# Eulerian integrals involving the multivariable I-function I

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ABSTRACT
In this paper, we derive a general Eulerian integral involving the multivariable I-function defined by Prasad [3], the Aleph-function of one variable, a general class of polynomials of several variables and a extension of the Hurwitz-lerch Zeta-function. Some of this key formula could provide useful generalizations of some known as well as of some new results concerning the multivariable H-function.

Keywords : Eulerian integral, Multivariable I-function, general class of polynomial, Aleph-function, multivariable H-function, a extension of the Hurwitz-lerch Zeta-function

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## 1.Introduction and preliminaries.

The object of this document is to evaluate a multiple Eulerian integrals involving the multivariable I-function defined by Prasad [3], the Aleph-function of one variable, a general class of polynomials of several variables and a extension of the Hurwitz-lerch Zeta-function.. These function generalize the multivariable H-function study by Srivastava et al [6], itself is a generalisation of G-function of several variables. The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,z_2,...z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha'_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta'_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b'_j,\beta'_j)_{1,q'};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(t_1,\cdots,t_r)\prod_{i=1}^{s}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [3]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}\Omega_i^{(k)}\pi$ , where

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}}\alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}}\beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)} + \left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right) +$$

$$+\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.3}$$

where $i=1,\cdots,r$

The complex numbers $z_i$ are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\gamma_1'},\cdots,|z_r|^{\gamma_r'}\,)\,,max(\,|z_1|,\cdots,|z_r|\,)\to 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|,\cdots,|z_r|^{\beta_s'}\,)\,,min(\,|z_1|\,,\cdots,|z_r|\,)\to\infty$$

where $k=1,\cdots,z:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m_k$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n_k$$

We will use these following notations in this paper :

$$U=p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};V=0,n_2;0,n_3;\cdots;0,n_{s-1} \tag{1.4}$$

$$W=(p',q');\cdots;(p^{(r)},q^{(r)});X=(m',n');\cdots;(m^{(r)},n^{(r)}) \tag{1.5}$$

$$A=(a_{2k},\alpha_{2k}',\alpha_{2k}'');\cdots;(a_{(r-1)k}),\alpha'_{(r-1)k},\alpha''_{(r-1)k},\cdots,\alpha_{(r-1)k}^{(r-1)}) \tag{1.6}$$

$$B=(b_{2k},\beta_{2k}',\beta_{2k}'');\cdots;(b_{(r-1)k}),\beta'_{(r-1)k},\beta''_{(r-1)k},\cdots,\beta_{(r-1)k}^{(r-1)}) \tag{1.7}$$

$$\mathfrak{A}=(a_{sk};\alpha'_{rk},\alpha''_{rk},\cdots,\alpha_{rk}^{r}):\mathfrak{B}=(b_{rk};\beta'_{rk},\beta''_{rk},\cdots,\beta_{rk}^{r}) \tag{1.8}$$

$$A'=(a_k',\alpha_k')_{1,p'};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};B'=(b_k',\beta_k')_{1,p'};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,p^{(r)}} \tag{1.9}$$

The multivariable I-function write :

$$I(z_1,\cdots,z_r)\;=\;I_{U:p_r,q_r;W}^{V;0,n_r;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\;;\mathfrak{A};\mathrm{A'} \\ \cdot & \\ \cdot & \\ \cdot & \mathrm{B};\;\mathfrak{B};\;\mathrm{B'} \\ \mathrm{z}_r & \end{array}\right) \tag{1.10}$$

The generalized polynomials defined by Srivastava [4], is given in the following manner :

$$S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t]\;=\;\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}$$

$$A[N_1',K_1;\cdots;N_t',K_t]y_1^{K_1}\cdots y_t^{K_t} \tag{1.11}$$

Where $M_1',\cdots,M_s'$ are arbitrary positive integers and the coefficients $A[N_1',K_1;\cdots;N_t',K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1=\frac{(-N_1')_{M_1'K_1}}{K_1!}\cdots\frac{(-N_t')_{M_t'K_t}}{K_t!}A[N_1',K_1;\cdots;N_t',K_t] \tag{1.12}$$

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(\mathrm{z}\ \middle|\ \begin{matrix}(\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r}\\(\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r}\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.13}$$

for all $z$ different to $0$ and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.14}$$

With $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0\,,i=1,\cdots,r$

For convergence conditions and other details of Aleph-function , see Südland et al [8].The serie representation of Aleph-function is given by Chaurasia et al [2].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_Gg!}z^{-s} \tag{1.15}$$

With $s=\eta_{G,g}=\frac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.14)

## 2. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z,S,a)$ is introduced by Srivastava et al ([6],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s},a)=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}\times\frac{z^n}{n!} \tag{2.1}$$

with : $p,q\in\mathbb{N}_0,\lambda_j\in\mathbb{C}(j=1,\cdots,p),a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^*\ (j=1,\cdots,q),\rho_j,\sigma_k\in\mathbb{R}^+$ $(j=1,\cdots,p;k=1,\cdots,q)$

where $\Delta>-1$ when $\mathfrak{s},z\in\mathbb{C}$; $\Delta=-1$ and $s\in\mathbb{C},when|z|<\bigtriangledown^*$ , $\Delta=-1$ and $Re(\chi)>\frac{1}{2}$ when $|z|=\bigtriangledown^*$

$$\nabla^* = \prod_{j=1}^{p}\rho_j^{\rho_j}\prod_{j=1}^{q}\sigma_j^{\sigma_j} \,;\, \Delta = \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j; \chi = \mathfrak{s} + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\lambda_j + \frac{p-q}{2}$$

We denote these conditions, the conditions (f).

## 3. Required formulas

We have : $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\ Re(\alpha) > 0, Re(\beta) > 0$ (3.1)

(2.1) can be rewritten in the form

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, Re(\alpha) > 0, Re(\beta) > 0, b \neq a \tag{3.2}$$

The binomial expansions for $t \in [a,b]$ yields :

$$(ut+v)^{\gamma} = (au+v)^{\gamma}\sum_{m=0}^{\infty}\frac{(-\gamma)_m}{m!}\left\{\frac{-u(a-t)}{au+v}\right\}^m \quad \text{where} \quad \left|\frac{(t-a)u}{au+v}\right| < 1 \tag{3.3}$$

With the help of (2.2) we obtain (see Srivastava et al [5])

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(at+v)^{\gamma}{}_2F_1\left(\begin{matrix}\alpha, -\gamma\\ \alpha+\beta\end{matrix}; -\frac{(b-a)u}{au+v}\right) \tag{3.4}$$

## 4. The general Eulerian integral

Let $g_1(t) = \frac{(t-a)^{\delta_1}(b-t)^{\eta_1}(ut+v)^{1-\delta_1-\eta_1}}{B(ut+v)+(A-B)(t-a)}$ ; $g_2(t) = \frac{(t-a)^{\delta_2}(b-t)^{\eta_2}(yt+z)^{1-\delta_2-\eta_2}}{D(yt+z)+(C-D)(t-a)}$

$$a' = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \text{ and } b_n = \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}}$$

**We have the following formula**

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{matrix}\mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1}\\ \cdot\ \cdot\ \cdot\\ \cdot\ \cdot\ \cdot\\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s}\end{matrix}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\left(x\,(g_1(t))^{c}\,(g_2(t))^{d}\right)\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z(g_1(t))^{c'}(g(t))^{d'};\mathfrak{s},a)$$

$$I_{U:p_r,q_r;W}^{V;0,n_r;X}\left(\begin{array}{c} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdots \\ \cdots \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r} \end{array}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}$$

$$\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}\frac{b_n z^n}{n!}a'\;\;Y_1^{K_1}\cdots y_s^{K_s}X^{\eta_{G,g}}\;\;Y^n\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}$$

$$I_{U:p_r+7,q_r+6;W}^{V;0,n_r+7;X}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdots \\ \cdots \\ \mathrm{Z}_r \end{array}\left|\begin{array}{c} \mathrm{A}\;;(1\text{-l-c}\eta_{G,g}-c'n-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r), \\ \cdots \\ \mathrm{B};(1\text{-c}\eta_{G,g}-c'n-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r), \end{array}\right.\right.$$

$$\begin{array}{c}(1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-(c'\delta_1+d'\delta_2)n-\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)K_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r), \\ \cdots \\ \mathrm{B}_1,\end{array}$$

$$\begin{array}{c}(1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-(c'\eta_1+d'\eta_2)n-\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)K_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r), \\ \cdots \\ (1+\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-(\delta_1+\eta_1)c'n-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r),\end{array}$$

$$\begin{array}{c}(1+\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-(\eta_2+\delta_2)d'n-\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r), \\ \cdots \\ (1+\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-(\eta_2+\delta_2)d'n-\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),\end{array}$$

$$\begin{array}{c}(1+\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-(\delta_1+\eta_1)c'n-\sum_{i=1}^{s}(\eta_1+\delta_1)c_iK_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r), \\ \cdots \\ \cdots\end{array}$$

$$\left.\begin{array}{c}(1\text{-m-d}\eta_{G,g}-d'n-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),A_1,\mathfrak{A};A' \\ \cdots \\ (1\text{-d}\eta_{G,g}-d'n-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),B_1,B_2,\mathfrak{B};B'\end{array}\right) \tag{4.1}$$

Where

$$\mathrm{B}_1=(1-\alpha-\beta-m-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i$$
$$-c'(\delta_1+\eta_1)n-d'(\delta_2+\eta_2)n;(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$$

$A_1 = (1-\alpha-\beta-l-m-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i$
$-c'(\delta_1+\eta_1)n-d'(\delta_2+\eta_2)n;(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$

$B_2 = (1-\alpha-\beta-m-l-k_1-k_2-c(\delta_1+\eta_1)\eta_{G,g}-d(\delta_2+\eta_2)\eta_{G,g}-\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i-$

$-\sum_{i=1}^{s}(\delta_2+\eta_2)d_iK_i-c'(\delta_1+\eta_1)n-d'(\delta_2+\eta_2)n;(\delta_1+\eta_1)u_1+(\delta_2+\eta_2)v_1,\cdots,$

$(\delta_1+\eta_1)u_r+(\delta_2+\eta_2)v_r)$

$$Z_i = \frac{z_i(b-a)^{(\delta_1+\eta_1)u_i+(\delta_2+\eta_2)v_i}}{B^{u_i}D^{v_i}(au+v)^{(\delta_1+\eta_1)u_i}(by+z)^{(\delta_2+\eta_2)v_i}}, i=1,\cdots,r$$

$$Y_i = \frac{y_i(b-a)^{(\delta_1+\eta_1)c_i+(\delta_2+\eta_2)d_i}}{B^{c_i}D^{d_i}(au+v)^{(\delta_1+\eta_1)c_i}(by+z)^{(\delta_2+\eta_2)d_i}}, i=1,\cdots,s$$

$$X = \frac{x(b-a)^{(\delta_1+\eta_1)c+(\delta_2+\eta_2)d}}{B^cD^d(au+v)^{(\delta_1+\eta_1)c}(by+z)^{(\delta_2+\eta_2)d}} \text{ and } Y = \frac{z(b-a)^{(\delta_1+\eta_1)c'+(\delta_2+\eta_2)d'}}{B^{c'}D^{d'}(au+v)^{(\delta_1+\eta_1)c'}(by+z)^{(\delta_2+\eta_2)d'}}$$

Provided that

a) $min\{c,d,c',d',c_i,d_i,u_j,v_j\} > 0, i=1,\cdots,s; j=1,\cdots,r$

b $min\{Re(\alpha),Re(\beta)\} > 0, b \neq a$

c) $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+z}\right|,\left|\frac{(t-a)(B-A)}{B(ut+v)}\right|,\left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\} < 1, t\in[a,b]$

d) $Re\big[\alpha+(c\delta_1+d\delta_2)\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+(c'\delta_1+d'\delta_2)n+\sum_{i=1}^{r}(\delta_1u_i+\delta_2v_i)\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

e) $Re\big[\beta+(c\eta_1+d\eta_2)\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+(c'\eta_1+d'\eta_2)n+\sum_{i=1}^{r}(\eta_1u_i+\eta_2v_i)\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}] > -1$

f) The conditions (f) are satisfied

g) $|argz_k| < \frac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is given in (1.3)

h) $|argx| < \frac{1}{2}\pi\Omega$ Where $\Omega = \sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

**Proof of (3.1)** Let $M = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$

We first replace the multivariable I-function on the L.H.S of (4.1) by its Mellin-barnes contour integral (1.1), the Aleph-function,the general class of polynomials of several variables and the extension of the Hurwitz-Lerch Zeta function in series using respectively (1.15), (1.11) and (2.1).Now we interchange the order of summation and integrations (which is permissible under the conditions stated) . We get :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a'\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}$$

$$(ut+v)^{\gamma}(yt+z)^{\rho}\Big\{M\Big\{(g_1(t))^{c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i}(g_2(t))^{d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i}\Big\}$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r\Big\}\mathrm{d}t \tag{4.2}$$

We change the order of $t-$integral and $(s_1,\cdots,s_r)-$integral, we obtain :

$$\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a'\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}x^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$M\Big\{B^{-(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}\,D^{-(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\int_a^b (t-a)^{\alpha+\delta_1(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\delta_2(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\eta_2(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(ut+v)^{\gamma-(\delta_1+\eta_1)(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}$$

$$(yt+z)^{\rho-(\delta_2+\eta_2)(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}\left(1-\frac{(D-C)(t-a)}{D(yt+z)}\right)^{-(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\left(1-\frac{(B-A)(t-a)}{B(ut+v)}\right)^{-(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}\mathrm{d}t\Big\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{4.3}$$

Using binomial expansion (2.3) provided that $max\left\{\left|\frac{(t-a)(B-A)}{B(ut+v)}\right|,\left|\frac{(t-a)(D-C)}{D(yt+z)}\right|\right\}<1, t\in[a,b]$

and also that the order of binomial summation and integration can be inversed, we get

$$\sum_{l,m=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a'\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})(B-A/B)^l\,(D-C/D)^m}{B_G g!\qquad l!m!}x^{\eta_{G,g}}$$

$$y_1^{K_1}\cdots y_s^{K_s}M\Bigg\{B^{-(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}D^{-(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\frac{\Gamma(l+c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)\Gamma(m+d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}{\Gamma(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)\Gamma(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$\int_a^b(t-a)^{\alpha+l+m+\delta_1(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\delta_2(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(b-t)^{\beta+\eta_1(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)+\eta_2(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)-1}$$

$$(yt+z)^{\rho-m-(\delta_2+\eta_2)(d\eta_{G,g}+d'n+\sum_{i=1}^{s}d_iK_i+\sum_{i=1}^{r}v_is_i)}$$

$$(ut+v)^{\gamma-l-(\delta_1+\eta_1)(c\eta_{G,g}+c'n+\sum_{i=1}^{s}c_iK_i+\sum_{i=1}^{r}u_is_i)}\mathrm{d}t\Bigg\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{4.4}$$

The inner integral in (3.4) can be evaluated by using the following extension of Eulerian integral of Beta function given by Hussain and Srivastava [5].

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}\mathrm{d}t=(b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\delta}B(\alpha,\beta)$$

$$\times F_3\left[\alpha,\beta,-\gamma,-\rho;\alpha+\beta;-\frac{(b-a)u}{au+v},\frac{(b-a)y}{by+z}\right] \tag{4.5}$$

where for convergence $min\{Re(\alpha),Re(\beta)\}>0, b\neq a$ and $max\left\{\left|\frac{u(b-a)}{au+v}\right|,\left|\frac{y(b-a)}{by+t}\right|\right\}<1$

and where $F_3$ denote the Appell function of two variables, see Appell et al [1]. Finally interpreting the resulting Mellin-Barnes contour integral as a multivariable I-function, we obtain the desired result (3.1).

## 5. Particular case

If $U=V=A=B=0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [6]. We have the following result.

$$\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}(ut+v)^{\gamma}(yt+z)^{\rho}S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1\,(g_1(t))^{c_1}\,(g_2(t))^{d_1}\\ \cdot\;\cdot\;\cdot\\ \cdot\;\cdot\;\cdot\\ \mathrm{y}_s\,(g_1(t))^{c_s}\,(g_2(t))^{d_s}\end{array}\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}\Big(x\,(g_1(t))^{c}\,(g_2(t))^{d}\Big)\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z(g_1(t))^{c'}(g(t))^{d'};\mathfrak{s},a)$$

$$H^{0,n_r;X}_{p_r,q_r;W}\left(\begin{array}{c} \mathrm{z}_1\,(g_1(t))^{u_1}\,(g_2(t))^{v_1} \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{z}_r\,(g_1(t))^{u_r}\,(g_2(t))^{v_r} \end{array}\right)\mathrm{d}t$$

$$= (b-a)^{\alpha+\beta-1}(au+v)^{\gamma}(by+z)^{\rho}\sum_{l,m,k_1,k_2=0}^{\infty}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-)^g\Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!}$$

$$\frac{(B-A/B)^l\,(D-C/D)^m}{l!m!k_1!k_2!}\frac{b_n z^n}{n!}a'\ \ Y_1^{K_1}\cdots y_s^{K_s}X^{\eta_{G,g}}Y^n\left\{-\frac{(b-a)u}{(au+v)}\right\}^{k_1}\left\{\frac{(b-a)y}{(by+z)}\right\}^{k_2}$$

$$H^{0,n_r+7;X}_{p_r+7,q_r+6;W}\left(\begin{array}{c} \mathrm{Z}_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ \mathrm{Z}_r \end{array}\left|\begin{array}{c} (1\text{-l-c}\eta_{G,g}-c'n-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r), \\ \cdot\cdot\cdot \\ (1\text{-c}\eta_{G,g}-c'n-\sum_{i=1}^{s}c_iK_i:u_1,\cdots,u_r), \end{array}\right.\right.$$

$$(1\text{-}\alpha-l-m-k_1-(c\delta_1+d\delta_2)\eta_{G,g}-(c'\delta_1+d'\delta_2)n-\textstyle\sum_{i=1}^{s}(\delta_1c_i+\delta_2d_i)K_i:\delta_1u_1+\delta_2v_1,\cdots,\delta_1u_r+\delta_2v_r),$$
$$\cdot\cdot\cdot$$
$$\mathrm{B}_1,$$

$$(1\text{-}\beta-k_2-(c\eta_1+d\eta_2)\eta_{G,g}-(c'\eta_1+d'\eta_2)n-\textstyle\sum_{i=1}^{s}(\eta_1c_i+\eta_2d_i)K_i:\eta_1u_1+\eta_2v_1,\cdots,\eta_1u_r+\eta_2v_r),$$
$$\cdot\cdot\cdot$$
$$(1\text{+}\gamma-l-(\delta_1+\eta_1)c\eta_{G,g}-(\delta_1+\eta_1)c'n-\textstyle\sum_{i=1}^{s}(\delta_1+\eta_1)c_iK_i;(\delta_1+\eta_1)u_1,\cdots,(\delta_1+\eta_1)u_r),$$

$$(1\text{+}\rho-m-k_2-(\eta_2+\delta_2)d\eta_{G,g}-(\eta_2+\delta_2)d'n-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$
$$\cdot\cdot\cdot$$
$$(1\text{+}\rho-m-(\eta_2+\delta_2)d\eta_{G,g}-(\eta_2+\delta_2)d'n-\textstyle\sum_{i=1}^{s}(\eta_2+\delta_2)d_iK_i:(\eta_2+\delta_2)v_1,\cdots,(\eta_2+\delta_2)v_r),$$

$$(1\text{+}\gamma-l-k_1-(\delta_1+\eta_1)c\eta_{G,g}-(\delta_1+\eta_1)c'n-\textstyle\sum_{i=1}^{s}(\eta_1+\delta_1)c_iK_i:(\eta_1+\delta_1)u_1,\cdots,(\eta_1+\delta_1)u_r),$$
$$\cdot\cdot\cdot$$
$$\cdot\cdot\cdot$$

$$\left.\begin{array}{c} (1\text{-m-d}\eta_{G,g}-d'n-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),A_1,\mathfrak{A};A' \\ \cdot\cdot\cdot \\ (1\text{-d}\eta_{G,g}-d'n-\sum_{i=1}^{s}d_iK_i:v_1,\cdots,v_r),B_1,B_2,\mathfrak{B};B' \end{array}\right) \qquad \textbf{(5.1)}$$

under the same notations and conditions that (4.1) with $U=V=A=B=0$

## 6. Conclusion

In this paper we have evaluated a generalized Eulerian integral involving the Aleph-function, a class of polynomials of several variables, a extension of the Hurwitz-Lerch Zeta-function and the multivariable I-function defined by Prasad [3]. The integral established in this paper is of very general nature as it contains multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

REFERENCES

[1]Appell P. and Kampe de Feriet J. Fonctions hypergeometriques et hypersphériques, polynôme d'Hermite, Gautier-Vilars. Paris (1926).

[2] Chaurasia V.B.L and Singh Y. New generalization of integral equations of fredholm type using Aleph-function Int. J. of Modern Math. Sci. 9(3), 2014, p 208-220.

[3] Y.N. Prasad , Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 ( 1986 ) , page 231-237.

[4] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), page183-191.

[5] H.M. Srivastava and Hussain M.A. Fractional integration of the H-function of several variables. Comp. Math. Appl. 30(9), (1995), page 73-85.

[6] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

[7] H.M. Srivastava, R.K. Saxena, T.K. Pogány, R. Saxena, Integral and computational representations of the extended Hurwitz–Lerch zeta function, Integr.Transf. Spec. Funct. 22 (2011) 487–506

[8] Südland N.; Baumann, B. and Nonnenmacher T.F. , Open problem : who knows about the Aleph-functions? Fract. Calc. Appl. Anal., 1(4) (1998): 401-402.

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