

Convexity in the 1st Class of the Rough Mathematical Programming Problems

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Abstract — Convex optimization is considered to be a reliable computational tool in engineering as it is used to solve many engineering problems in an efficient and fast method. The goal of this paper is to discuss, in depth, the convexity in the 1st class of rough mathematical programming and to present some related results.

Keywords — convexity, optimization, rough programming.

I. INTRODUCTION

Optimization (mathematical programming) is a sub-field of operations research and it has a widely grown in the last three decades. The goal of any optimization problem is to maximize (or minimize) one or more of objective functions under a determined set of conditions. Optimization can be applied to many fields like business, mining and engineering. Optimization is used in our daily life (e.g. moving from a place to another).

The model of a simple mathematical programming problem is:

$$\max_{x \in M} g(x)$$

where $g : U \rightarrow R$ is the crisp objective function and $M \subseteq U$ is the feasible set of the problem. U is the universe of the problem.

Convex optimization problem involves maximizing concave functions over convex sets. They can be converted into a minimization problem of convex functions by multiplying the objective function by minus one. One of the advantages of convex optimization is that it covers a broad range of practical optimization problems. Also, there are some non-convex problems that can be reformulated into convex problems. The other advantage is that if the decision set of the problem is convex, any local optimum is also a global optimum. There are some books that discuss convex optimization (e.g. Rockafellar [2], Stoer and Witzgall [3], Holmes [4], Bazaraa and Shetty [5], Ekeland and Temam [6], Ioffe and Tihomirov [7], Barbu and Precupanu [8] and Ponstein [9]). We assume that the reader has a previous knowledge of convex optimization.

In many actual optimization problems, the decision maker is not able to define the objective function and/or the set of constraints precisely but rather can define them in a "rough sense". Rough set theory (RST), introduced by Pawlak [1], provides a flexible mathematical tool to the decision maker to solve such problems.

Recently, Youness [10] combined rough set theory with mathematical programming. He described a new type of mathematical programming problems in which the feasible region is rough and called it RMPP. He defined new concepts, namely, "convex rough set", "local rough optimal solution" and "global rough optimal solution". Osman et al. [11] extended the previous work and demonstrated that the roughness may exist in the objective function, the feasible set or both of them. They classified rough programming problems into three classes according to the place of roughness. They discussed the convexity in the 1st class of the RMPPs in which the decision set is rough and the objective function is crisp. In their discussion, they showed that the lower and/or upper approximations of rough feasible set could be convex. They also introduced new concepts such as: "Upper convex" and "Lower convex".

In this paper, we extend the above mentioned works, and propose and prove some theorems related to the convexity in the 1st class of RMPPs.

II. ROUGH SET THEORY

RST has been proven to be an excellent mathematical tool dealing with vague or imprecise descriptions of objects. Therefore, many researchers applied RST to many domains such as pattern recognition, data mining, artificial intelligence, image processing, machine learning and medical applications [1].

The rough set methodology proposed by Pawlak [1], in 1982, assumed that any imprecise concept is characterized by a pair of precise concepts called the lower and the upper approximations. RST is based on equivalence relation that partitions the universe into classes of indiscernible objects.

RST expresses imprecision by employing a boundary region of a set. If the boundary region of a set is empty, then the set is crisp (exact) otherwise the set is rough (inexact). RST uses equivalence relation to group objects with similar characteristics into indiscernibility classes and any vague set is characterized by a pair of precise sets called the lower and the upper approximations. The lower approximation includes all objects that surely belong to the concept of interest, where the upper approximation includes all objects which possibly belong to that imprecise concept. The main advantage of using RST in handling imprecise concepts is that it does not need any additional information.

Let U be a non-empty finite set of objects, called the universe, and $E \subseteq U \times U$ be an equivalence relation on U . The ordered pair $A=(U, E)$ is called an approximation space generated by E on U . E generates a partition $U/E = \{Y_1, Y_2, \dots, Y_m\}$ where Y_1, Y_2, \dots, Y_m are the equivalence classes of the approximation space A . Based on the equivalence relation E , the mapping $[\cdot]_E : U \rightarrow 2^U$ is given by $[x]_E = \{y \in U \mid xEy\}$. Shortly, the subset $[x]_E \subseteq U$ is the equivalence class containing x .

In RST, any subset $M \subseteq U$ is defined in terms of the equivalence classes of the approximation space A by its lower and upper approximations (i.e. $E_*(M)$ and $E^*(M)$, respectively) as follows:

$$E_*(M) = \{x \in U \mid [x]_E \subseteq M\}$$

$$E^*(M) = \{x \in U \mid [x]_E \cap M \neq \emptyset\}$$

Therefore, $E_*(M) \subseteq M \subseteq E^*(M)$

The difference between the upper and the lower approximations is called the boundary of M and is denoted by $BN_E(M) = E^*(M) - E_*(M)$. For simplicity, let $E_*(M) = M_*$, $E^*(M) = M^*$ and $BN_E(M) = M_{BN}$.

III. THE 1ST CLASS OF RMPPS [11]

Let $A=(U, E)$ be an approximation space generated by an equivalence relation E on the universe U . Therefore, $U/E = \{Y_1, Y_2, \dots, Y_m\}$ is the partitioned universe generated by E on U where Y_1, Y_2, \dots, Y_m are the equivalence classes of the approximation space A .

A RMPP over the universe U takes the following form:

$$\begin{aligned} & \max_{x \in M} g(x) \\ & s.t. \quad M_* \subset M \subset M^* \end{aligned} \tag{1}$$

where $g : M^* \rightarrow R$ is the crisp objective function. $M \subset U$ is the set of constraints of the problem, that is roughly defined in the universe U by M_* and M^* , where:

$$M_* = \{x \in U \mid [x]_E \subseteq M\}$$

$$M^* = \{x \in U \mid ([x]_E \cap M) \neq \emptyset\}$$

Definition 3.1: In problem (1), the optimal value \bar{g} of the objective function is defined by its lower

and upper bounds \bar{g}_* and \bar{g}^* , respectively, such that:

$$\bar{g}_* = \max\{\alpha, \beta\}$$

$$\bar{g}^* = \max\{\alpha, \gamma\}$$

where

$$\alpha = \max_{x \in M_*} g(x)$$

$$\beta = \max_{Y \subseteq M_{BN}} \min_{x \in Y} g(x)$$

$$\gamma = \max_{x \in M_{BN}} g(x)$$

Definition 3.2: In problem (1), a point x is a surely-feasible solution, if and only if $x \in M_*$.

Definition 3.3: In problem (1), a point x is a possibly-feasible solution, if and only if $x \in M^*$.

Definition 3.4: In problem (1), a point x is a surely-not feasible solution, if and only if $x \notin M^*$.

Definition 3.5: In problem (1), a point x is a surely-optimal solution, if and only if $g(x) = \bar{g}^*$.

Definition 3.6: In problem (1), a point x is a possibly-optimal solution, if and only if $g(x) \geq \bar{g}_*$.

Definition 3.7: In problem (1), a point x is a surely-not optimal solution, if and only if $g(x) < \bar{g}_*$.

Definition 3.8: In problem (1), there are four optimal sets covering all possible degrees of feasibility and optimality, as follows:

- The set of all surely-feasible, surely-optimal solutions is denoted by $F_s O_s$, and it is defined by: $F_s O_s = \{x \in M_* \mid g(x) = \bar{g}^*\}$.
- The set of all surely-feasible, possibly-optimal solutions is denoted by $F_s O_p$, and it is defined by: $F_s O_p = \{x \in M_* \mid g(x) \geq \bar{g}_*\}$.
- The set of all possibly-feasible, surely-optimal solutions is denoted by $F_p O_s$, and it is defined by: $F_p O_s = \{x \in M^* \mid g(x) = \bar{g}^*\}$.
- The set of all possibly-feasible, possibly-optimal solutions is denoted by $F_p O_p$, and it is defined by: $F_p O_p = \{x \in M^* \mid g(x) \geq \bar{g}_*\}$.

IV. CONVEXITY IN 1ST CLASS OF RMPPS

Convex sets and concave functions have many attractive properties in mathematical programming. For example, any local maximum point of a concave function over a convex set is also a global maximum point. In this section, we present some significant

properties of RMPPs that have convex rough set and concave crisp function.

Definition 4.1: A rough set M is U -convex, if its upper approximation M^* is convex [11].

Definition 4.2: A rough set M is L -convex, if its lower approximation M_* is convex [11].

Definition 4.3: A rough set M is convex, if its upper and lower approximations (i.e. M^* and M_*) are convex [11].

Definition 4.4: A function $g(x)$ is concave on a convex set S , if $g(\lambda x_1 + (1-\lambda)x_2) \geq \lambda g(x_1) + (1-\lambda)g(x_2)$ for each $x_1, x_2 \in S$ and for each $\lambda \in (0,1)$ [12].

Definition 4.5: A function $g(x)$ is strictly concave on a nonempty convex set S , if $g(\lambda x_1 + (1-\lambda)x_2) > \lambda g(x_1) + (1-\lambda)g(x_2)$ for each $x_1, x_2 \in S$ and for each $\lambda \in (0,1)$ [12].

Definition 4.6: A function $g(x)$ is quasiconcave on a convex set S , if $g(\lambda x_1 + (1-\lambda)x_2) \geq \min\{g(x_1), g(x_2)\}$ for each $x_1, x_2 \in S$ and for each $\lambda \in (0,1)$ [12].

Definition 4.7: A function $g(x)$ is strictly quasiconcave on a nonempty convex set S , if for each $x_1, x_2 \in S$ with $g(x_1) \neq g(x_2)$, we have $g(\lambda x_1 + (1-\lambda)x_2) > \min\{g(x_1), g(x_2)\}$ for each $\lambda \in (0,1)$ [12].

Definition 4.8: A function $g(x)$ is strongly quasiconcave on a nonempty convex set S , if for each $x_1, x_2 \in S$ with $x_1 \neq x_2$, we have $g(\lambda x_1 + (1-\lambda)x_2) > \min\{g(x_1), g(x_2)\}$ for each $\lambda \in (0,1)$ [12].

Theorem 4.1: In problem (1), if M^* is a nonempty convex set and $\bar{x} \in M^*$ is a local optimal solution then:

- 1) If $g(x)$ is a strictly quasiconcave function, then \bar{x} is a surely-global optimal solution.
- 2) If $g(x)$ is a strongly quasiconcave function, then \bar{x} is the unique surely-global optimal solution.
- 3) If $g(x)$ is a strongly quasiconcave function and $\bar{x} \in M_*$, then \bar{x} is the unique global optimal solution (i.e. $F_s O_s = F_s O_p = F_p O_s = F_p O_p = \{\bar{x}\}$).

Proof:

- 1) Suppose, on the contrary, that there is an $\hat{x} \in M^*$ with $g(\hat{x}) > g(\bar{x})$. By convexity of M^* , $\lambda \hat{x} + (1-\lambda)\bar{x} \in M^*$, $\forall \lambda \in (0,1)$. Since \bar{x} is a local maximum by assumption, then $g(\bar{x}) \geq g(\lambda \hat{x} + (1-\lambda)\bar{x})$, $\forall \lambda \in (0, \delta)$ for some $\delta \in (0,1)$. But since $g(x)$ is strictly quasiconcave and $g(\hat{x}) > g(\bar{x})$, then $g(\lambda \hat{x} + (1-\lambda)\bar{x}) > g(\bar{x})$, $\forall \lambda \in (0,1)$. This contradiction shows that \hat{x} does not exist.
- 2) Since \bar{x} is a local optimal solution, then there is an ε -neighborhood $N_\varepsilon(\bar{x})$ around \bar{x} where $g(\bar{x}) \geq g(x)$ for $\forall x \in M^* \cap N_\varepsilon(\bar{x})$. Assume by contradiction to the conclusion of the theory that there is a point $\hat{x} \in M^*$ such that $\hat{x} \neq \bar{x}$ and $g(\hat{x}) > g(\bar{x})$. By strong quasiconcavity, it follows that $g(\lambda \hat{x} + (1-\lambda)\bar{x}) > \min\{g(\hat{x}), g(\bar{x})\} = g(\bar{x})$, $\forall \lambda \in (0,1)$. But for λ small enough, $\lambda \hat{x} + (1-\lambda)\bar{x} \in M^* \cap N_\varepsilon(\bar{x})$ and hence local optimality of \bar{x} is violated.
- 3) If $g(x)$ is a strongly quasiconcave function and $\bar{x} \in M_*$, then $\bar{g}_* = g(\bar{x}) = \bar{g}^*$. Thus, \bar{x} is a unique surely and possibly optimal solution. Hence, $F_s O_s = F_s O_p = F_p O_s = F_p O_p = \{\bar{x}\}$.

Theorem 4.2: In problem (1), if M^* is a nonempty convex set and $\bar{x} \in M^*$ is a local optimal solution then:

- 1) If $g(x)$ is a concave function, then \bar{x} is a surely-global optimal solution.
- 2) If $g(x)$ is a strictly concave function, then \bar{x} is the unique surely-global optimal solution.
- 3) If $g(x)$ is a strictly concave function and $\bar{x} \in M_*$, then \bar{x} is the unique global optimal solution (i.e. $F_s O_s = F_s O_p = F_p O_s = F_p O_p = \{\bar{x}\}$).

Proof: It is similar to the above proof.

Theorem 4.3: In problem (1), if M is a nonempty U -convex set and $g(x)$ is a concave function on M^* , then the point $\bar{x} \in M^*$ is a surely-optimal solution to this problem if and only if $g(x)$ has a subgradient ξ at \bar{x} such that $\xi^t(\bar{x} - x) \geq 0$ for all $x \in M^*$.

Proof: Assume that $\bar{x} \in M^*$ where $\xi'(\bar{x} - x) \geq 0, \forall x \in M^*$. ξ is a subgradient of g at \bar{x} . By concavity of g , $g(\bar{x}) \geq g(x) + \xi'(\bar{x} - x) \geq g(x), \forall x \in M^*$ and therefore \bar{x} is a surely-optimal solution to the problem.

To show the converse, assume that \bar{x} is a surely-optimal solution to the problem, and form the following two sets in U :

$$S_1 = \{(\bar{x} - x, y) \mid x \in U, y > g(\bar{x}) - g(x)\}$$

$$S_2 = \{(\bar{x} - x, y) \mid x \in M^*, y \leq 0\}$$

It is easy to prove that both S_1 and S_2 are convex sets. Also $S_1 \cap S_2 = \emptyset$ because otherwise there would be a point (x, y) where $x \in M^*, 0 \geq y > g(\bar{x}) - g(x)$ contradicting the assumption that \bar{x} is an optimal solution of the problem. Since $S_1 \cap S_2 = \emptyset$, then there is a hyperplane that separates S_1 and S_2 . Thus, there is a nonzero vector (ξ_0, μ) and a scalar α such that:

$$\xi_0'(\bar{x} - x) + \mu y \leq \alpha, x \in U, y > g(\bar{x}) - g(x) \quad (1)$$

$$\xi_0'(\bar{x} - x) + \mu y \geq \alpha, x \in M^*, y \leq 0 \quad (2)$$

If we let $\bar{x} = x$ and $y = 0$ in (2), then $\alpha \leq 0$. Next, letting $\bar{x} = x$ and $y = \varepsilon > 0$ in (1) makes $\mu\varepsilon \leq \alpha$. Since this is true for $\forall \varepsilon > 0$, then $\mu \leq 0$ and $\alpha \geq 0$. Briefly, we conclude that $\mu \leq 0$ and $\alpha = 0$. If $\mu = 0$, then from (1) $\xi_0'(\bar{x} - x) \leq 0, \forall x \in U$. If we let $\bar{x} = x + \xi_0$, then $0 \geq \xi_0'(\bar{x} - x) = \|\xi_0\|^2$ and thus $\xi_0 = 0$. Since $(\xi_0, \mu) \neq (0, 0)$, then $\mu < 0$. Dividing (1) and (2) by $-\mu$ and denoting $-\xi_0 / \mu$ by ξ , we obtain the following inequalities:

$$0 \geq \xi'(\bar{x} - x) - y, x \in U, y > g(\bar{x}) - g(x) \quad (3)$$

$$\xi_0'(\bar{x} - x) - y \geq 0, x \in M^*, y \leq 0 \quad (4)$$

By letting $y = 0$ in (4), we obtain $\xi'(\bar{x} - x) \geq 0, \forall x \in M^*$. From (3), it is clear that $g(\bar{x}) \geq g(x) + \xi'(\bar{x} - x), \forall x \in U$.

Thus, ξ is a subgradient of g at \bar{x} such that $\xi'(\bar{x} - x) \geq 0, \forall x \in M^*$.

Theorem 4.4: In problem (1), if M is a nonempty U -convex set and $g(x)$ is a concave function on M^* where M^* is open, then the point $\bar{x} \in M^*$ is a surely-optimal solution to this problem if and only if there is a zero subgradient of $g(x)$ at \bar{x} .

Proof: By the previous theorem, \bar{x} is a surely-optimal solution if and only if $\xi'(\bar{x} - x) \geq 0, \forall x \in M^*$ where ξ is a subgradient of g at \bar{x} . Since M^* is open, then $\bar{x} = x - \lambda\xi \in M^*$ for some positive λ . Hence, $-\lambda \|\xi\|^2 \geq 0$. This means that $\xi = 0$.

Theorem 4.5: In problem (1), if M is a nonempty U -convex set and $g(x)$ is a differentiable concave function on M^* , then the point $\bar{x} \in M^*$ is a surely-optimal solution to this problem if and only if $\nabla g'(\bar{x})(\bar{x} - x) \geq 0, \forall x \in M^*$. Furthermore, if M^* is open then the point $\bar{x} \in M^*$ is a surely-optimal solution to this problem if and only if $\nabla g(\bar{x}) = 0$.

Proof: It is straightforward.

Theorem 4.6: Consider the problem: $\min g(x)$ subject to $x \in M$, where M^* (the upper approximation of the rough set M) is a nonempty convex set and $g(x)$ is a concave function on M^* . If $\bar{x} \in M^*$ is a local optimal solution then $\xi'(\bar{x} - x) \leq 0$ for all $x \in M^*$ where ξ is a subgradient of g at \bar{x} .

Proof: Assume that $\bar{x} \in M^*$ is a local optimal solution. Then there is an ε -neighborhood $N_\varepsilon(\bar{x})$ where $g(\bar{x}) \leq g(x), \forall x \in M^* \cap N_\varepsilon(\bar{x})$. Let $x \in M^*$, and notice that there is $\bar{x} - \lambda(\bar{x} - x) \in M^* \cap N_\varepsilon(\bar{x})$ for $\lambda > 0, \lambda \approx 0$. Thus, $g(\bar{x} - \lambda(\bar{x} - x)) \geq g(\bar{x})$.

Let ξ be a subgradient of g at \bar{x} and by concavity of g , we have $g(\bar{x}) - g(\bar{x} - \lambda(\bar{x} - x)) \geq \lambda\xi'(\bar{x} - x)$.

The above two inequalities imply that $\lambda\xi'(\bar{x} - x) \leq 0$, and dividing by $\lambda > 0$, we get the required result.

Theorem 4.7: Consider the problem: $\min g(x)$ subject to $x \in M$, where M^* (the upper approximation of the rough set M) is a convex set and $g(x)$ is a differentiable concave function on M^* . If $\bar{x} \in M^*$ is a local optimal solution then $\nabla g'(\bar{x} - x) \leq 0, \forall x \in M^*$ where ξ is a subgradient of g at \bar{x} .

Proof: It is straightforward.

Theorem 4.8: Consider the problem: $\min g(x)$ subject to $x \in M$, where M^* (the upper

approximation of the rough set M) is a nonempty compact polyhedral set and $g(x)$ is a concave function on M^* . Then, there is an optimal solution $\bar{x} \in M^*$ to the problem, where \bar{x} is an extreme point of M^* .

Proof: Since M^* is compact, g assumes a minimum at $\bar{x} \in M^*$. If \bar{x} is an extreme point of M^* , then the result is acquired. Otherwise, $\bar{x} = \sum_{j=1}^k \lambda_j x_j$ where $\sum_{j=1}^k \lambda_j = 1, \lambda_j > 0$, and x_j is an extreme point of M^* for $j = 1, 2, \dots, k$. By the concavity of g , we have

$$g(\bar{x}) = g\left(\sum_{j=1}^k \lambda_j x_j\right) \geq \sum_{j=1}^k \lambda_j g(x_j)$$

But since $g(\bar{x}) \leq g(x_j)$, for $j = 1, 2, \dots, k$, the above inequality implies that $g(\bar{x}) = g(x_j)$ for $j = 1, 2, \dots, k$. Hence, the extreme points x_1, x_2, \dots, x_k are optimal solutions to the problem and the proof is complete.

Theorem 4.9: Consider the problem: $\min g(x)$ subject to $x \in M$, where M^* (the upper approximation of the rough set M) is a nonempty compact polyhedral set and $g(x)$ is a quasiconcave function on M^* . Then, there is an optimal solution $\bar{x} \in M^*$ to the problem, where \bar{x} is an extreme point of M^* .

Proof: Since g is a function on M^* and hence gets a minimum at $\bar{x} \in M^*$. If there is an extreme point whose objective is equal to $g(\bar{x})$, then the result is acquired. Otherwise, let x_1, x_2, \dots, x_k be extreme points of M^* , and suppose that $g(\bar{x}) < g(x_j)$ for $j = 1, 2, \dots, k$. \bar{x} can be represented as $\bar{x} = \sum_{j=1}^k \lambda_j x_j$ where $\sum_{j=1}^k \lambda_j = 1, \lambda_j > 0, j = 1, 2, \dots, k$.

Since $g(\bar{x}) < g(x_j)$ for each j , then $g(\bar{x}) < \min_{1 \leq j \leq k} g(x_j) = \alpha$ (1)

Now consider the set $M_\alpha^* = \{x \mid g(x) \geq \alpha\}$. Notice that $x_j \in M_\alpha^*$ for $j = 1, 2, \dots, k$ and M_α^* is a convex set. Hence, $\bar{x} = \sum_{j=1}^k \lambda_j x_j$ belongs to M_α^* . By quasiconcavity of g , $g(\bar{x}) \geq \alpha$, which contradicts (1). This contradiction shows that $g(\bar{x}) = g(x_j)$ for some extreme point x_j and the result is obtained.

V. CONCLUSIONS

In this paper, we provided some essentials of convex sets, convex functions, and convex optimization problems in a rough environment.

REFERENCES

- [1] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences*, 11, 341-356, 1982.
- [2] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
- [3] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, Springer-Verlag, 1970.
- [4] Holmes, R. B., *A Course on Optimization and Best Approximation*, Springer-Verlag, Berlin and New York, 1972.
- [5] Bazaraa, M. S., and Shetty, C. M., *Foundations of Optimization*, Springer-Verlag, Berlin and New York, 1976.
- [6] Ekeland, I., and Temam, R., *Convex Analysis and Variational Problems*, North Holland Publisher, Amsterdam, 1976.
- [7] Ioffe, A. D., and Tihomirov, Y. M., *Theory of External Problems*, North Holland Publisher, Amsterdam, 1979.
- [8] Barbu, V., and Precupanu, T., *Convexity and Optimization in Banach Spaces*, Sijthoff and Noordhoff, Alphen aan de Rijn, 1978.
- [9] Ponstein, J., "Approaches to the Theory of Optimization," Cambridge University, Press, London and New York, 1980.
- [10] E.A. Youness, Characterizing solutions of rough programming problems, *European Journal of Operational Research*, vol.168, no.3, pp.1019-1029, 2006.
- [11] M.S. Osman, E.F. Lashein, E.A. Youness, T.E.M. Atteya, Mathematical programming in rough environment, *Optimization*, vol.60, no.5, pp.603-611, 2011.
- [12] Mokhtar S. Bazaraa, Hanif D. Sherali, C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley and Sons Inc., 3rd ed.; 2006.