On Fixed Fuzzy Point Theorem in Fuzzy Length Space and Finite Dimension Fuzzy Length Spaces

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Abstract

In this paper we recall the definition of fuzzy length space on a fuzzy set after that we recall basic definitions and properties of this space. Then we prove the fixed fuzzy point theorems for a fuzzy complete fuzzy length spaces on a fuzzy set. Are finite dimensional fuzzy length spaces simpler than infinite dimensional ones in this direction we proved some properties that does not satisfied in the infinite case.

Keywords

Fuzzy length space on fuzzy set, Fuzzy fixed point, uniformly fuzzy continuous operator, Fuzzy contractive operator.

1.INTRODUCTION

The theory of fuzzy set was introduced by Zadeh in 1965[1]. In 1984[2], Katsaras is the first one who introduced the notion of fuzzy norm on a linear space during his studying the notion fuzzy topological vector spaces. In 1984 Kaleva and Seikkala [3] introduced a fuzzy metric space. In 1992 Felbin [4] introduced the notion of fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kaleva and

Seikkala type Kramosil and Michalek introduced another idea of fuzzy metric space [5]. In 1994 Cheng and Mordeson [6] introduced the notion fuzzy norm on a linear space so that the corresponding fuzzy metric is of Kramosil and Michalek type. Bag and Samanta [7] in 2003 studied finite dimensional fuzzy normed linear spaces. In 2005 Saadati and Vaezpour [8] studied some results on fuzzy complete fuzzy normed spaces. In 2005 Bag and Samanta [9] studied fuzzy bounded linear proved the fixed point theorems on fuzzy normed linear spaces. In 2009 Sadeqi and Kia [10] studied fuzzy normed linear space and its topological structure. In 2010 Si, Cao and Yang [11] studied the continuity in an intuitionistic fuzzy normed space. In 2015 Nadaban [12] studied properties of fuzzy continuous mapping on a fuzzy normed linear spaces. The concept of fuzzy norm has been used in developing the fuzzy functional analysis and its applications and a large number of papers by different authors have been published for reference please see [13,14,15,16,17,18,19,20,21,22].

In the present paper we recall the definition of fuzzy length on a fuzzy set. The structure of this paper is as follows: In section two we recall basic properties of fuzzy length space on a fuzzy set that's will be needed later. In section three we try to prove fixed fuzzy point theorem in fuzzy length spaces. Finally in the last section we study finite dimensional fuzzy length spaces because there are some properties satisfied in a finite dimensional fuzzy length spaces but it is not true in infinite case.

2. Basic Concept about fuzzy set

Definition 2.1:[1]

Let V be a nonempty set of elements, a fuzzy set $\tilde{A}$ in V is characterized by a membership function, $\mu_{\tilde{A}}(x)$: $V \rightarrow [0,1]$. Then we can write $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in V, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}$.

Definition 2.2:[11] Suppose that $\tilde{D}$ and $\tilde{B}$ be two fuzzy sets in $V \neq \emptyset$ and $W \neq \emptyset$ respectively then $\tilde{D} \times \tilde{B}$ is a fuzzy set whose membership is defined by: $\mu_{\tilde{D} \times \tilde{B}}(d, b) = \mu_{\tilde{D}}(d) \square \mu_{\tilde{B}}(b)$ $\forall (d, b) \in V \times W$.

Definition 2.3:[18]

A fuzzy point p in U is a fuzzy set with single element and is denoted by x_α or (x, α).

Two fuzzy points x_α and y_β are said to be different if and only if $x \neq y$.

Definition 2.4: [7]

Suppose that d_β is a fuzzy point and $\tilde{D}$ is a fuzzy set in U. then d_β is said to belong to $\tilde{D}$ which is written by $d_\beta \in \tilde{D} \Leftrightarrow \mu_{\tilde{D}}(x) > \beta$.

Proposition 2.5:[22]
Suppose that h: V→ W is a function. Then the image of the fuzzy point d_β in V, is the fuzzy point h(d_β) in W with h(d_β)=(h(d), β).

Definition 2.6:[23]

A binary operation $*$: $[0,1]^2 \to [0,1]$ is said to be t-norm (or continuous triangular norm) if $\forall$ p, q, t, r ∈ [0, 1] the conditions are satisfied : (i) $p*q = q*p$ (ii) $p*1 = p$ (iii) $(p*q)*t = p*(q*t)$ (iv) If $p \leq q$ and $t \leq r$ then $p*t \leq q*r$.

Examples 2.7:[23]

When $p*q = p.q$ and $p*q = p \wedge q$ $\forall$p, q ∈[0, 1] then $*$ is a continuous t-norm.

Remark 2.8:[23]

$\forall$ p>q, there is t such that $p*t \geq q$ and for every r, there is e such that $r*r \geq e$, where p, q, t, r and e belongs to [0, 1].

First we recall the main definition in this paper

Definition 2.9:[24]

Let U be a linear space over field $\mathbb{F}$ and let $\tilde{A}$ be a fuzzy set in X. let $*$ be a t-norm and $\tilde{F}$ be a fuzzy set from $\tilde{A}$ to [0,1] such that:

(FL_1) $\tilde{F}(x_\alpha) > 0$ for all $x_\alpha \in \tilde{A}$.

(FL_2) $\tilde{F}(x_\alpha) = 1$ if and only if $x_\alpha = 0$.

(FL_3) $\tilde{F}(cx, \alpha) = \tilde{F}(x, \frac{\alpha}{|c|})$, where $0 \neq c \in \mathbb{F}$.

(FL_4) $\tilde{F}(x_\alpha + y_\beta) \geq \tilde{F}(x_\alpha) * \tilde{F}(y_\beta)$.

(FL_5) $\tilde{F}$ is a continuous fuzzy set for all $x_\alpha, y_\beta \in \tilde{A}$ and α, β ∈[0,1].

Then the triple $(\tilde{A}, \tilde{F}, *)$ is called a fuzzy length space on the fuzzy set $\tilde{A}$.

Definition 2.10:[24]

Suppose that $(\tilde{D}, \tilde{F}, *)$ is a fuzzy length space on the fuzzy set $\tilde{D}$ then $\tilde{F}$ is continuous fuzzy set if whenever $\{(x_n, \alpha_n)\} \to x_\alpha$ in $\tilde{D}$ then $\tilde{F}\{(x_n, \alpha_n)\} \to \tilde{F}(x_\alpha)$ that is $\lim_{n\to\infty} \tilde{F}[(x_n, \alpha_n)] = \tilde{F}(x_\alpha)$.

Proposition 2.11:[24]

Let $(U, \|.\|)$ be a normed space, suppose that $\tilde{D}$ is a fuzzy set in U. Put $\|x_\alpha\| = \|x\|$. Then $(\tilde{D}, \|.\|)$, is a normed space.

Example 2.12:[24]

Suppose that $(U, \|.\|)$ is a normed space and assume that $\tilde{D}$ is a fuzzy set in U. Put $p * q = p.q$ for all p, q $\in [0,1]$. Define $\tilde{F}_{\|.\|}(x_\alpha) = \frac{\alpha}{\alpha + \|x\|}$.Then $(\tilde{D}, \tilde{F}_{\|.\|}, *)$ is a fuzzy length space on the fuzzy set $\tilde{D}$, is called the fuzzy length induced by $\|.\|$

Definition 2.13:[24]

Let $\tilde{A}$ be a fuzzy set in U, and assume that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space on the fuzzy set $\tilde{A}$ .let $\tilde{B}(x_\alpha, r)$= $\{y_\beta \in \tilde{A} : \tilde{F}(y_\beta - x_\alpha) > (1-p)\}$.So $\tilde{B}(x_\alpha, p)$ is said to be a fuzzy open fuzzy ball of center $x_\alpha \in \tilde{A}$ and radius r.

Definition 2.14:[24]

The sequence $\{(x_n, \alpha_n)\}$ in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$ on the fuzzy set $\tilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if for a given ε, $0 < \varepsilon < 1$, then there exists a positive number K such that $\tilde{F}[(x_n, \alpha_n) - x_\alpha] > (1-\varepsilon)$ $\forall$ n ≥ K.

Theorem 2.15:[24]

The sequence $\{(x_n, \alpha_n)\}$ in a fuzzy length space $(\tilde{A}, \tilde{F}, *)$ on the fuzzy set $\tilde{A}$ is fuzzy converges to a fuzzy point $x_\alpha \in \tilde{A}$ if and only if $\lim_{n\to\infty} \tilde{F}[(x_n, \alpha_n) - x_\alpha] = 1$.

Lemma 2.16:[24]

Suppose that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space on the fuzzy set $\tilde{A}$. Then $\tilde{F}(x_\alpha - y_\beta) = \tilde{F}(y_\beta - x_\alpha)$, for all $x_\alpha, y_\beta \in \tilde{A}$.

Definition 2.17:[24]

Suppose that $(\tilde{A}, \tilde{F}, *)$ is a fuzzy length space and $\tilde{D} \subseteq \tilde{A}$ then $\tilde{D}$ is called fuzzy open if for every y_β

$\in \tilde{D}$ there is $\tilde{B}(y_\beta,q) \subseteq \tilde{D}$. A subset $\tilde{E} \subseteq \tilde{A}$ is called fuzzy closed if $\tilde{E}^c = \tilde{A} - \tilde{E}$ is fuzzy open.

Theorem 2.18:[24]

Any $\tilde{B}(y_\beta,q)$ in a fuzzy length space $(\tilde{A},\tilde{F},*)$ is a fuzzy open.

Definition 2.19: [24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space, and assume that $\tilde{D} \subseteq \tilde{A}$.Then the fuzzy closure of $\tilde{D}$ is denoted by $\bar{\tilde{D}}$ or FC($\tilde{D}$) and is defined by $\bar{\tilde{D}}$ is the smallest fuzzy closed fuzzy set that contains $\tilde{D}$.

Definition 2.20:[24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space, and assume that $\tilde{D} \subseteq \tilde{A}$.Then $\tilde{D}$ is said to be fuzzy dense in $\tilde{A}$ if $\bar{\tilde{D}} = \tilde{A}$ or $FC(\tilde{D}) = \tilde{A}$.

Lemma 2.21:[24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space, and assume that $\tilde{D} \subseteq \tilde{A}$, Then $d_\alpha \in \bar{\tilde{D}}$ if and only if we can find $\{(d_n, \alpha_n)\}$ in $\tilde{D}$ such that $(d_n, \alpha_n) \to d_\alpha$.

Theorem 2.22:[24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space and let that $\tilde{D} \subseteq \tilde{A}$, then $\tilde{D}$ is fuzzy dense in $\tilde{A}$ if and only if for any $a_\alpha \in \tilde{A}$ we can find $d_\beta \in \tilde{D}$ with $\tilde{F}[a_\alpha - d_\beta] > (1-\varepsilon)$ for some $0 < \varepsilon < 1$.

Definition 2.23:[24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space. A sequence of fuzzy points $\{(x_n, \alpha_n)\}$ is said to be a fuzzy Cauchy if for any given ε, $0 < \varepsilon < 1$, there is a positive number K such that $\tilde{F}[(x_n,\alpha_n) - (x_m,\alpha_m)] > (1-\varepsilon)$ for all n, m $\geq$K.

Definition 2.24:[24]

Suppose that $(\tilde{A},\tilde{F},*)$ is a fuzzy length space and $\tilde{D} \subseteq \tilde{A}$. Then $\tilde{D}$ is said to be fuzzy bounded if we can find q ,$0 < q < 1$ such that, $\tilde{F}(x_\alpha) > (1-q)$, $\forall x_\alpha \in \tilde{A}$.

Definition 2.25:[24]

Let $(\tilde{A},\tilde{F}_{\tilde{A}},*)$ and $(\tilde{D},\tilde{F}_{\tilde{D}},*)$ be two fuzzy length space on fuzzy set $\tilde{A}$ and $\tilde{D}$ respectively, let $\tilde{E} \subseteq \tilde{A}$ then The operator T: $\tilde{E} \to \tilde{D}$ is said to be fuzzy continuous at $a_\alpha \in \tilde{E}$, if for every $0 < \varepsilon < 1$, there exist $0 < \delta < 1$, such that $\tilde{F}_{\tilde{D}}[T(x_\beta) - T(a_\alpha)] > (1-\varepsilon)$ whenever $x_\beta \in \tilde{E}$ satisfying $\tilde{F}_{\tilde{A}}(x_\beta - a_\alpha) > (1-\delta)$. If T is fuzzy continuous at every fuzzy point of $\tilde{E}$, then T it is said to be fuzzy continuous on $\tilde{E}$.

Theorem 2.26:[24]

Let $(\tilde{A},\tilde{F}_{\tilde{A}},*)$ and $(\tilde{D},\tilde{F}_{\tilde{D}},*)$ be two fuzzy length space, let $\tilde{E} \subseteq \tilde{A}$ The operator T: $\tilde{E} \to \tilde{D}$ is fuzzy continuous at $a_\alpha \in \tilde{E}$ if and only if whenever a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{E}$ fuzzy converge to a_α, then the sequence of fuzzy points $\{(T(x_n), \alpha_n)\}$ fuzzy converges to $T(a_\alpha)$.

Theorem 2.27:[24]

An operator $T: \tilde{A} \to \tilde{D}$ is fuzzy continuous if and only if $T^{-1}(\tilde{G})$ is fuzzy open in $\tilde{A}$ for all fuzzy open $\tilde{G}$ of $\tilde{D}$ where $(\tilde{A},\tilde{F}_{\tilde{A}},*)$ and $(\tilde{D},\tilde{F}_{\tilde{D}},*)$ are fuzzy length space

Theorem 2.28:[24]

The fuzzy length space $(\tilde{A},\tilde{F},*)$ is fuzzy compact if and only if every sequence of fuzzy points in $\tilde{A}$ has a subsequence fuzzy converging to a fuzzy point in $\tilde{A}$.

Definition 2.29:[24]

Let $(\tilde{A},\tilde{F}_{\tilde{A}},*)$ and $(\tilde{D},\tilde{F}_{\tilde{D}},*)$ be two fuzzy length space. An operator $T: \tilde{A} \to \tilde{D}$ is said to be uniformly fuzzy continuous on $\tilde{A}$ if for every $\varepsilon, 0 < \varepsilon < 1$ there exist $\delta, 0 < \delta < 1$ [depending on ε alone] such that

$\tilde{F}_{\tilde{B}}[T(y_\beta)-T(x_\alpha)] > (1-\varepsilon)$ whenever $\tilde{F}_{\tilde{A}}\,[y_\beta - x_\alpha] > (1-\delta)$.

3. Fixed Fuzzy Point Theorem in Fuzzy Length Space.

Definition 3.1:

Let $\tilde{A}$ be a fuzzy set in X and let $f\colon \tilde{A} \to \tilde{A}$ be a function. The fuzzy point $x_\alpha \epsilon \tilde{A}$ is said to be fixed fuzzy point if $f(x_\alpha) = x_\alpha$.

Proposition 3.2:

Let $(\tilde{A}, \tilde{F}\ ,*)$ be a fuzzy length space and let $f\colon \tilde{A} \to \tilde{A}$ be an operator. Then f is uniformly fuzzy continuous if and only if for each $\delta > 0$ there exists $\sigma > 0$ such that $\frac{1}{\tilde{F}[x-y,\lambda\,]} - 1 \le \sigma$ implies $\frac{1}{\tilde{F}[f(x)-f(y),\lambda\,]} - 1 \le \delta$ for each $x_\alpha, y_\beta \in \tilde{A}$ with $\lambda = \alpha \wedge \beta$.

Proof:

Assume that f is uniformly fuzzy continuous that is for each $\varepsilon, 0 < \varepsilon < 1$ there exists $\delta, 0 < \delta < 1$ such that $\tilde{F}[x-y,\lambda\,] \ge (1-\delta)$ implies that $\tilde{F}[f(x)-f(y),\lambda\,] \ge (1-\varepsilon)$ for each $x_\alpha, y_\beta \in \tilde{A}$ with $\lambda = \alpha \wedge \beta$.

Now $\frac{1}{\tilde{F}[x-y,\lambda\,]} - 1 \le \left(\frac{1}{1-\delta} - 1\right)$ implies $\frac{1}{\tilde{F}[f(x)-f(y),\lambda\,]} - 1 \le \left(\frac{1}{1-\varepsilon} - 1\right)$ for each $x_\alpha, y_\beta \in \tilde{A}$ .put $\left(\frac{1}{1-\delta} - 1\right) = \sigma$ and $\left(\frac{1}{1-\varepsilon} - 1\right) = \mu$. Hence $\frac{1}{\tilde{F}[x-y,\lambda\,]} - 1 \le \sigma$ implies $\frac{1}{\tilde{F}[f(x)-f(y),\lambda\,]} - 1 \le \mu$ for each $x_\alpha, y_\beta \in \tilde{A}$.

Conversely, suppose that $\frac{1}{\tilde{F}[x-y,\lambda\,]} - 1 \le \sigma$ implies $\frac{1}{\tilde{F}[f(x)-f(y),\lambda\,]} - 1 \le \delta$ for each $x_\alpha, y_\beta \in \tilde{A}$, $\lambda = \alpha \wedge \beta$, now $\tilde{F}[x-y,\lambda\,] \ge \frac{1}{1+\sigma}$, $0 < \sigma < 1$

Implies $\tilde{F}[f(x)-f(y),\lambda\,] \ge \frac{1}{1+\delta}$, $0 < \delta < 1$ for all $x_\alpha, y_\beta \in \tilde{A}$. Put $\frac{1}{1+\sigma} = (1-\varepsilon)$, $0 < \varepsilon < 1$ and $\frac{1}{1+\delta} = (1-\mu)$, $0 < \mu < 1$. Hence $\tilde{F}[x-y,\lambda\,] \ge (1-\varepsilon)$ implies $\tilde{F}[f(x)-f(y),\lambda\,] \ge (1-\mu)$ for all $x_\alpha, y_\beta \in \tilde{A}$. That is f is uniformly fuzzy continuous.

Definition 3.3:

Let $(\tilde{A}, \tilde{F}\ ,*)$ be a fuzzy length space then the operator $f\colon \tilde{A} \to \tilde{A}$ is said to be fuzzy contractive if there exists $0 < k < 1$ such that $\frac{1}{\tilde{F}[f(x_\alpha)-f(y_\beta)\,]} - 1 \le k\left[\frac{1}{\tilde{F}[x_\alpha - y_\beta\,]} - 1\right]$ for each $x_\alpha, y_\beta \in \tilde{A}$. [k is called contractive constant].

Proposition 3.4:

Let $(\tilde{A}, \tilde{F}\ ,*)$ be a fuzzy length space. If $f\colon \tilde{A} \to \tilde{A}$ is fuzzy contractive operator then f is uniformly fuzzy continuous.

Proof:

Since f is fuzzy contractive we have $\frac{1}{\tilde{F}[f(x_\alpha)-f(y_\beta)\,]} - 1 \le k\left[\frac{1}{\tilde{F}[x_\alpha - y_\beta\,]} - 1\right]$ for any $x_\alpha, y_\beta \in \tilde{A}$ .Then $\tilde{F}[f(x_\alpha)-f(y_\beta)\,] - 1 \ge \frac{1}{k}[\tilde{F}(x_\alpha - y_\beta) - 1]$. Now if there is $0 < r < 1$ such that $\tilde{F}[x_\alpha - y_\beta] \ge (1-r)$ $\tilde{F}[x_\alpha - y_\beta] - 1 \ge -r$ so $\frac{1}{k}[\tilde{F}(x_\alpha - y_\beta) - 1] \ge -\frac{r}{k}$. Hence $\tilde{F}[f(x_\alpha)\ - f(y_\beta)] - 1 \ge -\frac{r}{k}$ or $\tilde{F}[f(x_\alpha)\ - f(y_\beta)] \ge 1 - \frac{r}{k}$. $Put\ 1 - \frac{r}{k} = 1 - \varepsilon$ for some

$0 < \varepsilon < 1$.

Therefore $\tilde{F}\left[f(x_\alpha) - f(y_\beta)\right] \geq (1-\varepsilon)$ for all $x_\alpha, y_\beta \in \tilde{A}$. Hence f is uniformly fuzzy continuous.

Definition 3.5:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space. We will say that the sequence of fuzzy points $\{(x_n, \alpha_n)\}$ is fuzzy contractive if there exists $0 \leq k \leq 1$ such that

$\frac{1}{\tilde{F}[x_{n+1} - x_{n+2}, \lambda]} - 1 \leq k\left[\frac{1}{\tilde{F}[x_n - x_{n+1}, \lambda]}\right]$. For all $n \in \mathbb{N}$ where $\lambda = \min\{\alpha_n, n \in \mathbb{N}\}$

Theorem 3.6:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy complete fuzzy length space in which fuzzy contractive sequence of fuzzy points are fuzzy Cauchy. Let $T: \tilde{A} \to \tilde{A}$ be a fuzzy contractive operator with K the contractive constant. Then T has unique fixed fuzzy point.

Proof:

Fix $x_\alpha \epsilon \tilde{A}$, let $(x_n, \alpha_n) = T^n(x_\alpha), n \in \mathbb{N}$ .we have

$\frac{1}{\tilde{F}[T(x_\alpha) - T^2(x_\alpha))]} - 1 \leq k\left[\frac{1}{\tilde{F}[x - x_1, \lambda]} - 1\right]$. And by induction

$$\frac{1}{\tilde{F}[x_{n+1} - x_{n+2}, \lambda]} - 1 \leq k\left[\frac{1}{\tilde{F}[x_n - x_{n+1}, \lambda]}\right], n \in \mathbb{N}$$

Then $\{(x_n, \alpha_n)\}$ is a fuzzy contractive sequence of fuzzy points, so it is fuzzy Cauchy and hence $\{(x_n, \alpha_n)\}$ fuzzy converges to y_β for some $y_\beta \in \tilde{A}, 0 < \beta < 1$. We will see that y_β is a fixed fuzzy point for T. Now we have

$\frac{1}{\tilde{F}[T(y) - T(x_n), \lambda]} - 1 \leq k\left[\frac{1}{\tilde{F}[y - x_n, \lambda]} - 1\right] \to 0$

as $n \to \infty$

$.\lim_{n\to\infty} \tilde{F}[T(y) - T(x_n), \lambda] = 1$ and therefore

$\lim_{n\to\infty} T(x_n, \alpha_n) =$
$T(y_\beta)$ $i.e$ $\lim_{n\to\infty}(x_{n+1}, \alpha_{n+1}) = T(y_\beta)$

then $T(y_\beta) = y_\beta$.To show uniqueness, assume that $T(z_\beta) = z_\beta$ for some $z_\beta \in \tilde{A}$ then

$\frac{1}{\tilde{F}[y_\beta - z_\beta]} - 1 = \left[\frac{1}{\tilde{F}[T(y_\beta) - T(z_\beta)]}\right] - 1 \leq$
$k\left[\frac{1}{\tilde{F}[y_\beta - z_\beta]} - 1\right] \leq k\left[\frac{1}{\tilde{F}[T(y_\beta) - T(z_\beta)]} - 1\right]$

$\leq k^2\left[\frac{1}{\tilde{F}[y_\beta - z_\beta]} - 1\right] \leq \cdots \leq k^n\left[\frac{1}{\tilde{F}[y_\beta - z_\beta]} - 1\right] \to 0$ as $n \to \infty$

Hence $\tilde{F}[y_\beta - z_\beta] = 1$ and $y_\beta = z_\beta$.

Definition 3.7:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space, we say that the mapping $f: \tilde{A} \to \tilde{A}$ is fuzzy contractive if there exists $0 < k < 1$ such that $\tilde{F}[T(x) - T(y), kt] \geq \tilde{F}[x - y, t]$ where $t = \alpha \wedge \beta$.

Notation 3.8:

The infinite product

(1)$\tilde{F}[x - y, t] * \tilde{F}[x - y, t] * \ldots *$ in the fuzzy length space $(\tilde{A}, \tilde{F}, *)$ by $\prod_{i=1}^{\infty} \tilde{F}[x - y, t_i]$, $t_i = t$.

(2)Recall that the usual infinite product of real numbers t_n , $\prod_{i=1}^{\infty} t_i$ is converges if the sequence of the successive products $s_n = \prod_{m=1}^{n} t_m$ is convergent $i.e$ (s_n) converges as $n \to \infty$, to a non zero real number. In this case the sequence of remainders $r_m = \prod_{n=m+1}^{\infty} t_n$ converges to 1, as $m \to \infty$.

Theorem 3.10:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy complete fuzzy length space such that for each $\varepsilon > 0$ and an s-increasing sequence (t_n) there exists $k \in \mathbb{N}$ such that $\prod_{n \geq k} \tilde{F}[x - y, t_n] > (1 - \varepsilon)$. Let

$k, 0 < k < 1$ and let $T: \tilde{A} \to \tilde{A}$ be a mapping satisfies $\tilde{F}[T(x) - T(y), kt] \geq \tilde{F}[x - y, t]$ for all $x_\alpha, y_\beta \in \tilde{A}$, $t = \alpha \wedge \beta$. Then T has a unique fixed fuzzy point.

Proof:

Fix $x_\alpha \epsilon \tilde{A}$ let $(x_n, \alpha_n) = T^n(x_\alpha), n = 1,2, \dots$, we have

$\tilde{F}[x_1 - x_2, \alpha] = \tilde{F}[T(x) - T^2(x), \alpha] \geq \tilde{F}\left[x - T(x), \frac{\alpha}{k}\right]$

and by induction

$\tilde{F}[x_n - x_{n+1}, \alpha] \geq \tilde{F}\left[x - x_1, \frac{\alpha}{k^n}\right], n = 1, 2, \dots$

let $t > 0, \varepsilon > 0$ for $n, m \in \mathbb{N}$, we suppose $n < m$ if we take $s_i > 0, i = n, \dots, m-1$ satisfying $s_n + \dots + s_{m-1} \leq 1$ then

$\tilde{F}[x_n - x_m, \alpha] \geq \tilde{F}[x_n - x_{n+1}, s_n \alpha] * \dots * \tilde{F}[x_{m-1} - x_m, s_{m-1}\alpha] \geq \tilde{F}\left[x - x_1, \frac{s_n \alpha}{k^n}\right] * \dots * \tilde{F}\left[x - x_1, \frac{s_{m-1}\alpha}{k^{m-1}}\right]$

. In the particular, since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, we can take $s_i = \frac{1}{i(i+1)}$ $i = n, \dots, m-1$, and then

$\tilde{F}[x_n - x_m, \alpha] \geq \tilde{F}\left[x - x_1, \frac{\alpha}{n(n+1)k^n}\right] * \dots * \tilde{F}\left[x - x_1, \frac{\alpha}{(m-1)mk^{m-1}}\right] \geq \prod_{n=1}^{\infty} \tilde{F}\left[x - x_1, \frac{\alpha}{n(n+1)k^n}\right]$ now if we write $t_n = \frac{\alpha}{n(n+1)k^n}$ it is an easy to prove that $(t_{n+1} - t_n) \to \infty$ as $n \to \infty$, so (t_n) is an s-increasing sequence and then there exists $k \in \mathbb{N}$ such that

$\prod_{n=k}^{\infty} \tilde{F}\left[x - x_1, \frac{\alpha}{n(n+1)k^n}\right] > (1 - \varepsilon)$ and therefore $\tilde{F}[x_n - x_m, \alpha] > (1 - \varepsilon)$ for all $n, m > K$. Hence $\{(x_n, \alpha)\}$ is fuzzy Cauchy sequence. Since $\tilde{A}$ is fuzzy complete there is $y_\alpha \in \tilde{A}$ such that $\lim_{n\to\infty}(x_n, \alpha) = y_\alpha$. We claim that y_α is a fixed fuzzy point for T .we have

$\tilde{F}[T(y) - y, \alpha] \geq \tilde{F}[T(y) - T(x_n), \alpha] * \tilde{F}[x_{n-1} - y, \alpha] \geq \tilde{F}\left[y - x_n, \frac{\alpha}{k}\right] * \tilde{F}[x_{n-1} - y, \alpha] \to 1 * 1$

as $n \to \infty$ so $\tilde{F}[T(y) - y, \alpha] = 1$ and we get $(T(y), \alpha) = y_\alpha$. To show that uniqueness, let $(T(z), \alpha) = z_\alpha$ for some $z_\alpha \in \tilde{A}$. Then

$1 \geq \tilde{F}[z - y, \alpha] = \tilde{F}\left[T(z) - T(y), \frac{\alpha}{k}\right] \geq \tilde{F}\left[z - y, \frac{1}{k^2}\right] \geq \dots \geq \tilde{F}\left[z - y, \frac{\alpha}{k^n}\right]$

. Now $\left(\frac{\alpha}{k^n}\right)$ is an s-increasing sequence then by assumption for a given $0 < \varepsilon < 1$ there exists $k \in \mathbb{N}$ such that $\prod_{n\geq k} \tilde{F}\left[z - y, \frac{\alpha}{k^n}\right] \geq (1 - \varepsilon)$.

Then $\lim_{n\to\infty} \tilde{F}\left[z - y, \frac{\alpha}{k^n}\right] = 1$. Hence $\tilde{F}[z - y, \alpha] = 1$ and thus $z_\alpha = y_\alpha$.

4. Finite Dimension Fuzzy Length Spaces

Theorem 4.1:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space and let $\{(x_1, \alpha_1), (x_2, \alpha_2), \dots, (x_n, \alpha_n)\}$ be linearly independent fuzzy set of fuzzy vectors in $\tilde{A}$. Then there exists $0 < \lambda < 1$ and $0 < r < 1$ such that for any set of scalars $\{\beta_1, \beta_2, \dots, \beta_n\}$ we have

$\tilde{F}[\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n, \lambda \sum_{j=1}^{n} |\beta_j|] < (1 - r) \dots (2.13)$

Proof:

Let $s = \sum_{j=1}^{n} |\beta_j|$. If $s = 0$ then $\beta_j = 0$ for all j $= 1,2, \dots, n$. So the relation (2.13) holds for any $\lambda = \min\{\beta_j: 1 \leq j \leq n\}$. Then (2.13) is equivalent to

$\tilde{F}[\gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n, \lambda] < (1 - r) \dots (2.14)$

for all γ_j with $\sum_{j=1}^{n} |\gamma_j| = 1$. Assume that (2.14) does not hold. Then there exists a sequence $\left\{\left(y_m, \frac{1}{m}\right)\right\}$ of vectors $y_m = \gamma_1^{(m)} x_1 + \gamma_2^{(m)} x_2 + \dots + \gamma_n^{(m)} x_n$ with

$\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$ such that $\tilde{F}\left[\left(y_m, \frac{1}{m}\right)\right] \geq (1 - \frac{1}{m})$

since $\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$, we have $0 \leq |\gamma_j^{(m)}| \leq 1$ for $j = 1,2,\dots,n$. So for each fixed j the sequence $(\gamma_j^{(m)})$ is bounded and hence $(\gamma_1^{(m)})$ has convergent subsequence, let γ_1 denote the limit of that subsequence and let $\left\{\left(y_{1,m}, \frac{1}{m}\right)\right\}$ denote the corresponding subsequence of $\left\{\left(y_m, \frac{1}{m}\right)\right\}$. By the same argument $\left\{\left(y_{1,m}, \frac{1}{m}\right)\right\}$ has a subsequence $\left\{\left(y_{2,m}, \frac{1}{m}\right)\right\}$ for which the corresponding subsequence of scalars $(\gamma_2^{(m)})$ convergent to γ_2 . Continuing in this way after n steps we obtain a subsequence $\left\{\left(y_{n,m}, \frac{1}{m}\right)\right\}$ where $y_{n,m} = \sum_{j=1}^{n} \sigma_j^{(m)} x_j$ with $\sum_{j=1}^{n} |\sigma_j^{(m)}| = 1$ and $\sigma_j^{(m)} \to \gamma_j$ as $m \to \infty$. Let $y = \gamma_1 x_1 + \cdots + \gamma x_n$ thus we have $\lim_{m\to\infty} \tilde{F}[y_{n,m} - y, \lambda] = 1$ and $\lim_{m\to\infty} \tilde{F}[y_m - y, \lambda]$ =1. Now

$\tilde{F}[y,\lambda] = \tilde{F}[y - y_m + y_m, \lambda] \geq \tilde{F}[y - y_m, \lambda] * \tilde{F}[y_m, \lambda] \geq \lim_{m\to\infty} \tilde{F}[y - y_m, \lambda] * \lim_{m\to\infty} \tilde{F}[y_m, \lambda] \geq 1 * 1 = 1$

. It follows that $= 0$. But since $\sum_{j=1}^{n} |\gamma_j^{(m)}| = 1$ and $\{(x_1, \alpha_1), (x_2, \alpha_2), \dots, (x_n, \alpha_n)\}$ are linearly independent set of fuzzy vectors so $y = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma x_n \neq 0$ this is a contradiction.

Theorem 4.2:

Let $(\tilde{A}, \tilde{F}_1, *)$ be a finite dimensional fuzzy length space. If $\tilde{F}_2$ is another fuzzy length on $\tilde{A}$ then $\tilde{F}_1$ is fuzzy equivalent to $\tilde{F}_2$.

Proof:

Let $\{(e_1, \alpha_1), (e_2, \alpha_2), \dots, (e_k, \alpha_k)\}$ be a basis for $\tilde{A}$ and let (v_m, λ_m) be a sequence in $\tilde{A}$ fuzzy converge to (v, λ) .Let $(v_n, \lambda_n) = (\beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \cdots + \beta_k^{(n)} e_k, \lambda_n)$, $\lambda_n = \min\left\{\beta_j^{(n)} : 1 \leq j \leq k\right\}$. Where $\beta_j^{(n)}$ are scalar, so $\lim_{m\to\infty} \tilde{F}_1[v_m - v, \lambda] = 1$, that is $\lim_{m\to\infty} \tilde{F}_1\left[\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| e_j, \lambda\right] = 1 \dots (2.14)$

. Now from Theorem 2.8.5

$\tilde{F}_1\left[\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| e_j, \lambda \sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right|\right] < (1 - r)$

for some $0 < r < 1$ from (2.14) it follows that there is a positive integer K such that $\tilde{F}_1\left[\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| e_j, \lambda\right] > (1 - r)$ for each $j > K$ .now $\tilde{F}_1\left[\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| e_j, \lambda \sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right|\right]$

for $j > K$. Now, $\lambda \sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| = \lambda$. So $\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| = 1$. Hence $\left|\beta_j^{(m)} - \beta_j\right| < 1$ for all $j > K$. Therefore $\lim_{m\to\infty} \left|\beta_j^{(m)} - \beta_j\right| = 0$. Hence $\left(\beta_j^{(m)}\right) \to \beta_j$ when $m \to \infty$. Now $\tilde{F}_2[v_m - v, \lambda] = \tilde{F}_2\left[\sum_{j=1}^{k} \left|\beta_j^{(m)} - \beta_j\right| e_j, \lambda\right]$, $\tilde{F}_2[v_m - v, \lambda] \geq \tilde{F}_2\left[\left|\beta_1^{(m)} - \beta_1\right| e_1, \lambda\right] * \tilde{F}_2\left[\left|\beta_2^{(m)} - \beta_2\right| e_2, \lambda\right] * \dots * \tilde{F}_2\left[\left|\beta_k^{(m)} - \beta_k\right| e_k, \lambda\right]$

when $m \to \infty$ then $\left(\beta_j^{(m)} - \beta_j\right) \to 0$, $\beta_j^{(m)} \to \beta_j$ for $j = 1,2,\dots,k$. $\lim_{m\to\infty} \tilde{F}_2[v_m - v, \lambda] > 1 * 1 * \dots * 1$. Hence

$\lim_{m\to\infty} \tilde{F}_2[v_m - v, \lambda] = 1$. The proof of the case $(v_m, \lambda_m) \to (v, \lambda)$ in $(\tilde{A}, \tilde{F}_2, *)$ implies $(v_m, \lambda) \to (v, \lambda)$ in $(\tilde{A}, \tilde{F}_1, *)$ is similar hence is omitted.

Theorem 4.3:

Let $(\tilde{A}, \tilde{F}, *)$ be a fuzzy length space. If $\tilde{A}$ is a finite dimensional then $\tilde{A}$ is fuzzy complete.

Proof:

Let $\dim \tilde{A} = K$. Let $\{(e_1, \alpha_1), (e_2, \alpha_2), \dots, (e_k, \alpha_k)\}$ be a basis for $\tilde{A}$. Let $\{(x_n, \alpha)\}$ be a fuzzy Cauchy sequence in $\tilde{A}$. Put $(x_n, \alpha) = \beta_1^{(n)} e_1 + \beta_2^{(n)} e_2 + \dots + \beta_k^{(n)} e_k, \alpha$ where $\alpha = \min\{\alpha_1, \alpha_2, \dots, \alpha_k\}$. Where $\beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_k^{(n)}$ are scalars. Then $\lim_{m,n\to\infty} \tilde{F}[x_m - x_n, \alpha] = 1$. Also by Theorem 4.1 there is $r,\ 0 < r < 1$ $\tilde{F}\left[\sum_{j=1}^k (\beta_j^{(m)} - \beta_j^{(n)}) e_j, \alpha \sum_{j=1}^k \left|\beta_j^{(m)} - \beta_j^{(n)}\right|\right] < (1-r)$. Also there is K such that $\tilde{F}\left[\sum_{j=1}^k (\beta_j^{(m)} - \beta_j^{(n)}) e_j, \alpha\right] > (1-r)$ for all $n \geq K$. Now $\tilde{F}\left[\sum_{j=1}^k (\beta_j^{(m)} - \beta_j^{(n)}) e_j, \alpha\right] \geq (1-r) > \tilde{F}\left[\sum_{j=1}^k \left(\beta_j^{(m)} - \beta_j^{(n)}\right) e_j, \alpha \sum_{j=1}^k \left|\beta_j^{(m)} - \beta_j^{(n)}\right|\right]$ for all $m, n \geq K$. This implies that $\alpha \sum_{j=1}^k \left|\beta_j^{(m)} - \beta_j^{(n)}\right| = \alpha$ or $\sum_{j=1}^k \left|\beta_j^{(m)} - \beta_j^{(n)}\right| = 1$ or thus $\lim_{m,n\to\infty} \left|\beta_j^{(m)} - \beta_j^{(n)}\right| = 0 < 1$ for all $m, n \geq K$ so $\{\beta_j^{(n)}\}$ is a Cauchy sequence of scalars for each $j = 1,2,\dots,k$. Let $\lim_{n\to\infty} \beta_j^{(n)} = \beta_j$ for each $j = 1,2,\dots,k$. Put $(x, \alpha) = (\sum_{j=1}^k \beta_j e_j, \alpha)$. Now for all $n \geq K$ $\tilde{F}[x_n - x, \alpha] = \tilde{F}\left[\sum_{j=1}^k (\beta_j^{(n)} - \beta_j) e_j, \alpha\right] \geq \tilde{F}\left[(\beta_1^{(n)} - \beta_1) e_1, \alpha\right] * \tilde{F}\left[(\beta_2^{(n)} - \beta_2) e_2, \alpha\right] * \dots * \tilde{F}\left[(\beta_k^{(n)} - \beta_k) e_k, \alpha\right]$

$\lim_{n\to\infty} \tilde{F}[x_n - x, \alpha] \geq \lim_{n\to\infty} \tilde{F}\left[(\beta_1^{(n)} - \beta_1) e_1, \alpha\right] * \lim_{n\to\infty} \tilde{F}\left[(\beta_2^{(n)} - \beta_2) e_2, \alpha\right] * \dots * \lim_{n\to\infty} \tilde{F}\left[(\beta_k^{(n)} - \beta_k) e_k, \alpha\right]$. Hence $\lim_{n\to\infty} \tilde{F}[x_n - x, \alpha] \geq 1 * 1 * \dots * 1$ or $\lim_{n\to\infty} \tilde{F}[x_n - x, \alpha] = 1$. This implies that $(x_n, \alpha_n) \to (x, \alpha)$. Thus $\tilde{A}$ is fuzzy complete.

Theorem 4.5:

Let $(\tilde{A}, \tilde{F}, *)$ be a finite dimensional fuzzy length space. A subset $\tilde{C}$ of $\tilde{A}$ is fuzzy compact if and only if $\tilde{C}$ is fuzzy closed and fuzzy bounded.

Proof:

First we assume that $\tilde{C}$ is fuzzy compact and let $x_\alpha \in \bar{\tilde{C}}$ then there exists a sequence of fuzzy points $\{(x_n, \alpha_n)\}$ in $\tilde{C}$ such that $(x_n, \alpha_n) \to x_\alpha$ by lemma (2.22). But $\tilde{C}$ is fuzzy compact then there exists a subsequence $\{(x_{n_k}, \alpha_{n_k})\}$ of $\{(x_n, \alpha_n)\}$ fuzzy converges to a fuzzy point of $\tilde{C}$. Again $(x_n, \alpha_n) \to x_\alpha$ so $\tilde{C}$ is fuzzy closed. Now suppose that $\tilde{C}$ is not fuzzy bounded then there exists $r,\ 0 < r < 1$ such that for each positive integer n there exists $\left(x_n, \frac{n}{1+n}\right) \in \tilde{C}$ such that $\tilde{F}\left[x_n, \frac{n}{1+n}\right] \leq (1-r)$. Since $\tilde{C}$ is fuzzy compact there is a subsequence $\left\{\left(x_{n_k}, \frac{n_k}{1+n_k}\right)\right\}$ of $\left\{\left(x_n, \frac{n}{1+n}\right)\right\}$ fuzzy converges to some fuzzy element $x_\alpha \in \tilde{C}$ .Thus

$\lim_{k\to\infty}\tilde{F}\left[x_{n_k}-x_{\alpha},\frac{n_k}{1+n_k}\right]=1$. Also $\tilde{F}\left[x_{n_k},\frac{n_k}{1+n_k}\right]\leq(1-r)$. Now $(1-r)\geq\tilde{F}\left[x_{n_k},\frac{n_k}{1+n_k}\right]=\tilde{F}\left[x_{n_k}-x+x,\frac{n_k}{1+n_k}\right]\geq\tilde{F}\left[x_{n_k}-x,\frac{n_k}{1+n_k}\right]*\tilde{F}\left[x,\frac{n_k}{1+n_k}\right]\geq\lim_{k\to\infty}\tilde{F}\left[x_{n_k}-\alpha,\frac{n_k}{1+n_k}\right]*\lim_{k\to\infty}\tilde{F}\left[x,\frac{n_k}{1+n_k}\right]\geq 1*1=1$

Which implies $r\leq 0$ which is contradiction. Hence $\tilde{C}$ is fuzzy bounded.

Conversely, suppose that $\tilde{C}$ is fuzzy closed and fuzzy bounded. Let $\dim\tilde{A}=n$ and $\{(e_1,\alpha_1),(e_2,\alpha_2),\dots,(e_n,\alpha_n)\}$ be fuzzy basis for $\tilde{A}$ choose a sequence $\{(x_k,\gamma)\}$ in $\tilde{C}$ so $(x_k,\gamma)=\left(\beta_1^{(k)}e_1+\beta_2^{(k)}e_2+\cdots+\beta_n^{(k)}e_n,\gamma\right)$ where $\gamma=\min\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ where $\beta_1^{(k)},\beta_2^{(k)},\dots,\beta_n^{(k)}$ are scalars.

Now by Theorem 4.1 there exists r, $0<r<1$ such that

$\tilde{F}\left[\sum_{j=1}^{n}\beta_j^{(k)}e_j,\gamma\sum_{j=1}^{n}\left|\beta_j^{(k)}\right|\right]<(1-r)$.

Since $\tilde{C}$ is fuzzy bounded we have $\tilde{F}[x,\alpha]>(1-r)$ for each $x_\alpha\in\tilde{C}$. So $\tilde{F}\left[\sum_{j=1}^{n}\beta_j^{(k)}e_j,\gamma\right]>(1-r)$

Now we have

$\tilde{F}\left[\sum_{j=1}^{n}\beta_j^{(k)}e_j,\gamma\sum_{j=1}^{n}\left|\beta_j^{(k)}\right|\right]<(1-r)<\tilde{F}\left[\sum_{j=1}^{n}\beta_j^{(k)}e_j,\gamma\right]$

which implies that $\sum_{j=1}^{n}\left|\beta_j^{(k)}\right|=1$, $\gamma\sum_{j=1}^{n}\left|\beta_j^{(k)}\right|\leq\sum_{j=1}^{n}\left|\beta_j^{(k)}\right|$ so $\left|\beta_j^{(k)}\right|<1$ for each $j=1,2,\dots,n$. Hence for each j = 1,2,…,n the sequence are bounded. By using Bolzano-Weierstrass theorem. It follows that $\{\beta_j^{(k)}\}$ has a subsequence $\{\beta_j^{(k_m)}\}$ converges to β_j Put $(x_{k_m},\gamma)=\left(\beta_1^{(k_m)}e_1+\beta_2^{(k_m)}e_2+\cdots+\beta_n^{(k_m)}e_n,\gamma\right)$ and (x, γ) = $(\beta_1 e_1+\beta_2 e_2+\cdots+\beta_n e_n,\gamma)$. Now

$\tilde{F}[x_{k_m}-x,\gamma]=\tilde{F}\left[\sum_{j=1}^{n}\left(\beta_j^{(k_m)}-\beta_j\right)e_j,\gamma\right]\geq\tilde{F}\left[\left(\beta_1^{(k_m)}-\beta_1\right)e_1,\gamma\right]*\tilde{F}\left[\left(\beta_2^{(k_m)}-\beta_2\right)e_2,\gamma\right]*\dots*\tilde{F}\left[\left(\beta_n^{(k_m)}-\beta_n\right)e_n,\gamma\right]$

$\lim_{m\to\infty}\tilde{F}[x_{k_m}-x,\gamma]\geq\lim_{m\to\infty}\tilde{F}\left[\left(\beta_1^{(k_m)}-\beta_1\right)e_1,\gamma\right]*\lim_{n\to\infty}\tilde{F}\left[\left(\beta_2^{(k_m)}-\beta_2\right)e_2,\gamma\right]*\dots*\lim_{n\to\infty}\tilde{F}\left[\left(\beta_n^{(k_m)}-\beta_n\right)e_n,\gamma\right]=1*1*\dots*1=1$

It follows that $\{(x_{k_m},\gamma)\}$ is a fuzzy convergent subsequence of $\{(x_k,\gamma)\}$ and fuzzy converges to (x,γ). Since $\tilde{C}$ is fuzzy closed so $(x,\gamma)\in\tilde{C}$. Hence $\tilde{C}$ is fuzzy compact. Since $\{(x_k,\gamma)\}$ was arbitrary sequence of fuzzy points in $\tilde{C}$.

Theorem 4.6:

Let $\tilde{Y}$ and $\tilde{Z}$ be a subsequence of the fuzzy length space $(\tilde{A},\tilde{F},*)$ and $\tilde{Y}$ is a fuzzy closed and proper subset of $\tilde{Z}$. Then there exists $z_\alpha\in\tilde{Z}$ such that $\tilde{F}[z_\alpha]>0$ and $\tilde{F}[z-y,\lambda]\leq\theta$ for all $y_\beta\in\tilde{Y}$ and for some $0<\theta<1$, where $\lambda=\alpha\wedge\beta$.

Proof:

Let $\tilde{Y}$ be a proper subset of $\tilde{Z}$ then there is $x_\alpha\in\tilde{Z}-\tilde{Y}$. Let $a=\sup\{\tilde{F}[x-y,\lambda]>0\}$, $\lambda=\alpha\wedge\beta$

Now we can take $b_\beta\in\tilde{Y}$ such that $\sup\{\tilde{F}[x-b,\lambda]>0\}<a$ where $\lambda=\alpha\wedge\beta$ ... (2.15) Let $z_\alpha=(x-b,\lambda)$. From (2.15) $\tilde{F}[z_\alpha]>0$. Now for any $y_\beta\in\tilde{Y}$, we have

$\sup\{\tilde{F}[z-y,\lambda]>0\}=\sup\{\tilde{F}[x-b-y,\lambda]>0\}$,$\{b+y\in\tilde{Y}\}<a\leq\frac{a}{\lambda}=\theta$

Theorem 4.7:

Let $(\tilde{A},\tilde{F},*)$ be a fuzzy length space. If the fuzzy set $\tilde{C}=\{z_\alpha:\tilde{F}[z_\alpha]>0\}$ is fuzzy compact then $\tilde{A}$ is finite dimensional.

Proof:

Let $\tilde{C}$ be fuzzy compact but $\dim\tilde{A}=\infty$ take any $(z_1,\alpha_1)\in\tilde{C}$ and let $\tilde{A}_1$ be a subspace of $\tilde{A}$ whose basis is $\{(z_1,\alpha_1)\}$ then $\tilde{A}_1$ is fuzzy closed and proper subset of $\tilde{A}$ since $\dim\tilde{A}=\infty$ so by Theorem 4.6 there is $(z_2,\alpha_2)\in\tilde{C}$ such that $\tilde{F}[z_2-z_1,\lambda]\leq\theta=\frac{1}{2}$ with $\lambda=\alpha_1\wedge\alpha_2$. Let $\tilde{A}_2$ be the subspace of $\tilde{A}$ whose basis is $\{(z_1,\alpha_1),(z_2,\alpha_2)\}$ so $\tilde{A}_2$ is a proper subset of $\tilde{A}$ and fuzzy closed. Again by lemma 2.8.9 there is (z_3,α_3) in $\tilde{C}$ such that $\tilde{F}[z_3-z_1,\lambda]\leq\frac{1}{2}$ and $\tilde{F}[z_3-z_2,\lambda]\leq\frac{1}{2}$ with $\lambda=\alpha_1\wedge\alpha_2\wedge\alpha_3$. Hence by induction we obtain a sequence of fuzzy points $\{(z_n,\alpha_n)\}$ whose element $(z_n,\alpha_n)\in\tilde{C}$ such that $\tilde{F}[z_m-z_n,\lambda]\leq\frac{1}{2}$ with $\lambda=\alpha_n\wedge\alpha_m$. It follows that neither the sequence $\{(z_n,\alpha_n)\}$ nor its any subsequence fuzzy converges. This contradicts the fuzzy compactness of $\tilde{C}$. Hence $\dim\tilde{A}$ is finite.

Conclusion

In the present paper our aim is to prove the fixed fuzzy point theorems for fuzzy complete fuzzy length spaces. Also we proved some properties for finite dimensional fuzzy length spaces on a fuzzy set.

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