

Fixed Point Theorem for six self mappings on a Fuzzy Metric Space using JCLR property

I.H.Nagaraja Rao^{#1} and S.Rajesh^{#2}

^{#1} Sr. Prof., GVP-LIAS, G.V.P. College of Engineering, Madhurawada, Visakhapatnam-530048, India.

^{#2} Asst. Prof., G.V.P. College for Degree and P.G. Courses, School of Engineering, Rushikonda, Visakhapatnam-530045, India.

Abstract - In this paper, we prove a unique common fixed point theorem for six self mappings on a fuzzy metric space using mainly weakly compatible and JCLR properties. This result extends a result of Chauhan et. al. Examples are provided in support of the results.

Keywords - Fuzzy metric space, common fixed point, weakly compatible mappings, JCLR property.

1. INTRODUCTION

The concept of fuzzy set is initiated by Zadeh[1]. The notion of fuzzy metric space is introduced by Kramosil and Michalek[2]. George and Veeramani[3] modified the above notion to get a Hausdorff topology on this space. Sintunavarat and Kumam[4] coined the notion "common limit in the range" property (CLR-property) and obtained common fixed point for a pair of self mappings. Recently, Chauhan along with the above two authors[5] defined the generalized notion "joint common limit in the ranges of mappings" property (JCLR-property) and established common fixed point results for two pairs of self mappings.

In this paper, we establish similar result for six self mappings under suitable hypothesis on the self mappings.

We deduce the results of Chauhan et. al. Further, we gave supporting examples.

We hereunder give the necessary definitions and results needed for a clear understanding of our findings.

2. PRELIMINARIES

Definition 2.1 A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is said to be a continuous triangular (t-) norm if and only if (iff) it satisfies the following:

- (i) $*$ is commutative and associative;
- (ii) $*$ is continuous;
- (iii) $a*1=a$, for all $a \in [0,1]$ and
- (iv) $a*b \leq c*d$ whenever one of a, b is \leq one of c, d and the other of a, b is \leq the other of c, d for all $a, b, c, d \in [0,1]$.

Examples of continuous triangular norms are $a*_1 b = \min\{a,b\}$, $a*_2 b = ab$ and $a*_3 b = \max\{a+b-1, 0\}$.

Definition 2.2 An ordered triple $(X, M, *)$ is said to be a fuzzy metric space iff X is a non empty set, $*$ is a continuous triangular norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following:

- (i) $M(x, y, t) > 0$, for all $t > 0$;
- (ii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;
- (iii) $M(x, y, t) = M(y, x, t)$, for all $x, y \in X$ and for all $t > 0$,
- (iv) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y \in X$ and
- (v) $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$ for all $x, y, z \in X$ and for all $t, s > 0$.

Definition 2.3 $(X, M, *)$ is a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ iff $\lim_{n \rightarrow \infty} M(x_n, y_n, t_n)$ exists and $= M(x, y, t)$ whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ converging to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(y_n, y, t) = 1$$

and $\lim_{n \rightarrow \infty} M(x, y, t_n) = M(x, y, t)$.

Result 2.4 $(X, M, *)$ is a fuzzy metric space. Then $M(x, y, \cdot)$ is monotonic increasing for all $x, y \in X$.

Result 2.5 $(X, M, *)$ is a fuzzy metric space. If there is a $\lambda \in (0, 1)$ such that

$M(x, y, \lambda t) \geq M(x, y, t)$ for all $x, y \in X$ and $t > 0$ then $y = x$.

Definition 2.6 Two self mappings f and g on a non-empty set X are said to be weakly compatible (or coincidentally commuting) iff they commute at their coincidence points; i.e, if $fz = gz$ for some $z \in X$ then $fgz = gzf$.

Definition 2.7 $(X, M, *)$ is a fuzzy metric space and f, g, a, b be self mappings on X . The pairs (f, b) and (a, g) are said to satisfy the "joint common limit in the range of b and g " ($JCLR_{bg}$) property iff there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} b x_n = \lim_{n \rightarrow \infty} a y_n = \lim_{n \rightarrow \infty} g y_n = bu = gu \text{ for some } u \in X.$$

Chauhan et.al[5] proved the following:

Result 2.8 $(X, M, *)$ is a fuzzy metric space, where $*$ is a continuous triangular norm and f, g, a and b are self mappings on X . Further the pairs (f, b) , (a, g) are weakly compatible and there exists a constant

$$k \in (0, \frac{1}{2}) \text{ such that } M(fx, ay, kt) \geq \phi(M(bx, gy, t), M(fx, bx, t), M(ay, gy, t), M(fx, gy, at), M(ay, bx, 2t - at))$$

holds for all $x, y \in X$, $\alpha \in (0, 2)$, $t > 0$ and $\phi: [0, 1]^5 \rightarrow [0, 1]$ is continuous and monotonic increasing function in any coordinate and $\phi(t, t, t, t, t) \geq t$ for all $t \in [0, 1]$. If (f, b) and (a, g) satisfy the $JCLR_{bg}$ -property, then f, g, a and b have a unique common fixed point in X .

Remark 2.9 In the above result:

(i) $\lambda \in (0, \frac{1}{2})$ is not necessary, λ can be in $(0, 1)$;

(ii) $\alpha \in (0, 2)$ is not necessary; we can take 1 in the place of α in the inequality.

Now, we prove a unique common fixed point result for six self mappings on a fuzzy metric space; we deduce Result (2.8) from our extended result.

3. MAIN RESULT

Theorem 3.1 $(X, M, *)$ is a fuzzy metric space, where $*$ is a continuous triangular norm and f, g, h, k, p and q be self mappings on X satisfying the following:

(i) the ordered pairs (fh, q) and (p, gk) satisfies $JCLR_{q(gk)}$ property;

(ii) the pairs $\{fh, q\}$ and $\{p, gk\}$ are weakly compatible mappings;

(iii) $fh = hf$ and either $fq = qf$ or $hq = qh$;

(iv) $gk = kg$ and either $gp = pg$ or $kp = pk$;

(v) there is a $\lambda \in (0, 1)$ such that

$$M(fhx, py, \lambda t) \geq \phi(M(qx, gky, t), M(fhx, qx, t), M(py, gky, t), M(fhx, gky, t), M(py, qx, t))$$

for all $x, y \in X$ and $t > 0$, where $\phi: [0, 1]^5 \rightarrow [0, 1]$ is continuous, increasing in each variable and $\phi(t, t, t, t, t) \geq t$ for all $t \in [0, 1]$.

Then f, g, h, k, p and q have a unique common fixed point in X .

Proof: Under the given hypothesis (i), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} q x_n = \lim_{n \rightarrow \infty} p y_n = \lim_{n \rightarrow \infty} g k y_n = qu = gku = z(\text{say}) \text{ for some } u \in X$$

We now show that $pu = z$.

Taking $x = x_n$ and $y = u$ in (v), we get that

$$M(fhx_n, pu, \lambda t) \geq \phi(M(qx_n, gku = z, t), M(fhx_n, qx_n, t), M(pu, gku = z, t), M(fhx_n, gku = z, t), M(pu, qx_n, t))$$

Now, as $n \rightarrow \infty$, the above inequality becomes

$$M(z, pu, \lambda t) \geq \phi(M(z, z, t), M(z, z, t), M(pu, z, t), M(z, z, t), M(pu, z, t))$$

$$M(z, pu, \lambda t) \geq \phi(1, 1, M(pu, z, t), 1, M(pu, z, t))$$

$$M(z, pu, \lambda t) \geq \phi(M(z, pu, t), M(z, pu, t), M(z, pu, t), M(z, pu, t), M(z, pu, t))$$

$$\geq M(z, pu, t) \text{ (by the property of } \phi), \text{ for all } t > 0.$$

By Result (2.5), it follows that $pu = z (= qu = gku)$.

Similarly, by taking $x = u$ and $y = y_n$ in (v), we get that $fhu = z$. Thus $fhu = gku = pu = qu = z$.

By (ii), it follows that $(fh)qu=q(fhu)$ and $(gk)pu=p(gku)$. i.e, $fhz=qz$ and $gkz=pz$.

We now show that $fhz = z$. Taking $x = z$ and $y = u$ in (v), we get that

$$\begin{aligned} M(fhz, pu = z, \lambda t) &\geq \phi(M(qz, gku = z, t), M(fhz, qz, t), M(pu = z, gku = z, t), \\ &\quad M(fhz, gku = z, t), M(pu = z, qz, t)) \\ &= \phi(M(fhz, z, t), 1, 1, M(fhz, z, t), M(z, fhz, t)) \\ &\geq \phi(M(fhz, z, t), M(fhz, z, t), M(fhz, z, t), \\ &\quad M(fhz, z, t), M(fhz, z, t)) \\ &\geq M(fhz, z, t). \end{aligned}$$

$\Rightarrow fhz = z$. Hence $qz = fhz = z$.

Since $fh = hf$, we have $z = fhz = hfz$. i.e, $hfz = fhz = qz = z$.

Suppose f commutes with q ; so $fq = qf$. Hence $qfz = fqz = fz$.

Since $fh = hf$, we have $fhfz = f(hfz) = fz$. We now show that $fz = z$.

Taking $x = fz$ and $y = u$ in (v), we get that

$$\begin{aligned} M(fz, z, \lambda t) &\geq \phi(M(fz, z, t), M(fz, fz, t), M(z, z, t), \\ &\quad M(fz, z, t), M(z, fz, t)) \\ &= \phi(M(fz, z, t), 1, 1, M(fz, z, t), M(z, fz, t)) \\ &\geq \phi(M(fz, z, t), M(fz, z, t), M(fz, z, t), \\ &\quad M(fz, z, t), M(fz, z, t)) \\ &\geq M(fz, z, t). \end{aligned}$$

$\Rightarrow fz = z$. Hence $hz = fz = qz = z$.

Similarly, if h commutes with q , then by taking $x = hz$ and $y = u$ in (v), we get that $hz = fz = qz = z$.

We now show that $gkz = z$. By taking $x = u$ and $y = z$ in (v), we get that $gkz = z$.

Since g commutes with k , $z = gkz = kgz$. i.e, $z = kgz = gkz = pz$.

Suppose g commutes with p , so $gp = pg$. Hence $pgz = gpz = gz$.

Since $gk = kg$, we have $gkgz = g(gkz) = gz$. Taking $x = u$ and $y = gz$ in (v), we get that $gz = z$. Since $gkz = z$, $gkz = kgz = kz = z$. Thus $gz = kz = pz = z$.

Similarly, if k commutes with p , then by taking $x = u$ and $y = kz$ in (v), we get that $kz = z$. Therefore $fz = gz = hz = kz = pz = qz = z$.

Hence z is a common fixed point of f, g, h, k, p and q in X .

If w is also a common fixed point of f, g, h, k, p and q in X , then by taking $x = z$ and $y = w$ in (v), we get that $w = z$.

This completes the proof of the theorem.

Remark 3.2 Take $h = k = I$ (the identity mapping on X) and $p = a, q = b$, we get Result(2.8) as a Corollary of our Theorem(3.1).

We hereunder give an example in support of our Theorem.

Example 3 $(X, M, *)$ is a fuzzy metric space, where $X = [0, \infty)$ with the usual metric and $M: X^2 \times [0, \infty) \rightarrow [0, 1]$ is

$$\text{defined by } M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X, t > 0$ and $*$ is the min t -norm, i.e, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let f, g, h, k, p and q be the self mappings on X , defined by

$$\begin{aligned} f(x) &= \begin{cases} 0 & \text{if } x \leq 9 \\ 1 & \text{if } x > 9 \end{cases}, \quad gx = x^2, \quad hx = 0, \quad kx = x, \\ p(x) &= \begin{cases} 0 & \text{if } x \leq 9 \\ 4 & \text{if } x > 9 \end{cases} \text{ and } qx = x \text{ for all } x \in X. \end{aligned}$$

Define $\phi: [0, 1]^5 \rightarrow [0, 1]$ by

$$\phi\{x_1, x_2, x_3, x_4, x_5\} = \min\{x_1, x_2, x_3, x_4, x_5\}.$$

(a) $x \in X$ and $y \leq 9$.

Since $hx = 0$ and $py = 0$, we get that the L.H.S of (v) in the Theorem is $M(0, 0, \lambda t) = 1$.

Clearly L.H.S of (v) is \geq the R.H.S of (v).

(b) $x \in X$ and $y > 9$.

Since $hx = 0$ and $py = 4$, we get that

the L.H.S of (v) in the Theorem is

$$M(0, 4, \lambda t) = \frac{t}{t + \frac{4}{\lambda}}$$

R.H.S is

$$\begin{aligned} \min\{M(x, y^2, t), M(0, x, t), M(4, y^2, t), M(0, y^2, t), M(4, x, t)\} \\ \leq M(0, y^2, t) < \frac{1}{t+81} \quad (Q \ y > 9). \end{aligned}$$

In order to show that the L.H.S of the inequality is \geq R.H.S, it is sufficient to

show that $\frac{t}{t + \frac{4}{\lambda}} \geq \frac{t}{t + 81}$ and this happens if

$$81 \geq \frac{4}{\lambda} \Leftrightarrow \lambda \geq \frac{4}{81}.$$

Taking $\lambda \in [\frac{4}{81}, 1)$, we get that (v) of

Theorem (3.1) is satisfied. Clearly f, g, h, k, p and q satisfy the hypothesis of Theorem(3.1) and 0 is the unique common fixed point of f, g, h, k, p and q in X .

Remark 3.4 For the Theorem (3.1) of Chauhan et. al[5], they have not provided an example. We hereunder provide an example to enhance the value of the Result.

Example 3.5 X, M and $*$ are as in the Example (3.3). f, g, p and q are defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{cases}, \quad gx = x^2,$$

$$p(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 4 & \text{if } x > 2 \end{cases}, \quad \text{and } qx = x \text{ for all } x \in X.$$

Define $\phi : [0, 1]^5 \rightarrow [0, 1]$ by

$$\phi\{x_1, x_2, x_3, x_4, x_5\} = \min\{x_1, x_2, x_3, x_4, x_5\}.$$

REFERENCES

- [1] L.A.Zadeh, *Fuzzy sets*, Inform and Control, 8(1965), 338-353.
- [2] O.Kramosil and J.Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika, 11(1975), 336-344.
- [3] A. George and P. Veeramani, *On some result in Fuzzy metric space*, Fuzzy Sets and Systems, Vol. 64, No. 3, (1994), 395-399.
- [4] Sinturavarant and Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, Journal of Applied Mathematics, Article ID 637958 (2011).
- [5] Sunny Chauhan, Wutiphol Sintunavarat and Poom Kumam, *Common fixed point theorems for weakly compatible mappings in fuzzy metric space using (JCLR) property*, Applied Mathematics, 3(2012), 976-982.