A study of unified results involving a product of generalized Legendre's function, a general class of polynomials, Aleph-function with the multivariable Aleph-function

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Paper dedicated to Professor M.A. Pathan on the occasion of his 75th birthday

ABSTRACT
In this paper an integral involving general class of polynomials, Legendre's associated function, Aleph-function and Aleph-function of several variables has been evaluated and an expansion formula for product of the general class of polynomials, Legendre's associated function, Aleph-function and Aleph-function of several variables has been established with the application of this integral. The result established in this paper are of general nature and hence encompass several particular cases.

Keywords :Multivariable Aleph-function, Aleph-function, Legendre's associated function, general class of polynomials, expansion formula

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1. Introduction and preliminaries.

The Aleph- function , introduced by Südland [9] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z) = \aleph^{M,N}_{P_i,Q_i,c_i;r}\left(z \left| \begin{array}{l} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array} \right.\right) = \frac{1}{2\pi\omega}\int_L \Omega^{M,N}_{P_i,Q_i,c_i;r}(s) z^{-s} \mathrm{d}s \quad (1.1)$$

for all z different to 0 and

$$\Omega^{M,N}_{P_i,Q_i,c_i;r}(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\sum_{i=1}^{r} c_i \prod_{j=N+1}^{P_i}\Gamma(a_{ji} + A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} - B_{ji}s)} \quad (1.2)$$

With :

$$|argz| < \frac{1}{2}\pi\Omega \quad \text{Where } \Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i\Big(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}\Big) > 0 \quad \text{with } i = 1, \cdots, r$$

For convergence conditions and other details of Aleph-function , see Südland et al [9].

Serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph^{M,N}_{P_i,Q_i,c_i;r}(z) = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(s)}{B_G g!} z^{-s} \quad (1.3)$$

With $s = \eta_{G,g} = \dfrac{b_G + g}{B_G}, P_i < Q_i, |z| < 1$ and $\Omega^{M,N}_{P_i,Q_i,c_i;r}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [7], is given in the following manner :

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \quad (1.5)$$

Where $M_1, \cdots, M_s$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_s, K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a = \frac{(-N_1)_{M_1 K_1}}{K_1!} \cdots \frac{(-N_s)_{M_s K_s}}{K_s!} A[N_1, K_1; \cdots; N_s, K_s] \tag{1.6}$$

The Aleph-function of several variables generalize the multivariable h-function defined by H.M. Srivastava and R. Panda [8] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We have : $\aleph(z_1, \cdots, z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array} \middle| \right.$

$$\begin{array}{ll} [(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,\mathfrak{n}}] & ,[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : \\ \cdots\cdots\cdots\cdots\cdots\cdots & ,[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{m+1,q_i}] : \end{array}$$

$$\left. \begin{array}{l} [(c_j^{(1)}), \gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}, \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}]; \cdots ; ; [(c_j^{(r)}), \gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}, \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(d_j^{(1)}), \delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}, \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}]; \cdots ;; [(d_j^{(r)}), \delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}, \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, ds_1 \cdots ds_r \tag{1.7}$$

with $\omega = \sqrt{-1}$

For more details, see Ayant [1].
The reals numbers τ_i are positives for $i = 1, \cdots, R$, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1, \cdots, R^{(k)}$
The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \ , i^{(k)} = 1, \cdots, R^{(k)} \tag{1.8}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.
We may establish the the asymptotic expansion in the following convenient form :

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}) , max(|z_1|, \cdots, |z_r|) \to 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}) , min(|z_1|, \cdots, |z_r|) \to \infty$$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \ ; \ V = m_1, n_1; \cdots ; m_r, n_r \tag{1.9}$$

$$\mathrm{W} = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots , p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.10}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots , \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots , \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.11}$$

$$B = \{\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots , \beta_{ji}^{(r)})_{m+1,q_i}\} \tag{1.12}$$

$$C = \{(c_j^{(1)}; \gamma_j^{(1)})_{1,n_1}\}, \tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)}; \gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\}, \cdots ,\{(c_j^{(r)}; \gamma_j^{(r)})_{1,n_r}\}, \tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)}; \gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \tag{1.13}$$

$$D = \{(d_j^{(1)}; \delta_j^{(1)})_{1,m_1}\}, \tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)}; \delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\}, \cdots, \{(d_j^{(r)}; \delta_j^{(r)})_{1,m_r}\}, \tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)}; \delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \tag{1.14}$$

The multivariable Aleph-function write :

$$\aleph(z_1, \cdots , z_r) = \aleph_{U:W}^{0,\mathrm{n}:V}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} : \mathrm{C} \\ \cdot & \cdot \ \cdot \ \cdot \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \mathrm{B} : \mathrm{D} \end{array}\right) \tag{1.15}$$

2. Generalized Legendre's associated function

Formula 1

The following results are required in our investigation ([3],p.343, Eq(38); [3], p.340, Eq(26) and eq (27))

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_{k-\frac{m-n}{2}}^{m,n}(x)\, \mathrm{d}x = \frac{2^{\rho+\sigma-\frac{m-n}{2}}\Gamma(\rho-\frac{m}{2}+1)\Gamma(\sigma+\frac{n}{2}+1)}{\Gamma(1-m)\Gamma(\rho+\sigma-\frac{m-n}{2}+2)}$$

$$\times\, {}_3F_2\left(-k, n-m+k+1, \rho-\frac{m}{2}+1; 1-m, \rho-\sigma-\frac{m-n}{2}+2; 1\right) \tag{2.1}$$

Provided that $Re(\rho-\frac{m}{2}) > -1; Re(\sigma+\frac{n}{2}) > -1$

Formula 2

$$\int_{-1}^{1} P_{k-\frac{u-v}{2}}^{u,v}(x) P_{t-\frac{u-v}{2}}^{u,v}(x)\mathrm{d}x = \frac{2^{-u+v+1}k!\Gamma(k+v+1+1)}{(2k-u+v+1)\Gamma(k-u+1)\Gamma(k-u+v+1)}\delta_{k,t} \tag{2.2}$$

Where $\delta_{k,t} = 1$ if $k = t, 0$ else.

Provided $Re(u) < 1, Re(v) > -1$

3. Main integral formula

$$\int_{-1}^{1} (1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}} P_{k-\frac{u-v}{2}}^{u,v}(x)\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c} (\mathrm{a}_j, A_j)_{1,\mathrm{n}}, [c_i(a_{ji}, A_{ji})]_{\mathrm{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array}\right.\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1} \\ \cdot \ \cdot \ \cdot \\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s} \end{array}\right)\aleph_{U:W}^{0,\mathrm{n}:V}\left(\begin{array}{c} \mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdot \ \cdot \ \cdot \\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right)\mathrm{d}x$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{\infty}a\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}\frac{(-k)_l l^{v-u+k+1}}{l!\Gamma(1-u+l)}2^{\rho-u+v+\sigma+(\mathfrak{g}+w)\eta_{G,g}+1}y^{\eta_{G,g}}$$

$$\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c|c}\mathrm{z}_1 2^{h_1+k_1} & (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:k_1,\cdots,k_r),\\ \cdot & \cdots\\ \cdot & \cdots\\ \mathrm{z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c}(\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1,\cdots,h_r),A:C\\ \cdots\\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right)\qquad(3.1)$$

where $U_{21}=p_i+2,q_i+1,\tau_i;R$, provided that

a) $Re(u)>-1, Re(v)<1$

b) $g,w>0; g_i,w_i>0, i=1,\cdots,s; h_i,k_i>0, i=1,\cdots,r; k, k-(u-v)/2$ positive integer

c) $Re[\rho-u+\mathfrak{g}\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1$

d) $Re[\sigma+v+w\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{r}h_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>-1$

e) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.8)

f) $|argy|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

Proof

To prove the formula(3.1), we express the general polynomials, the Aleph-function of one variable with the help of equation (1.5) and (1.3) respectively and the multivariable Aleph-function in terms of Mellin-Barnes type contour integrals with the help of equation (1.7). Now interchanging the order of summation and integration (which permissible under the conditions stated), and then evaluate the inner x-integral by using the formula (2.1); we arrive at the desired result.

4. Expansion formula

In this section, we evaluate an expansion formula for product of the general class of polynomials, Legendre's associated function, Aleph-function and Aleph-function of several variables. We have.

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_Gg!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P^{u,v}_{k-\frac{u-v}{2}}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}y^{\eta_{G,g}}$$

$$\aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c|c} \mathrm{z}_1 2^{h_1+k_1} & (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:k_1,\cdots,k_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1,\cdots,h_r),A:C \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B:D \end{array}\right) \tag{4.1}$$

where $U_{21}=p_i+2,q_i+1,\tau_i;R$, which holds true under the same conditions as needed in (3.1)

Proof

Let $f(x)=(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\middle|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right)$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)=\sum_{k=0}^{\infty}c_kP^{u,v}_{k-\frac{u-v}{2}}(x) \tag{4.2}$$

The equation (4.2) is valid since $f(x)$is continuous and bounded variation in the interval (-1,1). Now , multiplying both the sides of (4.2) by$P^{m,n}_{t-\frac{m-n}{2}}(x)$and integrating with respect to x from -1 to 1 ; change the order of integration and summation (which is permissible) on the right,

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\middle|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right)$$

$$P^{u,v}_{t-\frac{u-v}{2}}(x)S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)\mathrm{d}x$$

$$=\sum_{k=0}^{\infty}c_k\int_{-1}^{1}P^{u,v}_{k-\frac{u-v}{2}}(x)P^{u,v}_{t-\frac{u-v}{2}}(x)\mathrm{d}x \tag{4.3}$$

using the orthogonality property for the generalized Legendre's associated function on the right (2.2) on the right-hand side and the result (3.1) on the left hand side of (4.3) ; we obtain

$$c_k=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_Gg!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)} P^{u,v}_{k-\frac{u-v}{2}}(x) 2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}} \prod_{j=1}^{s} y_j^{K_j} 2^{(g_j+w_j)K_j} y^{\eta_{G,g}}$$

$$\aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c} \mathrm{z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r 2^{h_r+k_r} \end{array}\middle| \begin{array}{c} (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s} w_i K_i : k_1,\cdots,k_r), \\ \cdots \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s} g_i K_i : h_1,\cdots,h_r), A:C \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i : h_1+k_1,\cdots,h_r+k_r), B:D \end{array}\right) \tag{4.4}$$

Now on substituing the value of c_k in (4.2), the result (4.1) follows.

5. Particular cases

a) If $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$, the Aleph-function of several variables degenere to the I-function of several variables. We have the following expansion formula for multivariable I-function defined by Sharma et al [3]

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\middle| \begin{array}{c} (\mathrm{a}_j, A_j)_{1,\mathfrak{n}}, [c_i(a_{ji}, A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j, B_j)_{1,m}, [c_i(b_{ji}, B_{ji})]_{m+1,q_i;r} \end{array}\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1} \\ \cdots \\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s} \end{array}\right) I\left(\begin{array}{c} \mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdots \\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r} \end{array}\right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_G g!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)} P^{u,v}_{k-\frac{u-v}{2}}(x) 2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}} \prod_{j=1}^{s} y_j^{K_j} 2^{(g_j+w_j)K_j} y^{\eta_{G,g}}$$

$$I^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c} \mathrm{z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r 2^{h_r+k_r} \end{array}\middle| \begin{array}{c} (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s} w_i K_i : k_1,\cdots,k_r), \\ \cdots \\ \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-g\eta_{G,g}+u-\sum_{i=1}^{s} g_i K_i : h_1,\cdots,h_r), A':C' \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(g+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i : h_1+k_1,\cdots,h_r+k_r), B':D' \end{array}\right) \tag{5.2}$$

Where $U_{21} = p_i+2, q_i+1; R$, which holds true under the same conditions and notations as needed in (3.1)

b) If $\tau_i = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = 1$and $R = R^{(1)} =, \cdots, R^{(r)} = 1$the Aleph-function of several variables degenere to the H-function of several variables. We have the following expansion formula for multivariable H-function defined by Srivastava et al [8].

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)H\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_G g!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P^{u,v}_{k-\frac{u-v}{2}}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}y^{\eta_{G,g}}$$

$$H^{0,\mathfrak{n}+2:V}_{p+2,q+1:W}\left(\begin{array}{c}\mathrm{z}_1 2^{h_1+k_1}\\ \cdot\\ \cdot\\ \mathrm{z}_r 2^{h_r+k_r}\end{array}\left|\begin{array}{c}(\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:k_1,\cdots,k_r),\\ \cdots\\ \cdots\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1,\cdots,h_r),A'':C''\\ \cdots\\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r),B'':D''\end{array}\right)\qquad(5.3)$$

which holds true under the same conditions and notations as needed in (3.1)

c) if $U = n = 0$, the Aleph-function of r variables degenere to product of r Aleph-functions of one variable, and we have

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r'}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)\prod_{u=1}^{r}\aleph^{m_u,n_u}_{p_{i(u)},q_{i(u)},\tau_{i(u)};r^{(u)}}(z_u(1-x)^{h_u}(1+x)^{k_u})$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_G g!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P^{u,v}_{k-\frac{u-v}{2}}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}y^{\eta_{G,g}}$$

$$\aleph^{0,2:V}_{2,1:W}\left(\begin{array}{c|c} \mathrm{z}_1 2^{h_1+k_1} & (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:h_1,\cdots,h_r), \\ \cdot & \cdots \\ \cdot & \cdots \\ \mathrm{z}_r 2^{h_r+k_r} & \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1,\cdots,h_r):C \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1,\cdots,h_r+k_r):D \end{array}\right) \tag{5.4}$$

d) If $r=2$, the Aleph-function of several variables degenere to Aleph-function of two variables defined by K.Sharma [5]. We get the following expansion formula.

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c}\mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1}\\ \cdots\\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s}\end{array}\right)\aleph^{0,\mathfrak{n}:V}_{U:W}\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\\ \cdots\\ \mathrm{z}_2(1-x)^{h_2}(1+x)^{k_2}\end{array}\right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_Gg!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P^{u,v}_{k-\frac{u-v}{2}}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}y^{\eta_{G,g}}$$

$$\aleph^{0,\mathfrak{n}+2:V}_{U_{21}:W}\left(\begin{array}{c|c} \mathrm{z}_1 2^{h_1+k_1} & (\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:k_1,k_2), \\ \cdot & \cdots \\ \mathrm{z}_2 2^{h_2+k_2} & \cdots \end{array}\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1,h_2),A_2:C_2 \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1,h_2+k_2),B_2:D_2 \end{array}\right) \tag{5.5}$$

Where $U_{21}=p_i+2,q_i+1,\tau_i;R$, which holds true under the same conditions as needed in (3.1) with $r=2$

e) If $r=1$, we obtain the Aleph-function of one variable defined by Südland [9]. And we have

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph^{M,N}_{P_i,Q_i,c_i;r}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1(1-x)^{g_1}(1+x)^{w_1} \\ \cdots \\ \mathrm{y}_s(1-x)^{g_s}(1+x)^{w_s} \end{array}\right)\aleph_{P_i',Q_i',c_i';r'}^{M',N'}\left(\ \mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1}\ \right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_G g!\qquad l!\Gamma(1-u+l)}a$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P_{k-\frac{u-v}{2}}^{u,v}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}\prod_{j=1}^{s}y_j^{K_j}2^{(g_j+w_j)K_j}y^{\eta_{G,g}}$$

$$\aleph_{P_i'+2,Q_i'+1,c_i'';r'}^{M',N'+2}\left(\mathrm{z}_1 2^{h_1+k_1}\left|\begin{array}{l} (\ \mathrm{a'}_j,A_j')_{1,\mathfrak{n}'},[c_i'(a_{ji},A_{ji}')]_{\mathfrak{n}'+1,p_i';r},(\text{-}\sigma-v-w\eta_{G,g}-\sum_{i=1}^{s}w_iK_i:k_1), \\ \cdots \qquad\qquad \cdots \\ (\ \mathrm{b'}_j,B_j')_{1,m},[c_i(b_{ji}',B_{ji}')]_{m'+1,q_i';r'}, \qquad \cdots \end{array}\right.\right.$$

$$\left.\begin{array}{c} (\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-\sum_{i=1}^{s}g_iK_i:h_1),A_1:C_1 \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-\sum_{i=1}^{s}(g_i+w_i)K_i:h_1+k_1),B_1:D_1 \end{array}\right) \tag{5.6}$$

f) If $y_2=\cdots=y_s=0$, then the class of polynomials $S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}(y_1,\cdots,y_s)$ defined of (1.14) degenere to the class of polynomials $S_{N'}^{M'}(y)$ defined by Srivastava [6] and we have

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(y(1-x)^{\mathfrak{g}}(1+x)^{w}\left|\begin{array}{c}(\ \mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\ \mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{array}\right.\right)$$

$$S_{N'}^{M'}\left(\ \mathrm{y(1\text{-}x)}^{g'}(1+x)^{w'}\ \right)\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1(1-x)^{h_1}(1+x)^{k_1} \\ \cdots \\ \mathrm{z}_r(1-x)^{h_r}(1+x)^{k_r}\end{array}\right)$$

$$=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K=0}^{[N'/M']}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})(-k)_l\Gamma(v-u+k+l+1)}{B_G g!\qquad l!\Gamma(1-u+l)}a'$$

$$\frac{(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}P_{k-\frac{u-v}{2}}^{u,v}(x)2^{\rho+\sigma+(\mathfrak{g}+w)\eta_{G,g}}y^{K}2^{(g'+w')K}y^{\eta_{G,g}}$$

$$\aleph_{U_{21}:W}^{0,\mathfrak{n}+2:V}\left(\begin{array}{c}\mathrm{z}_1 2^{h_1+k_1} \\ \cdot \\ \cdot \\ \mathrm{z}_r 2^{h_r+k_r}\end{array}\left|\begin{array}{c}(\text{-}\sigma-v-w\eta_{G,g}-w'K:k_1,\cdots,k_r), \\ \cdots \\ \cdots\end{array}\right.\right.$$

$$\left.\begin{array}{c}(\text{-}\rho-m-\mathfrak{g}\eta_{G,g}+u-g'K:h_1,\cdots,h_r),A:C \\ \cdots \\ (\text{u-v-}\rho-\sigma-m-(\mathfrak{g}+w)\eta_{G,g}-(g'+w')K:h_1+k_1,\cdots,h_r+k_r),B:D\end{array}\right) \tag{5.7}$$

Where $U_{21} = p_i + 2, q_i + 1, \tau_i; R$, which holds true under the same conditions as needed in (3.1)

6. Conclusion

The aleph-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions such as , multivariable H-function , defined by Srivastava et al [8] , the Aleph-function of two variables defined by K.sharma [5].

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