta;q)] = \frac{\partial^3\Phi(\eta;q)}{\partial\eta^3} + \frac{1}{2}\Phi(\eta;q)\,\frac{\partial^2\Phi(\eta;q)}{\partial\eta^2}, \tag{25}$$

$$N_\theta[\Theta(\eta;q)] = \frac{\partial^2\Theta(\eta;q)}{\partial\eta^2} + \frac{1}{2}\,\mathrm{Pr}\left[\Phi(\eta;q)\,\frac{\partial\Theta(\eta;q)}{\partial\eta}\right]. \tag{26}$$

Let h denote the non-zero auxiliary parameter. Then construct the zeroth-order deformation equations:

$$(1-q)\,L_f[\Phi(\eta;q) - f_o(\eta)] = q\,h\;H_f(\eta)\,N_f[\Phi(\eta;q)] \tag{27}$$

$$(1-q)\,L_\theta[\Theta(\eta;q) - \theta_o(\eta)] = q\,h\;H_\theta(\eta)\,N_\theta[\Theta(\eta;q)] \tag{28}$$

Subject to the boundary conditions:

$$\Phi(0;q) = 0\;,\quad \left.\frac{\partial\Phi(\eta;q)}{\partial\eta}\right|_{\eta=0} = 0\;, \tag{29}$$

$$\Theta(0;q) = \frac{a}{a+\mathrm{Pr}}\;,$$

$$\left.\frac{\partial\Theta(\eta;q)}{\partial\eta}\right|_{\eta=0} = -\,a\,(1-\Theta(0;q)) = \frac{-\,a\,\mathrm{Pr}}{a+\mathrm{Pr}}\,. \tag{30}$$

$$\left.\frac{\partial\Phi(\eta;q)}{\partial\eta}\right|_{\eta=\infty} = 1\;,\quad \Theta(\infty;q) = 0\,. \tag{31}$$

where $q \in [\,0\,,1\,]$ is an embedding parameter. When $q = 0$, it is straightforward that:

$$\Phi(\eta;0) = f_o(\eta)\,,\quad \Theta(\eta;0) = \theta_o(\eta)\,. \tag{32}$$

When $q = 1$ the zeroth-order deformation equations (27)–(31) are equivalent to the original equations (11)–(16), so that we have:

$$\Phi(\eta;q) = f(\eta)\;,\quad \Theta(\eta;q) = \theta(\eta)\,, \tag{33}$$

respectively. Thus as q increases from 0 to 1, $\Phi(\eta;q)$ and $\Theta(\eta;q)$ vary from the initial guess $f_o(\eta)$ and $\theta_o(\eta)$ to the solutions $f(\eta)$ and $\theta(\eta)$ of the problem, respectively. So expanding $\Phi(\eta;q)$ and $\Theta(\eta;q)$ in Taylor's series about the embedding parameter q, we have:

$$\Phi(\eta;q) = \Phi(\eta;0) + \sum_{m=1}^{\infty} f_m(\eta)\;q^m\;, \tag{34}$$

$$\Theta(\eta;q) = \Theta(\eta;0) + \sum_{m=1}^{\infty} \theta_m(\eta)\;q^m\;, \tag{35}$$

where:

$$f_m(\eta) = \frac{1}{m!}\,\left.\frac{\partial^m\Phi(\eta;q)}{\partial q^m}\right|_{q=0}\,, \tag{36}$$

$$\theta_m(\eta) = \frac{1}{m!}\,\left.\frac{\partial^m\Theta(\eta;q)}{\partial q^m}\right|_{q=0}\,. \tag{37}$$

If h is properly chosen, the series (34) and (35) are convergent at $q = 1$, we have, using (32) and (33), the solution series:

$$f(\eta) = f_o(\eta) + \sum_{m=1}^{\infty} f_m(\eta)\;, \tag{38}$$

$$\theta(\eta) = \theta_o(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)\;. \tag{39}$$

3.2 Higher order deformation equations

Differentiating the zero-order deformation equations (27) and (28) m times about q, then setting $q = 0$, and finally dividing them by $m!$, we obtain the mth-order deformation equations:

$$L_f[\,f_m(\eta) - \chi_m\,f_{m-1}(\eta)\,] = h\;H_f(\eta)\;R_m(\eta)\,, \tag{40}$$

$$L_\theta[\,\theta_m(\eta) - \chi_m\,\theta_{m-1}(\eta)\,] = h\;H_\theta(\eta)\;S_m(\eta)\,, \tag{41}$$

subject to the boundary conditions:

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0\,, \tag{42}$$

$$\theta_m(0) = \theta_m(\infty) = 0\,, \tag{43}$$

where

$$\chi_m = \begin{cases} 0\;; & m \le 1 \\ 1\;; & m \ge 2 \end{cases} \tag{44}$$

and

$$R_m(\eta) = f'''_{m-1}(\eta) + \frac{1}{2}\sum_{k=0}^{m-1} f_k(\eta)\,f''_{m-1-k}(\eta)\,, \tag{45}$$

$$S_m(\eta) = \theta''_{m-1}(\eta) + \frac{1}{2}\,\mathrm{Pr}\sum_{k=0}^{m-1} f_k(\eta)\,\theta'_{m-1-k}(\eta)\,. \tag{46}$$

According to initial approximations and the auxiliary linear operators, we set:

$$H_f(\eta) = e^{-\eta}, \quad H_\theta(\eta) = e^{-\eta}. \tag{47}$$

The first order deformation equations:

$$L_f\,[f_1(\eta)] = h\; H_f(\eta)\; R_1(\eta)\,, \tag{48}$$

$$L_\theta[\,\theta_1(\eta)\,] = h\; H_\theta(\eta)\; S_1(\eta)\,, \tag{49}$$

and the boundary conditions:

$$f_1(0) = f_1'(0) = f_1'(\infty) = 0\,, \tag{50}$$

$$\theta_1(0) = \theta_1(\infty) = 0\,, \tag{51}$$

so that we have:

$$f_1(\eta) = \frac{-7h}{24}\, e^{-\eta} + \frac{h}{8}\, e^{-2\eta} - \frac{h}{36}\, e^{-3\eta} - \frac{h}{8}\,\eta\, e^{-2\eta} + \frac{7h}{36}, \tag{52}$$

$$\theta_1(\eta) = e^{-\eta}\left[-\frac{ah(-2+pr+6pr^2+2pr^3)}{2(1+pr)^2(2+pr)(a+pr)} - \frac{ae^{-(1+pr)\eta}h(pr^2(1+pr)}{2(1+pr)^2(2+pr)(a+pr)} + \frac{e^{\eta}(2+pr)(1+pr^2(-3+\eta)+pr(-1+\eta)))}{2(1+pr)^2(2+pr)(a+pr)}\right]. \tag{53}$$

Similarly, we obtain:

$$f_2(\eta) = e^{-\eta}\Big(-\frac{691h^2}{6912} + \frac{1}{69120}e^{-4\eta}(5999e^{5\eta}h^2 + h^2(96 + 840e^{3\eta}(-5+3\eta) + 15e^{\eta}(-103+42\eta) + 160e^{2\eta}(41-28\eta + 6\eta^2))))\Big) \tag{54}$$

$$\begin{aligned}
\theta_2(\eta) &= e^{-\eta}\Big(\frac{1}{144}\Big(-\frac{9ah^2(-4+pr)(-2+pr+6pr^2+2pr^3)}{(1+pr)^2(2+pr)(a+pr)} - \frac{6ah^2pr(-2+pr+6pr^2+2pr^3)}{(1+pr)^2(2+pr)(a+pr)} \\
&- \frac{1}{(1+pr)^3(a+pr)}ah^2(-14pr(1+pr)^2 + \frac{2pr^2(1+pr)^2(1+19pr)}{(3+pr)(4+pr)} + \frac{9pr(1+pr)^2(24-105pr-216pr^2-132pr^3-25pr^4)}{(2+pr)^2(3+pr)^2} \\
&+ \frac{3(-96-120pr+448pr^2+1188pr^3+1231pr^4+606pr^5+115pr^6)}{(2+pr)^3}\Big)\Big) \\
&+ \frac{1}{144}\Big(\frac{6ae^{-2\eta}h^2pr(-2+pr+6pr^2+2pr^3)}{(1+pr)^2(2+pr)(a+pr)} + \frac{9ae^{-\eta}h^2(-2+pr+6pr^2+2pr^3)(-4+pr+2pr\,\eta)}{(1+pr)^2(2+pr)(a+pr)} \\
&+ \frac{1}{(1+pr)^3(a+pr)}ae^{-(3+pr)\eta}h^2(-14e^{3\eta}pr(1+pr)^2 + \frac{2pr^2(1+pr)^2(1+19pr)}{(3+pr)(4+pr)} \\
&+ 1/((2+pr)^2(3+pr)^2)9e^{\eta}pr(1+pr)^2(24 + 15pr(-7+2\eta) + pr^4(-25+9\eta) + 2pr^3(-66 + 25\eta) + pr^2(-216+79\eta)) \\
&+ 1/(2+pr)^3 3e^{2\eta}(-96 - 24pr(5+2\eta) + 16pr^2(28-18\eta+3\eta^2) + pr^6(115 - 72\eta+12\eta^2) + 6pr^5(101-66\eta+12\eta^2) + 12pr^3(99 - 59\eta+12\eta^2) + pr^4(1231-792\eta+156\eta^2)))))
\end{aligned} \tag{55}$$

Since the solutions $f_3(\eta)$ and $\theta_3(\eta)$ are too long, so they are calculated and shown graphically.

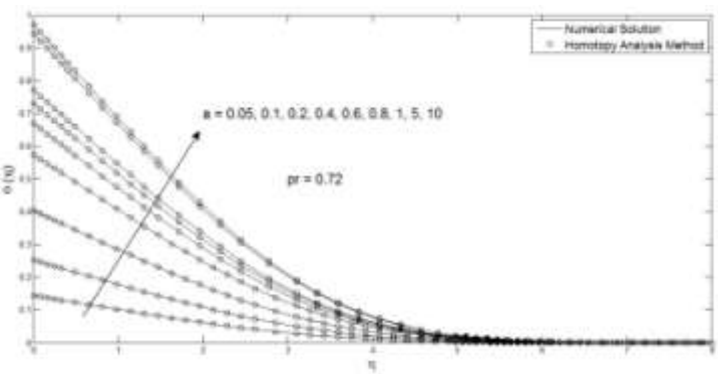

Fig.1 Temperature distribution on a flat plate with convective boundary condition for various values of parameter a .

Table 1 Analytical and numerical solutions for variation of temperature on the flat plate compared with previous work [1, 2, 3]

a	Pr	$\theta(0)$			$-\theta'(0)$		
		Analytical	Numerical	Previous Work	Analytical	Numerical	Previous Work
0.05	0.1	0.253733	0.253733	0.2536	0.0373134	0.0373134	0.0373
0.1	0.1	0.40456	0.40456	0.4046	0.059544	0.059544	0.0594
0.2	0.1	0.576067	0.576067	0.5761	0.0847866	0.0847866	0.848
0.4	0.1	0.731019	0.731019	0.7310	0.107593	0.107593	0.1076
0.6	0.1	0.803018	0.803018	0.8030	0.118189	0.118189	0.1182
0.8	0.1	0.844611	0.844611	0.8446	0.124311	0.124311	0.1243
1	0.1	0.871701	0.871701	0.8717	0.128299	0.128299	0.1283
5	0.1	0.971405	0.971405	0.9714	0.142973	0.142973	0.1430
10	0.1	0.985495	0.985495	0.9855	0.145047	0.145047	0.1450
0.05	0.72	0.145023	0.145023	0.1447	0.0427488	0.0427488	0.0428
0.1	0.72	0.252756	0.252756	0.2528	0.0747244	0.0747244	0.0747
0.2	0.72	0.403523	0.403523	0.4035	0.119295	0.119295	0.1193
0.4	0.72	0.575014	0.575014	0.5750	0.169994	0.169994	0.1700
0.6	0.72	0.669916	0.669916	0.6699	0.198051	0.198051	0.1981
0.8	0.72	0.730170	0.730170	0.7302	0.215864	0.215864	0.2159
1	0.72	0.771822	0.771822	0.7718	0.228178	0.228178	0.2282
5	0.72	0.944174	0.944174	0.9441	0.279131	0.279131	0.2791
10	0.72	0.971285	0.971285	0.9713	0.287146	0.287146	0.2871
0.05	10	0.0657465	0.0657465	0.0643	0.0467127	0.0467127	0.0468
0.1	10	0.120752	0.120752	0.1208	0.0879248	0.0879248	0.0879
0.2	10	0.215484	0.215484	0.2155	0.156903	0.156903	0.1569
0.4	10	0.354565	0.354565	0.3546	0.258174	0.258174	0.2582
0.6	10	0.451759	0.451759	0.4518	0.328945	0.328945	0.3289
0.8	10	0.523512	0.523512	0.5235	0.381191	0.381191	0.3812
1	10	0.578656	0.578656	0.5787	0.421344	0.421344	0.4213
5	10	0.872883	0.872883	0.8729	0.635583	0.635583	0.6356
10	10	0.932128	0.932128	0.9321	0.678721	0.678721	0.6787

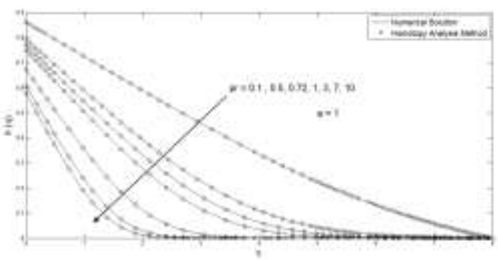

Fig. 2. Plots of the dimensionless temperature profiles $\theta(\eta)$ for the value $a = 1$ of the convective heat transfer parameter and the indicated values of the Prandtl number Pr

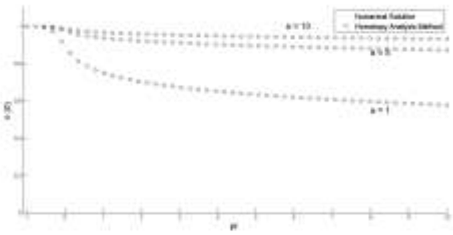

Fig. 3. Plots of the wall temperature $\theta(0)$ as a function of the Prandtl number Pr for the indicated values of the convective heat transfer parameter a.

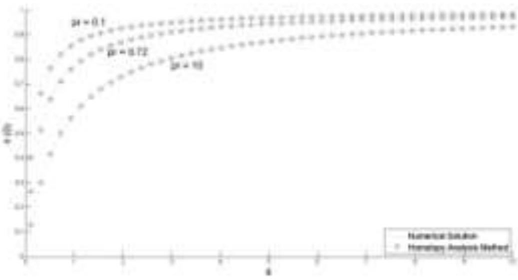

Fig. 4. Plots of the wall temperature $\theta(0)$ as a function of the convective heat transfer parameter a for the indicated values of the Prandtl number Pr.

4. Convergence of the HAM solution

For an analytic solution obtained by the homotopy analysis method, its convergent depends on the auxiliary parameter h. If this parameter is properly chosen, the given solution is valid, as verified in previous works [25, 26–29]. Since the interval for the admissible values of h corresponds to the line segments nearly parallel to the horizontal axis. Then we know that the admissible values for the parameter h is $-1.4 \leq h \leq -0.6$. In this paper we choose $h = -1.2$.

5. Results and discussion

In Table 1, the results of the problem for fixed Prandtl numbers of 0.05, 0.1, 0.2, 0.4, 0.6, 0.8, 1, 5 and 10. For each Prandtl number, both $\theta(0)$ and $\theta'(0)$ increase as a increases.

According to the results, as $a \rightarrow \infty$, the solution approaches the classical solution for the constant surface temperature. This can be seen from the boundary condition equation (15) which reduces to $\theta(0) = 1$ as $a \rightarrow \infty$. It can be observed that these results of previous work, numerical and analytical solutions agree up to four places of decimal.

Fig. 1 compares the Homotopy analysis method with the numerical solutions for a fixed Prandtl number of 0.72 and for a range of values of the parameter a. For each curve, the vertical intercept gives the plate surface temperature. The plate surface temperature increases as a increases.

In Fig. 2 the temperature profiles $\theta(\eta)$ are shown for the value $a = 1$ of the convective heat transfer parameter and seven selected values of the Prandtl number Pr for both numerical and analytical solutions. The larger Pr, the smaller temperature profiles $\theta(\eta)$.

As an illustration of the comparing between analytical and numerical results, in Fig. 3 the wall temperature $\theta(0)$ is plotted as a function of Pr for three selected values of a. The wall temperature $\theta(0)$ increases monotonically with increasing values of a for all specified values of Pr. When $\Pr \longrightarrow 0$

and $\theta(0)$ approaches the value 1 for any a, in a full agreement with Eq. (30).
In fig. 4 the numerical and analytical solutions for the wall temperature $\theta(0)$ is plotted as a function of a for three selected values of Pr. As it is seen in Fig. 4, the wall temperature decreases monotonically with increasing value of Pr for all specified values of a.

The temperature and heat transfer characteristics of the Blasius flow have been investigated if the convective heat transfer of the fluid heating the plate on its lower surface is proportional to $x^{-\frac{1}{2}}$. Numerical and analytical solutions were compared with previous work [1, 2, 3] in order to get an excellent agreement between results.

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