G-Clean rings

Shaaban Ali SafariSabet, MarziehFarmani

[1]*Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran*

[2]*(Corresponding author), Young Researchers and Elite Club, Roudehen Branch, Islamic Azad University, Roudehen, Iran*

Abstract

An element of a ring R is clean if it is the sum of an idempotent and a unit. A ring R is called clean if each of its elements is clean. In this paper we define a ring is G-Clean if each of its elements is the sum of a G-regular and an idempotent element. Finally we investigate some properties of G-clean rings.

Keywords: *Group, Ring, Von Neumann regular,G-regular, Clean, G-clean*

1. Introduction

In 1936 Von Neumann defined that an element x in R is regular if $x = xyx$, for some $y \in R$, the ring R is regular if each of its elements is regular and an element $x \in R$ is said to be strongly (Von Neumann) regular ifthere exists $y \in R$ such that $x = x^2 y$, the ring R is strongly regular if each of its elements is strongly regular. Some properties of regular rings and strongly regular has been studied in [1,7,10].A ring$R$ is said to be π- regular if for every element $a \in R$, there is an element $b \in R$ such that $a^n = a^n\, b\, a^n$,for some positive integer n. A ring R is said to be strongly π- regular if for every $x \in R$, there is an element $y \in R$ such that$x^m = x^{m+1}\, y$,for some positive integer m.In many papers concerned π-regular and strongly π-regular rings, see [3,4,5,6,9,13,14,15,17,20,21,23]. A ring R is abelian if every idempotent element of R is central. A ring R is called prime (resp. semiprime) if (0) is a prime (resp. semiprime) ideal. A ring R is called Jacobson semisimple (or J-semisimple for short) if $rad(R) = 0$.

Throughout the present article R is an associative ring with identity1, and G denote a group, $U(R)$ the group of units and $Id(R)$ the set of idempotents. Let X be a set, group action is a map

$$X \;\; G \rightarrow X$$

(If there is no fear of confusion, we write (x, g) simply as by x^g) such that

I. $(x^g)^h = x^{gh}$for all $x \in X$ and$g, h \in G$.
II. $x^1 = x$for all $x \in X$.

For every $g \in G$ we set $R^g = \{r^g | r \in R\}$.

We define an element $x \in R$ to be G-regular if there exist an element $g \in G$, depending on x, and $r \in R$ such that$x^g = x^g r x^g$.$R$is said to be G-regular if all of its elements are G-regular.

An element $x \in R$ is called clean if $x = u + e$, where $u \in U(R)$ and $e \in Id(R)$. We call an element x of ring R is G-clean if $x = r + e$,where r is G-regular and $e \in Id(R)$. Clearly G-regular rings and clean rings are G-clean.

2. Group-regular rings

Let G be a group. We defined G-regular and strongly G-regular rings in [21].

Definition 2.1: An element $x \in R$ is called to be$G$-regular if there exist an element $g \in G$, depending on x, and $r \in R$ such that $x^g = x^g r x^g$. R is said to be G-regular if all of its elements are G-regular.

Remark 2.2: For each $n \in \mathbb{N}$ by a^{ng}us mean$(a^g)^n$.

Definition 2.3: An element $x \in R$is called to bestrongly G-regular if there exist an element $g \in G$, depending on x, and $r \in R$ such that $x^g = x^{2g} r$ with this property that $(x^2)^g = (x^g)^2$. R is said to be strongly G-regular if all of its elements are strongly G-regular.

Here we give some examples of G-regular rings.

Definition 2.4:Let $G = U(R)$ (where $U(R)$ is the group of units of R). We call the action $(x, g) \rightarrow gx$and the action $\big((x, g) \rightarrow gxg^{-1}\big)$from$R \;\; G$ to R, regular action andconjugate action.

Example 2.5: Let $G = U(R)$. We define an element $x \in R$to be unitary regular(resp. strongly unitary regular) element if there exist an element $g \in G$, depending on x, and $r \in R$ such that $xg = xgrxg$(resp. $xg = (xg)^2r$). R is said to be unitary regular (resp. strongly unitary regular) if all of its elements are unitary regular (resp. strongly unitary regular).

Example 2.6: Let $G = U(R)$. We define an element $x \in R$ to be conjugate regular(resp. strongly conjugate regular) element if there exist an element $g \in G$, depending on x, and $r \in R$ such that $x = xg^{-1}rgx$ (resp.$xg = x^2gr$). R is said to be conjugate regular (resp. strongly conjugate regular) if all of its elements are conjugate regular (resp. strongly conjugate regular).

Example 2.7: Let $Aut(R)$ be automorphism group of R. We define an element $x \in R$ to be Automorphic-regular ((Aut)-regular) if there exist an element $\alpha \in Aut(R)$, depending on x, and $r \in R$ such that$x^\alpha = x^\alpha r x^\alpha$. R is said to be automorphic regular if all of its elements are automorphic regular . If choice of α is indepent of x we say that R is α-regular.

Also we define a G-regular ideals as follows:

Definition 2.8: A two sided ideal J in a ring R is G-regular provided that for each $x \in J$, there exist $y \in J$ and $g \in G$ such that $x^g = x^g r x^g$.

Let $\mu: R \times G \to R$ be a group action and I be a two sided ideal of R. Then G can acts naturallyon R by the rule $\mu(r + I, g) = \mu(r, g) + I$.

Theorem 2.9: Every factor ring of a G-regular (resp. strongly G-regular) ring is G-regular (resp. strongly G-regular). In particular a homomorphic image of a G-regular ring is G-regular.

:ProofSee [19].

Lemma 2.10: Let G be a group acts on the ring R by this property that $(xy)^g = x^g y^g$ for each $x, y \in R$. If $x, y \in R$, $g \in G$ and $x' = x^g - x^g y x^g$, and if $x'^h = x'^h a x' x'^h$for some $a \in R$and some $h \in G$. Then $x^g = x^g b x^g$for some $b \in R$.

:ProofSee [19].

Theorem 2.11: Let $J \leq K$ be two sided ideals in a ring R. If K is G-regular then K/J is regular.

:ProofIt is trivial.

We define action of G on $\prod_{i \in I} R_i$by the following manner:

For each $g \in G$ and $(x_i) \in \prod_{i \in I} R_i$we define $(x_i)_{i \in I}^g = (x_i^g)_{i \in I}$,thus we have:

Lemma 2.12: A finite direct product $\prod_{i \in I} R_i$ (I is a finite set) of G-regular rings $\{R_i\}_{i \in I}$ is G-regular, where G is an abelian group.

:ProofSee [19].

Theorem 2.13:(1) Let $x \in R$ be G-regular. Then there exist $g \in G$ and $r \in R$ such that $x^g r$ is idempotent.

(2)If an element $x \in R$ is Von Neumann regular, then it is G-regular by taking G to be trivial group.

(3)If an element $x \in R$ is π-regular, then it is $\mathbb{Z}$-regular in which $\mathbb{Z}$ acts on R by the rule $\mu(x, n) = x^n$.

(4) An element $x \in R$ is G-regular if there exist$g \in G$ such that x^gisVon Neumann regular.

:Proof(1) Since $x \in R$is G-regular thus there exist $g \in G$and $r \in R$such that $x^g = x^g r x^g$. Therefore$x^g r = x^g r x^g r = (x^g r)^2$.(2), (3), (4) are trivial.

Theorem 2.14: Let R be a commutative ring. Then the following statements are equivalent for $x \in R$:

(1) x is Von Neumann regular.

(2) $x^2 u = x$ fore some $u \in U(R)$.

(3) $x = ue$ for some $u \in U(R)$ and $e \in id(R)$.

(4) $xy = 0$ for some Von Neumann regular element $x \neq y \in R$ with $x + y \in U(R)$.

(5) $xy = 0$ for some $y \in R$ with $x + y \in U(R)$.

Proof: See [1].

Theorem 2.15: Let R be a commutative ring. Then the following statements are equivalent for $x \in R$:

(1) x is G-regular.

(2) x^gisVon Neumann regular for some $g \in G$.

(3) $x^g = ue$for some $u \in U(R)$\$, $e \in Id(R)$, and $g \in G$.

Proof: We deduce from theorem 2.14.

Theorem 2.16: Let S be the center of G-regular ring R with the property that $S^g \subset S$, $\forall g \epsilon G$. Then S is G-regular.

Proof: Let R be a ring with center S, and let $x \in R$. There exist$y \in R$ and $g \in G$ such that $x^g = x^g y x^g$, and we set $z = y x^g y$. Note that$x^g z x^g = x^g y x^g y x^g = x^g$given any $r \in R$, we have:

$zr = y x^g y r$

$= y^2 r x^g$

$$= y^2 r x^g y x^g$$

$= y x^g y x^g r y$

$= y x^g r y$

Similarly we have $rz = y r x^g y$, so

$rz = y r x^g y$

$= y x^g r y$

$$= zr$$

therefore $z \in S$. Thus S is also G-regular.

Theorem 2.17: Every strongly G-reguler ring is G-regular.

Proof: Assume R is a strongly G-regular ring. Then for any $a \in R$ there exist $r \in R$ and $g \in G$ such that $a^g = a^{2g} r$. $R^g$is reduced. Indeed, if $a \in R$ such that $(a^g)^2 = 0$, then we see that $a^g = 0$. then we have:

$$(a^g - a^g\, r a^g)^2 = a^{2g} - a^g a^g r a^g - a^g r a^g a^g + a^g r a^g a^g r a^g$$
$$= a^{2g} - a^{2g} - a^g r a^{2g} + a^g r a^{2g}$$
$$= 0$$

So $x^g = x^g r x^g$.

Theorem 2.18: If R is a G-regular domain, then R is a strongly G-regular.

Proof: Since R is G-regular, for each $x \in R$, there exist$r \in R$, $g \in G$ such that$x^g = x^g r x^g$.If$a^g = 0$ for some $g \in G$thentrivially $x^g = x^{2g} r$ for any $r \in R$ so R is strongly G-regular as we wants. Otherwise if $x^g \neq 0$for any $g \in G$then $x^{2g} = x^{2g} r x^g$,i.e., $(x^{2g} - x^{2g} r x^g) = 0$. So $(x^g - x^{2g} r) x^g = 0$. Since R is domain, this implies that $x^g = x^{2g} r$, therefore R is strongly G-regular.

3. G-clean rings

In this section first we define G-clean element and G-clean rings and we investigate some properties of G-clean rings.

Definition 3.1: Let G be a group. An element x of a ring R is G-clean if $x = r + e$, where r is a G-regular element of R and $e \in Id(R)$. R is G-clean if every element of R is G-clean.

Now we define strongly G-clean rings.

Definition 3.2: Let G be a group. An element x of a ring R is strongly G-clean if $x = r + e$, where r is a G-regular element of R and $e \in Id(R)$ such that $er = re$. R is strongly$G$-clean if every element of R is strongly G-clean.

Theorem 3.3: Every factor ring of a G-clean ring is G-clean.

Proof: let R be G-clean and $I \lhd R$. Now let $\bar{x} = x + I \epsilon R/I$. Since R is G-clean we have $x = r + e$where x is G-regular and $e \in Id(R)$.

Thus $\bar{x} = \bar{r} + \bar{e}$.By theorem 2.10, we conclude that $\bar{e}$ is G-regular. Since$\bar{e} \in Id(R/I)$, it follows that R/I is G-clean.

Corollary 3.4: The homomorphic image of G-clean (resp. strongly π-Clean) is G-clean (resp. strongly π-Clean).

Proof: We deduce from theorem 3.3 immediately.

Theorem 3.5: Let G be an abelian group. A finite direct product $\prod_{i\in I} R_i$($I$is a finite set)of rings $\{R_i\}_{i\in I}$ is G-Clean if and only if so is each $\{R_i\}_{i\in I}$.

Proof: One direction immediately follows from theorem 3.3. Conversely, let$R_i$be$G$-clean for each $i \in I$. Set $x = (x_i)_{i\in I} \in \prod_{i\in I} R_i$. For each i, write$x_i = r_i + e_i$, where r_i is G-clean and $e_i \in Id(R_i)$. By lemma 2.12, we conclude that $(r_i)_{i\in I} \in \prod_{i\in I} R_i$is G-regular. Since $(e_i)_{i\in I} \in Id(\prod_{i\in I} R_i)$, it follows that $\prod_{i\in I} R_i$ is G-clean.

Let R be a ring and H, G be two group, we shall denote the group ring of H over R as RH. The augmentation ideal of RH is the ideal of RH generated by $\{1 - h | \, h \in H\}$. We shall use Δ to denote the augmentation ideal of RH.It is known That R is a homomorphic image of RH since $RH/\Delta \cong R$.If $RH$is G-clean, then R is G-clean by theorem 2.9.

Theorem3.6 :Let R be a ring in which 2 be invertible in $R$and $H = \{1, h\}$be a group. Then RH is G-clean if and only if R is G-clean, where G is an abelian group.

Proof: If RH is G-clean then by what we said in above R is G-clean.

Conversely, if R is G-clean and 2 is invertible in R then $RH \cong R \times R$ via the mapping $a + bg \to (a + b, a - b)$,[12].

Hence RH is G-clean by theorem 3.5.

References

[1] Anderson, D. F., A. Badawi.(2012). Von Neumann regular and related elements incommutative rings.Algebra Colloq,19, 1017-1041.

[2] Ara, P. (1996).π-regular rings have stable rangeone.Proc. Amer. Math. Soc., 124, No. 11, 3293-3298.

[3] Azumaya, G. (1954). Stronglyπ-regular rings. J. Fac. Sci. Hokkaido Univ.,13. 34-39.

[4] Badawi, A. (1993). Semicommutativeπ-regular rings.Commun. Algebra, 22, No. 1, 151-157.

[5] Badawi, A.(1997).Abelian π-regular rings.Commun. Algebra, 25, No. 4, 1009-1021.

[6] Burgess, W. D., Menal.P .(1988). Strongly π-regular rings and homomorphisms into them.Commun. Algebra, 16, 1701-1725.

]7] Camillo, V.,Xiao, Y. (1994).Regular rings.Commun. Algebra, 22, No. 10, 4095-4112.

]8[Chacron, M. (1969).Algebraic rings.Bull. Austral. Math. Soc., 1, 385-389.

[9] Chin,A. Y. M., Chen, H. V. (2002) On strongly π-regular, Southeast Asian Bulletin of Mathematices, 26, 387-390.

[10] Goodearl, K. R. (1979). Von Neumann regular rings.Pitman, London.

[11] Gupta, V. (1984).Generalization of strongly regular rings.Acta Math. Hung., 43, 57-61.

[12] Han, J.,Nicholson, W. K. (2001). Extensions of clean rings. Comm. Algebra, 29 (6),2589-2595.

]13] Hirano, Y. (1978). Studies on strongly π-regular rings.Math. J. Okayama Univ., 20, No. 2, 141-149.

[14] Hirano, Y. (1985). π-regular rings with involution.Math. J. Okayama Univ., 27, 45-47.

[15] Hirano, Y., J.Park, J. P.(1993).Self-injective strongly π-regularrings.Commun.Algebra, 21, No. 1, 85-91.

[16] Lam, T. Y.(1991) A first course in Non-commutative rings.Gradate texts in Mathematics, No. 131,springer-verlag.

]17] Ohori, M. (1997). Stronglyπ-regular rings and periodic rings.Math. J. Okayama Univ., 19, No. 2, 123-127.

[18] Safari sabet, Sh. A., Farmani, M., Karami, M. (2013).On strongly commuting regular rings.J. Basic. Appl. Sci. Res, 3. No. 2.

[19] Safari sabet.Sh. A.Karami, M. (2014). G-regular and strongly G-regular Rings.IJONS. 26(5), 1953-1958.

]20] Steger. A.)1966 Elementary .(factorization in π-regular rings.Canad. J. Math., 18, 307-313.

]21] Xue, W.(1993).Three questions on strongly π-regular rings and regular rings. Commun. Algebra21, No. 2, 699-704.

]22] Yamini, A. H.,Safari Sabet, Sh. A.)2004 .(Commuting regular rings.Int. J. Appl. Math, 14(4), 357-364.

]23] Yu, H. P. (1995).On strongly π-regular rings of stable rangeone.Bull. Austral. Math.Soc., 51, No. 3, 433-437.