On some multidimensional integral transforms of multivariable Aleph-function III

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ABSTRACT
The object of the paper is to obtain few general multiple integral transformations of the multivariable Aleph-function, as kernel product with Fox's H-function and general class of polynomials. Several cases are also included.

Keywords :Multivariable Aleph-function, multidimensional integral transforms ,General class of polynomials, Multivariable I-function, Aleph-function of two variables,Laguerre polynomials.

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1.Introduction and preliminaries.

In this document, we obtain few general multiple integral transformations involving the Fox's h-function of one variable ,see [1], a class of polynomials of one variables and the multivariable Aleph-function. The Aleph-function of several variables generalize the multivariable I-function defined by H.M. Sharma and Ahmad [2] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.The generalized Aleph-function of several variables is defined as following.

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}} \left(\begin{array}{c|} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\right.$

$[(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}]$ $,[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}]:$
$\cdots\cdots\cdots$ $,[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}]:$

$$\left.\begin{array}{c} [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}],[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;\ ;[(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}],[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}],[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;;[(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}],[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.2}$$

and $$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

where $j=1$ to r and $k=1$ to r

Suppose , as usual , that the parameters

$a_j,j=1,\cdots,p;b_j,j=1,\cdots,q;$

$c_j^{(k)}, j = 1, \cdots, n_k; c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, p_{i^{(k)}};$

$d_j^{(k)}, j = 1, \cdots, m_k; d_{ji^{(k)}}^{(k)}, j = m_k + 1, \cdots, q_{i^{(k)}};$

with $k = 1 \cdots, r, i = 1, \cdots, R \;, i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha' s, \beta' s, \gamma' s$ and $\delta' s$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} + \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$-\tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0 \tag{1.4}$$

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}} \alpha_j^{(k)} - \tau_i \sum_{j=\mathfrak{n}+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{m_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \text{ with } k = 1 \cdots, r, i = 1, \cdots, R \;, i^{(k)} = 1, \cdots, R^{(k)} \tag{1.5}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.
We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1} \ldots |z_r|^{\alpha_r}), max(|z_1| \ldots |z_r|) \rightarrow 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1} \ldots |z_r|^{\beta_r}), min(|z_1| \ldots |z_r|) \rightarrow \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)} / \delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i; R \;;\; V = m_1, n_1; \cdots; m_r, n_r \tag{1.6}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}, \cdots, p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \tag{1.7}$$

$$A = \{(a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,n}\}, \{\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{n+1,p_i}\} \tag{1.8}$$

$$B=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}\} \quad (1.9)$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\},\cdots,\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\} \quad (1.10)$$

$$D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\},\cdots,\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\} \quad (1.11)$$

The multivariable Aleph-function write :

$$\aleph(z_1,\cdots,z_r)=\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\left|\begin{array}{c}\mathrm{A}:\mathrm{C}\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{array}\right.\right) \quad (1.12)$$

Srivastava [5] introduced the general class of polynomials :

$$S_N^M(x)=\sum_{k=0}^{[N/M]}\frac{(-N)_{Mk}}{k!}A_{N,k}x^k, N=0,1,2,... \quad (1.19)$$

Where M is an arbtrary positive integer and the coefficient $A_{N,k}$ are arbitrary constants, real or complex.

By suitably specialized the coefficient $A_{N,k}$ the polynomials $S_N^M(x)$ can be reduced to the classical orthogonal polynomials such as Jacobi, Hermite, Legendre and Laguerre polynomials etc.

2. Main Result

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^\sigma S_\alpha^\beta\left[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^h\right]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})\left|\begin{array}{c}(\mathrm{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\ (\mathrm{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{array}\right.\right]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1X_1\\ \cdot\\ \cdot\\ \mathrm{z}_rX_r\end{array}\left|\begin{array}{c}\mathrm{A}:\mathrm{C}\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{array}\right.\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$=\zeta^{-s}\Psi(k_1,\cdots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\zeta^{-hk}\aleph_{U_{r+q,p+1}:W}^{0,\mathfrak{n}+r+m:V}\left(\begin{array}{c}\mathrm{Z}_1\\ \cdot\\ \cdot\\ \mathrm{Z}_r\end{array}\left|\begin{array}{c}(1\text{-}\rho_j/\sigma_j;\zeta_j'/\sigma_j,\cdots,\zeta_j^{(r)}/\sigma_j)_{1,r},\\ \cdots\\ (1\text{-S}+\sigma;N_1-n_1,\cdots,N_r-n_r),\end{array}\right.\right.$$

$$\left.\begin{array}{c}(1\text{-g}_j-(S+hk)\gamma_j;N_1\gamma_j,\cdots,N_r\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-e}_j-(S+hk)\epsilon_j;N_1\epsilon_j,\cdots,N_r\epsilon_j)_{1,p},B:D\end{array}\right) \quad (2.1)$$

where $X_i=x_1^{\zeta_1^{(i)}}\cdots x_r^{\zeta_r^{(i)}}(k_1x_1^{\rho_1}+\cdots+k_rx^{\rho_r})^{\eta_i}$ (2.2)

$$S=\sigma+\frac{\sigma_1}{\rho_1}+\cdots+\frac{\sigma_r}{\rho_r} \quad (2.3)$$

$$\Psi(k_1,\cdots,k_r)=(\sigma_1\cdots\sigma_r)^{-1}k_1^{-\sigma_1/\rho_1}\cdots k_r^{-\sigma_r/\rho_r} \tag{2.4}$$

$$N_i=n_i+\frac{\zeta_1^{(i)}}{\rho_1}+\cdots+\frac{\zeta_r^{(i)}}{\rho_r} \tag{2.5}$$

$$Z_i=z_i\zeta^{-N_i}k_1^{-\zeta_1^{(i)}/\rho_1}\cdots k_r^{-\zeta_r^{(i)}/\rho_r} \tag{2.6}$$

and $U_{r+q,p+1}=p_i+r+q,q_i+p+1,\tau_i;R$ (2.7)

Provided

a) $k_i>0,\rho_i>0,n_i\geqslant 0,\zeta_j^{(i)}>0,i,j=1,\cdots,r$

b) $Re(\sigma_i)>0,i=1,\cdots,r$ and

c) $Re[S+\min_{1\leqslant j\leqslant m}\frac{g_j}{\gamma_j}+\sum_{i=1}^{r}N_i\min_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

d) $|arg(\zeta)|<\frac{1}{2}\pi\left(\sum_{j=1}^{m}\gamma_j-\sum_{j=m+1}^{q}\gamma_j-\sum_{j=1}^{p}\epsilon_j\right)$

e) $|argz_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is given in (1.5)

Proof : Let $M=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}$

and $S_r=\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}f[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{array}{c}\mathrm{z}_1X_1\\ \cdot\\ \cdot\\ \mathrm{z}_rX_r\end{array}\middle|\begin{array}{c}\mathrm{A}:\mathrm{C}\\ \cdot\ \cdot\ \cdot\\ \mathrm{B}:\mathrm{D}\end{array}\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$

where the X_i is defined by (2.2) and the function f is such that the multiple integral converges. On replacing the multivariable Aleph-function occuring here by contour integral given by (1.1), under the various conditions stated with (2.1), we find that

$$S_r=M\left\{\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1+\sum_{i=1}^r s_i\zeta_1^{(i)}-1}\cdots x_r^{\sigma_r+\sum_{i=1}^r s_i\zeta_r^{(i)}-1}\ (k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^{\sum_{i=1}^r n_is_i}\right.$$

$$\left.f[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})]\mathrm{d}x_1\cdots\mathrm{d}x_r\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.8}$$

Now we evaluate the innermost $(x_1,\cdots,x_r)$-integral by using the following form

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}\ (k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^\sigma f[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})]\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$=\Psi(k_1,\cdots,k_r)\frac{\Gamma(\sigma_1/\rho_1)\cdots\Gamma(\sigma_r/\rho_r)}{\Gamma(\sigma_1/\rho_1+\cdots+\sigma_r/\rho_r)}\int_0^\infty z^{\sigma_1/\rho_1+\cdots+\sigma_r/\rho_r+\sigma-1}f(z)\mathrm{d}z \tag{2.9}$$

where $\Psi(k_1,\cdots,k_r)$ is given by (2.4) and $min\{k_i,\rho_i,Re(\sigma_i)\}>0,i=1,\cdots,r$ then (2.8) reduces in the following form

$$S_r = \Psi(k_1,\cdots,k_r)M\left[\frac{\Gamma(\sigma_1^*/\rho_1)\cdots\Gamma(\sigma_r^*/\rho_r)}{\Gamma(\sigma_1^*/\rho_1+\cdots+\sigma_r^*/\rho_r)}Y_1^{s_1}\cdots Y_r^{s_r}\left\{\int_0^\infty z^{\sigma_1/\rho_1+\cdots+\sigma_r/\rho_r+\sigma-1}f(z)\mathrm{d}z\right\}\right]$$

$$\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.10}$$

where $\Psi(k_1,\cdots,k_r)$, N_i and S are given by (2.4), (2.5) and (2.3) and $Y_i = zk_i^{\sum_{i=1}^r \zeta_j^{(i)}/\rho_j}$,

$$\sigma_j^* = \sigma_j + \sum_{i=1}^r \zeta_j^{(i)} s_i, j = 1,\cdots,r$$

Now in the integral (2.10) , we get $f(z) = S_\alpha^\beta[(\gamma z^h]H_{p,q}^{\mathfrak{m},o}\left[z\zeta\left|\begin{matrix}(\mathrm{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(\mathrm{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right.\right]$ (2.11)

and evaluate the z-integral by following formula (when n = 0), expressing the Mellin transform of Fox's H-function [5,p.311,eq(3.3)].

$$\mathfrak{M}\{H_{p,q}^{m,n}(zx):s\} = \frac{\prod_{j=1}^m\Gamma(\beta_j+B_js)\prod_{j=1}^n\Gamma(1-\alpha_j-A_js)}{\prod_{j=m+1}^q\Gamma(1-\beta_j-B_js)\prod_{j=n+1}^p\Gamma(\alpha_j+A_js)}z^{-s} \tag{2.12}$$

Interpret the resulting $(s_1,\cdots,s_r)$-integral as a multivariable Aleph-function, we obtain the desired result (2.1)

3. Particular cases

For the general class of polynomials, we take the cases of Hermite polynomials [9,p.106,eq.(5.54)] and [8,p.158] by setting $S_\beta^2 = z^{\beta/2}H_\beta\left[\frac{1}{2\sqrt{z}}\right]$ in which case $\alpha = 2, A_{\beta,k} = (-)^k$

the result (2.1) reduces in the following form

a) $$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^{\sigma+\beta h/2}H_\beta\left[\frac{1}{2\sqrt{\eta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^h}}\right]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})\left|\begin{matrix}(\mathrm{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(\mathrm{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right.\right]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1X_1\\ \cdot\\ \cdot\\ \mathrm{z}_rX_r\end{matrix}\left|\begin{matrix}\mathrm{A}:\mathrm{C}\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right.\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$= \zeta^{-s}\Psi(k_1,\cdots,k_r)\sum_{k=0}^{[\beta/2]}\frac{\beta!(-)^k}{(\beta-2k)!k!}\eta^k\zeta^{-hk}\aleph_{U_{r+q,p+1}:W}^{0,\mathfrak{n}+r+m:V}\left(\begin{matrix}\mathrm{Z}_1\\ \cdot\\ \cdot\\ \mathrm{Z}_r\end{matrix}\left|\begin{matrix}(1\text{-}\rho_j/\sigma_j;\zeta_j'/\sigma_j,\cdots,\zeta_j^{(r)}/\sigma_j)_{1,r},\\ \cdots\\ (1\text{-S}+\sigma;N_1-n_1,\cdots,N_r-n_r),\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1\text{-g}_j-(S+hk)\gamma_j;N_1\gamma_j,\cdots,N_r\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-e}_j-(S+hk)\epsilon_j;N_1\epsilon_j,\cdots,N_r\epsilon_j)_{1,p},B:D\end{matrix}\right) \tag{3.1}$$

valid under the same conditions and notations that (2.1)

b) If we set $\alpha = 1$and $A_{\beta,k} = \binom{\beta+\upsilon}{\beta}\frac{1}{(\upsilon+1)_k}$, the general class of polynomials reduces in Laguerre

polynomials [9, p.106,eq.(15,16)] and [8,p.159] where Laguerre polynomials is given by

$$L_\beta^\upsilon(z) = \sum_{k=0}^{\beta}\binom{\beta+\upsilon}{\beta-k}\frac{(-z)^k}{k!}$$

the result (2.1) reduces in the following form

b) $$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^\sigma L_\beta^\upsilon\big[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^h\big]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})\middle| \begin{matrix}(\mathrm{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(\mathrm{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1X_1\\ \cdot\\ \cdot\\ \mathrm{z}_rX_r\end{matrix}\middle|\begin{matrix}\mathrm{A}:\mathrm{C}\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$= \zeta^{-s}\Psi(k_1,\cdots,k_r)\sum_{k=0}^{\beta}\binom{\beta+\upsilon}{\beta-k}\frac{(-\eta)^k}{k!}\zeta^{-hk}\aleph_{U_{r+q,p+1}:W}^{0,\mathfrak{n}+r+m:V}\left(\begin{matrix}\mathrm{Z}_1\\ \cdot\\ \cdot\\ \mathrm{Z}_r\end{matrix}\middle|\begin{matrix}(1\text{-}\rho_j/\sigma_j;\zeta_j'/\sigma_j,\cdots,\zeta_j^{(r)}/\sigma_j)_{1,r},\\ \cdots\\ (1\text{-S}+\sigma;N_1-n_1,\cdots,N_r-n_r),\end{matrix}\right.$$

$$\left.\begin{matrix}(1\text{-g}_j-(S+hk)\gamma_j;N_1\gamma_j,\cdots,N_r\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-e}_j-(S+hk)\epsilon_j;N_1\epsilon_j,\cdots,N_r\epsilon_j)_{1,p},B:D\end{matrix}\right) \tag{3.2}$$

valid under the same conditions and notations that (2.1)

For the Jacobi polynomials [9, p.68,eq.(15,16)] and [8,p.159] by setting

$S_\beta^1(z) = P_\beta^{(s,t)}(1-2z)$ in which case $\alpha = 1$ and $A_{\beta,k} = \binom{\beta+s}{\beta}\frac{(s+t+\beta+1)_k}{(s+1)_k}$

the result (2.1) reduces in the following form

c) $$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^\sigma P_\beta^{(s,t)}\big[1-2\eta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^h\big]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})\middle| \begin{matrix}(\mathrm{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(\mathrm{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\mathrm{z}_1X_1\\ \cdot\\ \cdot\\ \mathrm{z}_rX_r\end{matrix}\middle|\begin{matrix}\mathrm{A}:\mathrm{C}\\ \cdots\\ \mathrm{B}:\mathrm{D}\end{matrix}\right)\mathrm{d}x_1\cdots\mathrm{d}x_r$$

$$= \zeta^{-s}\Psi(k_1,\cdots,k_r)\sum_{k=0}^{\beta}\binom{\beta+s}{\beta}\binom{\beta+t+k+s}{k}\zeta^{-hk}\ \aleph_{U_{r+q,p+1}:W}^{0,\mathfrak{n}+r+m:V}\left(\begin{matrix}\mathrm{Z}_1\\ \cdot\\ \cdot\\ \mathrm{Z}_r\end{matrix}\middle|\right.$$

$$\left. \begin{matrix} (1\text{-}\rho_j/\sigma_j;\zeta_j'/\sigma_j,\cdots,\zeta_j^{(r)}/\sigma_j)_{1,r},(1\text{-}g_j-(S+hk)\gamma_j;N_1\gamma_j,\cdots,N_r\gamma_j)_{1,q},A:C \\ \cdots \\ (1\text{-}S+\sigma;N_1-n_1,\cdots,N_r-n_r),(1\text{-}e_j-(S+hk)\epsilon_j;N_1\epsilon_j,\cdots,N_r\epsilon_j)_{1,p},B:D \end{matrix} \right) \tag{3.3}$$

valid under the same conditions and notations that (2.1)

4. Multivariable I-function

If $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$, the Aleph-function of several variables degenere to the I-function of several variables. The multiple integral have been derived in this section for multivariable I-functions defined by Sharma et al [2].

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^\sigma S_\alpha^\beta[(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})^h]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r})\middle| \begin{matrix}(e_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(g_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right]I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}z_1X_1\\ \cdot\\ \cdot\\ z_rX_r\end{matrix}\middle|\begin{matrix}A:C\\ \cdots\\ B:D\end{matrix}\right)dx_1\cdots dx_r$$

$$=\zeta^{-s}\Psi(k_1,\cdots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\zeta^{-hk}I_{U_{r+q,p+1}:W}^{0,\mathfrak{n}+r+m:V}\left(\begin{matrix}Z_1\\ \cdot\\ \cdot\\ Z_r\end{matrix}\middle|\begin{matrix}(1\text{-}\rho_j/\sigma_j;\zeta_j'/\sigma_j,\cdots,\zeta_j^{(r)}/\sigma_j)_{1,r},\\ \cdots\\ (1\text{-}S+\sigma;N_1-n_1,\cdots,N_r-n_r),\end{matrix}\right.$$

$$\left.\begin{matrix}(1\text{-}g_j-(S+hk)\gamma_j;N_1\gamma_j,\cdots,N_r\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-}e_j-(S+hk)\epsilon_j;N_1\epsilon_j,\cdots,N_r\epsilon_j)_{1,p},B:D\end{matrix}\right) \tag{4.1}$$

valid under the same conditions and notations that (2.1) with $\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$

5. Aleph-function of two variables

If $r=2$, we obtain the Aleph-function of two variables defined by K.Sharma [4], and we have the following multiple integral.

$$\int_0^\infty\int_0^\infty x_1^{\sigma_1-1}x_r^{\sigma_2-1}(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})^\sigma S_\alpha^\beta[(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})^h]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})\middle| \begin{matrix}(e_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(g_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right]\aleph_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}z_1X_1\\ \cdot\\ \cdot\\ z_2X_2\end{matrix}\middle|\begin{matrix}A:C\\ \cdots\\ B:D\end{matrix}\right)dx_1\cdots dx_r$$

$$=\zeta^{-s}\Psi(k_1,k_2)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\zeta^{-hk}\aleph_{U_{2+q,p+1}:W}^{0,\mathfrak{n}+2+m:V}\left(\begin{matrix}Z_1\\ \cdot\\ \cdot\\ Z_2\end{matrix}\middle|\begin{matrix}(1\text{-}\rho_j/\sigma_j;\zeta_j^{(1)}/\sigma_j,\zeta_j^{(2)}/\sigma_j)_{1,2},\\ \cdots\\ (1\text{-}S+\sigma;N_1-n_1,N_2-n_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1\text{-g}_j-(S+hk)\gamma_j;N_1\gamma_j,N_2\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-e}_j-(S+hk)\epsilon_j;N_1\epsilon_j,N_2\epsilon_j)_{1,p},B:D\end{matrix}\right) \tag{5.1}$$

valid under the same conditions and notations that (2.1) with $r=2$

6. I-function of two variables

If $\tau_i,\tau_i',\tau_i''\rightarrow 1$, then the Aleph-function of two variables degenere in the I-function of two variables defined by sharma et al [3] and we obtain the same formulae with the I-function of two variables.

$$\int_0^\infty\int_0^\infty x_1^{\sigma_1-1}x_r^{\sigma_2-1}(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})^\sigma S_\alpha^\beta[(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})^h]$$

$$H_{p,q}^{\mathfrak{m},o}\left[\zeta(k_1x_1^{\rho_1}+k_2x_2^{\rho_2})\middle|\begin{matrix}(\text{e}_1,\epsilon_1),\cdots,(e_p,\epsilon_p)\\(\text{g}_1,\gamma_1),\cdots,(g_q,\gamma_q)\end{matrix}\right]I_{U:W}^{0,\mathfrak{n}:V}\left(\begin{matrix}\text{z}_1X_1\\ \cdot\\ \cdot\\ \text{z}_2X_2\end{matrix}\middle|\begin{matrix}\text{A : C}\\ \cdots\\ \text{B : D}\end{matrix}\right)\text{d}x_1\cdots\text{d}x_r$$

$$=\zeta^{-s}\Psi(k_1,k_2)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\zeta^{-hk}I_{U_{2+q,p+1}:W}^{0,\mathfrak{n}+2+m:V}\left(\begin{matrix}\text{Z}_1\\ \cdot\\ \cdot\\ \text{Z}_2\end{matrix}\middle|\begin{matrix}(1\text{-}\rho_j/\sigma_j;\zeta_j^{(1)}/\sigma_j,\zeta_j^{(2)}/\sigma_j)_{1,2},\\ \cdots\\ (1\text{-S+}\sigma;N_1-n_1,N_2-n_2),\end{matrix}\right.$$

$$\left.\begin{matrix}(1\text{-g}_j-(S+hk)\gamma_j;N_1\gamma_j,N_2\gamma_j)_{1,q},A:C\\ \cdots\\ (1\text{-e}_j-(S+hk)\epsilon_j;N_1\epsilon_j,N_2\epsilon_j)_{1,p},B:D\end{matrix}\right) \tag{6.1}$$

valid under the same conditions and notations that (2.1) with $r=2$

Remark

If a$\tau_i,\tau_{i^{(1)}},\cdots,\tau_{i^{(r)}}\rightarrow 1$nd $R=R^{(1)}=\cdots=R^{(r)}=1$the Aleph-function of several variables degenere to the H-function of several variables defined by Srivastava et al [6]. The results have been by Srivastava et al [7].

7. Conclusion

In this paper we have evaluated several multiple integrals involving the multivariable Aleph-function, a class of polynomials and Fox's H-function of one variable.The multiple integrals established in this paper is of very general nature as it contains Multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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