Fixed Point Theorems for Multivalued Contractive mappings in b-Metric Space

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Abstract

In this paper,we prove a fixed point theorem for multivalued contractive mappings in b-metric spaces.This results offers a generalization of Swati Agarawal,K.Qureshi and Jyoti Nema theorem in [1].An example to support our result is presented

Key words:b-Metric space;Contraction;Multivalued mappings;Fixed point.

1 Introduction and preliminaries

In many branches of science,economics,computer science,engineering and the development of nonlinear analysis,the fixed point theory is one of the most important tool.In 1989,Backhtin [2] introduced the concept of b-metric space.In 1993,Czerwik [6] extended the results of b-metric spaces.Using this idea many researcher presented generalization of the renowned banach fixed point theorem in the b-metric space.Mehmet Kir [8],Boriceanu [5],Czerwik [6],Bota [4],Pacurar [9] extended the fixed point theorem in b-metric space.Various problems of the convergence of measurable functions with respect to measure,Czerwik [6] first presented a generalization of banach fixed point theorem in b-metric spaces.The following definitions will be needed in the sequel:

Definition 1.1. [6] *Let X be a nonempty set and let $s \geq 1$ be a given real number.A function $d\colon X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied :*
(1)$d(x, y) = 0$ if and only if $x = y$;
(2)$d(x, y) = d(y, x)$;
(3)$d(x, z) \leq s[d(x, y) + d(y, z)]$.
Then,the triplet (X, d, s) is called a b-metric space.

It is an obvious fact that a metric space is also a b-metric space with $s = 1$,but the converse is not generally true.To support this fact,we have the following example.

Example 1.1. [5] *The set* $l_p(\mathbb{R})$ *(with* $0 < p < 1$*),where* $l_p(\mathbb{R}) := \{(x_n) \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$*,together with the function* $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$*,*
$d(x, y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ *where* $x = x_n, y = y_n \in l_p(\mathbb{R})$ *is a b-metric space.By an elementary calculation we obtain that* $d(x, z) \leq 2^{\frac{1}{p}}[d(x, y) + d(y, z)]$.

Example 1.2. [5] *Let* $X = \{0, 1, 2\}$ *and* $d(2, 0) = d(0, 2) = m \geq 2$,$d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ *and* $d(0, 0) = d(1, 1) = d(2, 2) = 0$.*then* $d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.*If* $m > 2$ *then the triangle inequality does not hold.*

Example 1.3. [5] *The space* $L_p(0 < p < 1)$, *for all real function* $x(t)$,$t \in [0, 1]$ *such that* $(\int_0^1 |x(t) - y(t)|^p dt)^{\frac{1}{p}}$.

Definition 1.2. [5] *Let* (X, d) *be a b-metric space.Then a sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if and only if for all* $\epsilon > 0$ *there exist* $n(\epsilon) \in \mathbb{N}$ *such that for each* $n, m \geq n(\epsilon)$ *we have* $d(x_n, x_m) < \epsilon$.

Definition 1.3. [5] *Let* (X, d) *be a b-metric space.Then a sequence* $\{x_n\}$ *in* X *is called convergent sequence if and only if there exists* $x \in X$ *such that for all* $\epsilon > 0$ *there exists* $n(\epsilon) \in \mathbb{N}$ *such that for all* $n \geq n(\epsilon)$ *we have* $d(x_n, x) < \epsilon$.*In this case we write* $\lim_{n\to\infty} x_n = x$.

Definition 1.4. [10] *Let* X *and* Y *be nonempty sets.*T *is said to be multivalued mapping from* X *to* Y *if* T *is a function for* X *to the power set of* Y.*we denote a multivalued map:*
$T : X \to 2^Y$.

Definition 1.5. [10] *A point of* $x_0 \in X$ *is said to be a fixed point of the multivalued mappings* T *if* $x_0 \in Tx_0$.

Definition 1.6. [3] *Let* (X, d) *be a metric space.A map* $T : X \to X$ *is called Contraction if there exists* $0 \leq \lambda < 1$ *such that* $d(Tx, Ty) \leq \lambda d(x, y)$,*for all* $x, y \in X$.

Definition 1.7. [3] *Let* (X, d) *be a metric space.We define the Hausdorff metric on* $CB(X)$ *induced by* d.*That is*
$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$ *for all* $A, B \in CB(X)$,*where* $CB(X)$ *denotes the family of all nonempty closed and bounded subsets of* X *and* $d(x, B) = \inf\{d(x, b) : b \in B\}$,*for all* $x \in X$.

Definition 1.8. [3] *Let* (X, d) *be a metric space.A map* $T : X \to CB(X)$ *is said to be multivalued contraction if there exists* $0 \leq \lambda < 1$ *such that* $H(Tx, Ty) \leq \lambda d(x, y)$,*for all* $x, y \in X$.

In 2016,Swati Agarawal,K.Qureshi and Jyoti Nema proved the following theorem in [1].

Theorem 1.4. *Let (X, d) be a complete b-metric space.Let T be a mappings $T: X \to X$ such that*

$$d(Tx, Ty) \leq a \max\{d(x, Tx), d(y, Ty), d(x, y)\} + b\{d(x, Ty) + d(y, Tx)\}$$

where $a, b > 0$ such that $a + 2bs \leq 1$ for all $x, y \in X$ and $s \geq 1$ then T has a unique fixed point.

In 1996,B.E.Rhoades define the contractive definition as follows:
Let $F : X \to CB(X)$.For each $x, y \in X$,

$$H(Fx, Fy) \leq \alpha d(x, y) + \beta \max\{d(x, Fx), d(y, Ty)\} + \gamma[d(x, Fy) + d(y, Fy)], \quad (1)$$

where $\alpha, \beta, \gamma \geq 0$ and such that

$$s := (\frac{1 + \alpha + \gamma}{1 - \beta - \gamma})(\frac{\alpha + \beta + \gamma}{1 - \gamma}) < 1.$$

In this paper,we study the existence of fixed point on above contractive mapping under b-metric space.Our results generalization of [1].

2 Main results

Theorem 2.1. *Let (X, d) be a complete b-metric space with constant $s \geq 1$ and the mapping $T: X \to CB(X)$ be multivalued map satisfying*

$$H(Tx, Ty) \leq ad(x, y) + b \max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Ty)] \quad (2)$$

for all $x, y \in X$,and $a, b, c \in [0, 1)$ are constant such that $a + b + c < 1$.Then T has a unique fixed point in X.

Proof. For every $x_0 \in X$ and $n \geq 1$,$x_1 \in Tx_0$ and $x_{n+1} \in Tx_n$.we have

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq H(Tx_n, Tx_{n-1}) \\
&\leq ad(x_n, x_{n-1}) + b \max\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})\} \\
&+ c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1})] \\
&\leq ad(x_n, x_{n-1}) + b \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\
&+ c[d(x_n, x_n) + d(x_{n-1}, x_n)] \\
&\leq (a + c)d(x_n, x_{n-1}) + b \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \\
&\leq (a + c)d(x_n, x_{n-1}) + bM_1
\end{aligned}$$

where $M_1 = \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}$
Now two cases arises,
Case I: If suppose that $M_1 = d(x_n, x_{n+1})$then we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq (a+c)d(x_n, x_{n-1}) + bd(x_n, x_{n+1}) \\ &\leq (\frac{a+c}{1-b})d(x_n, x_{n-1}) \\ &\leq kd(x_n, x_{n-1}) \end{aligned} \tag{3}$$

where $k = (\frac{a+c}{1-b}) < 1$

$$\therefore d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),$$
$$d(x_n, x_{n+1}) \leq k^2 d(x_{n-1}, x_{n-2})$$

Continuing this process,we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1).$$

Case II: If suppose that $M_1 = d(x_{n-1}, x_n)$

$$\begin{aligned} d(x_n, x_{n+1}) &\leq (a+c)d(x_n, x_{n-1}) + bd(x_{n-1}, x_n) \\ &\leq (a+b+c)d(x_n, x_{n-1}) \\ &\leq kd(x_n, x_{n-1}) \end{aligned} \tag{4}$$

where $k = a+b+c < 1$

$$\therefore d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n),$$
$$d(x_n, x_{n+1}) \leq k^2 d(x_{n-1}, x_{n-2})$$

Continuing this process,we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1).$$

Let $m, n \in N$,$m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)], \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)], \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+}) + \ldots\ldots \\ &\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_{n+1}, x_{n+2}) + s^3 k^{n+2} d(x_0, x_1) + \ldots\ldots \\ &\leq sk^n d(x_0, x_1)[1 + sk + (sk)^2 + (sk)^3 + \ldots\ldots] \\ &\leq \frac{sk^n}{1-sk} d(x_0, x_1). \end{aligned} \tag{5}$$

Since $k < 1$, $\lim_{n\to\infty} \frac{sk^n}{1-sk} d(x_0, x_1) = 0$ as $n, m \to \infty$. Then $\{x_n\}$ is Cauchy sequence in (X, d).Since (X, d) is a complete b-metric space,there exists $z \in X$

such that $x_n \to z$ as $n \to \infty$.
Now,

$$\begin{aligned} d(z,Tz) &\leq s[d(z,x_{n+1}) + d(x_{n+1},Tz)] \\ &\leq sd(z,Tx_n) + sH(Tx_n,Tz) \\ &\leq sd(z,x_{n+1}) + s[ad(x_n,z) + b\max\{d(x_n,Tx_n),d(z,Tz)\} \\ &+ c(d(x_n,Tz) + d(z,Tz))] \\ &\leq sd(z,x_{n+1}) + s[ad(x_n,z) + b\max\{d(x_n,x_{n+1}),d(z,Tz)\} \\ &+ c(d(x_n,Tz) + d(z,Tz))] \\ &\leq sd(z,x_{n+1}) + sad(x_n,z) + sb\max\{d(x_n,x_{n+1}),d(z,Tz)\} \\ &+ sc(d(x_n,Tz) + d(z,Tz)) \\ &\leq sd(z,x_{n+1}) + sad(x_n,z) + sbM_2 + sc(d(x_n,Tz) + d(z,Tz)) \qquad (6) \end{aligned}$$

where $M_2 = \max\{d(x_n,x_{n+1}),d(z,Tz)\}$.
Case I: If suppose that $M_2 = d(x_n,x_{n+1})$ then we have,

$$\begin{aligned} d(z,Tz) &\leq sd(z,x_{n+1}) + sad(x_n,z) + sbd(x_n,x_{n+1}) + sc(d(x_n,Tz) + d(z,Tz)) \\ &\leq sd(z,x_{n+1}) + sad(x_n,z) + s^2bd(x_n,z) + s^2bd(z,x_{n+1}) \\ &+ s^2cd(x_n,z) + s^2cd(z,Tz) + scd(z,Tz) \\ &\leq (\frac{s+s^2b}{1-s^2c-sc})d(z,x_{n+1})) + (\frac{sa+s^2b+s^2c}{1-s^2c-sc})d(x_n,z) \qquad (7) \end{aligned}$$

Letting $n \to \infty$,we get $d(z,Tz) = 0$.
Thus,$Tz = z$.Therefore z is the fixed point of T.
Case II: If suppose that $M_2 = d(z,Tz)$ then we have,

$$\begin{aligned} d(z,Tz) &\leq sd(z,x_{n+1}) + sad(x_n,z) + sd(z,Tz) + sc(d(x_n,Tz) + d(z,Tz)) \\ &\leq sd(z,x_{n+1}) + sad(x_n,z) + sd(z,Tz) \\ &+ s^2cd(x_n,z) + s^2cd(z,Tz) + scd(z,Tz) \\ &\leq (\frac{s}{1-s-s^2c-sc})d(z,x_{n+1})) + (\frac{sa+s^2b}{1-s-s^2c-sc})d(x_n,z) \qquad (8) \end{aligned}$$

Letting $n \to \infty$,we get $d(z,Tz) = 0$.
Thus,$Tz = z$.Therefore z is the fixed point of T. Now we show that z is the unique fixed point of T.Assume that u is another fixed point of T.Then we have $Tv = v$ and

$$\begin{aligned} d(u,v) &= d(Tz,Tv) \\ &\leq s[d(u,Tv) + d(v,Tu)] \end{aligned}$$

we obtain,$d(u,v) \leq 2sd(u,v)$.This implies that $u = v$.Therefore T has a fixed point in X.

Example 2.2. *Let* $X = [0,1]$*.We define* $d\colon X \times X \to X$ *by* $d(x,y) = |x-y|^2$ *for all* $x, y \in X$*.Then* (X,d) *is a complete b-metric space.*
Define $T\colon X \to CB(X)$ *by* $Tx = \frac{x}{7}$ *for all* $x, y \in X$*.Then,*

$$H(Tx,Ty) = \frac{1}{49} d(x,y)$$

[where,$b = c = 0, a = \frac{1}{49}$*].*
Therefore,$0 \in X$ *is the unique fixed point of* T*.*

□

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