Edge Deletion and Restrained Sets in Graphs

D.K.Thakkar[1] and S.M.Badiyani[2]
[1]Department of Mathematics, Saurashtra University Campus, University Road, Rajkot – 360005 India
[2]V.V.P Engineering College, Kalawad Road, VirdaVagdi,Rajkot – 360005, India.

Abstract: *The concept of a Restrained set was defined in [1]. A set S of a vertices of a graph G is called a 'Restrained set' if for every vertex v not is S there is a vertex u not in S such that u is adjacent to v. In this paper we consider the effect of removing an edge from the graph on the restrainedness number. We prove necessary and sufficient conditions under which this number increases or decreases when an edge is removed from the graph. Also we show that when this number decreases, it decreases by 2 and when it increases, it increases by 2.*

Keywords:*Restrained Set, Maximal Restrained Set, RE Set, RE Number.*

AMS Mathematics Subject Classification (2010): *05C69.*

I. INTRODUCTION

Restrainedness is a concept which was introduced in [1]. The Restrainedness number(RE number) was also defined and several theorems were proved about the RE number of a graph. In this paper we continue the study and prove that the RE number of a graph may increase or decrease when an edge is removed from the graph.

II. PRELIMINARIES AND NOTATIONS

If G is a graph then $V(G)$ and $E(G)$ will denote the vertex set and edge set of the graph respectively. If S is any set then $|S|$ will denote the cardinality of S.

We will consider only simple graphs having finite vertex sets.

Definition 2.1.[1]: Let G be a graph. A subset S of $V(G)$ is called a '*Restrained Set*' if for every $v \in V(G) \setminus S$, there is u in $V(G) \setminus S$ such that u is adjacent to v.

Definition 2.2.[1]: A restrained set S of a graph G is said to be a '*Maximal Restrained Set'* if for every $v$in $V(G) \setminus S$, $S \cup \{v\}$ is not a restrained set.

Definition 2.3.[1]: Amaximal restrained set with minimum cardinality is called a '*RE Set*' of G.

Definition 2.4.[1]: The cardinality of a RE set is called the 'Restrainedness Number' or 'RE Number' of the graph G.

The following results were proved in [1].

Theorem 2.5.

A restrained set S is a maximal restrained set if and only if for every vertex $v \in V(G) \setminus S$ there exists a vertex $w \in V(G) \setminus S$ such that w is adjacent to only one vertex of $V(G) \setminus S$namely v. This vertex w is unique in $V(G) \setminus S$ for the given vertex v.

Corollary 2.6.

If S is amaximal restrained set then $V(G) \setminus S$ has even number of vertices.

We begin with the following proposition.

Proposition 2.7.

Let G be a graph and e be an edge of G.

1) If $RE(G \setminus e) < RE(G)$then$RE(G \setminus e) = RE(G) - 2$.
2)If $RE(G \setminus e) > RE(G)$then$RE(G \setminus e) = RE(G) + 2$.

Proof: 1) Let$S$ be an RE set of G and Let T be an RE set of$G \setminus e$.

Suppose$|T| = |S| - 1$.

Now $|V(G) \setminus S|$ is an even number and $|V(G) \setminus T|$ is also an even number.

Therefore $|S| - |T|$ must be an even number;which is a contradiction.

Thus $|T| = |S| - 2$.

Therefore$RE(G \setminus e) = RE(G) - 2$.

2) The proof of this part is similar to proof of part-1. ■

Now we prove a necessary and sufficient condition under which the RE number of a graph decreases.

Theorem 2.8.

Let G be a graph with $RE(G) > 2$and $e = uv$be an edge of Gthen $RE(G \setminus e) < RE(G)$if and only if the following two conditions are satisfied:

i) For every RE set $S$of$G \setminus e$, $u \notin S$and $v \notin S$.

ii)There is an RE set T of G such that $v \in T$and $u \notin T$ (OR $u \in T$ and $v \notin T$).

Proof: First suppose that $RE(G \setminus e) < RE(G)$.

i) Suppose there is an RE set S of $G \setminus e$ such that $u \in S$ or $v \in S$.

Then S is a maximal set of $G$and therefore $RE(G) \leq |S| = RE(G \setminus e)$;which is a contradiction.

Hence$u \notin S$ and $v \notin S$ for every RE set S of $G \setminus e$.

ii) Let S be any RE set of $G \setminus e$then $u \notin S$ and $v \notin S$.

Now there is a vertex x in $V(G) \setminus S$ which is adjacent to u only in $V(G) \setminus S$.Likewise there is a vertex y in $V(G) \setminus S$ which is adjacent to v only in$V(G) \setminus S$.

Note that $x \neq y$ because otherwise x will be adjacent to two vertices u and $v$outside S.

Let $T = S \cup \{u, x\}$ then T is an RE set of G because $RE(G \setminus e) < RE(G)$.

Obviously $u \in T$ but $v \notin T$.

(Note: $T' = S \cup \{v, y\}$ then T' is an RE set of G such that $v \in T$ and $u \notin T$.)

Conversely suppose condition i) and ii) are satisfied.

Let T be any RE set of G such that $u \in T$ and $v \notin T$ then T is a maximal subset of $G \setminus e$.

If $|T| = RE(G \setminus e)$ then we have a contradiction because according to the assumption u and v does not belongs to any RE set of G.

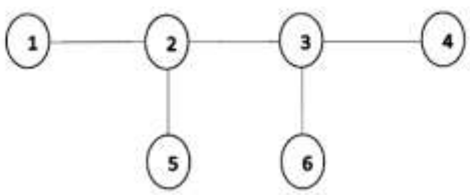
If $|T| < RE(G \setminus e)$ then we have a contradiction because any set whose cardinality is less than $RE(G \setminus e)$ cannot be a maximal subset of$G \setminus e$.

Therefore$|T| > RE(G \setminus e)$.

Therefore $RE(G) > RE(G \setminus e)$. ■

Example 2.9.

Consider the graph G as follows:



G

Figure 1

Here $S = \{1,4,5,6\}$ is an RE set of G and therefore $RE(G) = 4$.

Now consider theedge $e = 23$. Here $T = \{1,4\}$ is an RE set of $G \setminus e$ and therefore RE $(G \setminus e) = 2$.

Therefore $RE(G) > RE(G \setminus e)$. ■

Remark 2.10.

In theorem 14 of [1] it is proved that if $RE(G) > 1$ and $v \in V(G)$ then $RE(G \setminus v) < RE(G)$ if and only if $v \in S$ for some RE set $S$of G.

From Theorem 2.8. It is clear that if $e = uv$ is an edge of G, $RE\ (G) > 1$ and$RE(G \setminus e) < RE(G)$then $RE(G \setminus u) < RE(G)$ or $RE(G \setminus v) < RE(G)$.

Now we prove a necessary and sufficient condition under which the RE number of a graph increases.

Theorem 2.11.

Let G be a graph and $e = uv$ be an edge of G then $RE(G \setminus e) > RE(G)$if and only if the following two conditions are satisfied:

i) For every RE set T of$G$, $u \notin T$ and $v \notin T$.
ii) There is an RE set S of $G \setminus e$ such that $v \in S$ and $u \in S$.

Proof: First suppose that $RE(G \setminus e) > RE(G)$.

i) Suppose there is some RE set T of G such that $u \in T$ or $v \in T$.

Then T is a maximal set of $G \setminus e$ and therefore $RE(G \setminus e) \leq |T| = RE(G)$; which is a contradiction.

Hence $u \notin T$ and $v \notin T$ for every RE set T of G.

ii) Let T be a RE set of G so $u \notin T$ and $v \notin T$.

Let $S = T \cup \{u, v\}$. Then $|S| = |T| + 2$.

Let $x \in V(G) \setminus S$ then $x \in V(G) \setminus T$ and $x \neq u$ also $x \neq v$.

Since T is a maximal set in G, there is a unique vertex y in $V(G) \setminus T$ such that x is adjacent to y.

If $y = v$ then v would be adjacent to two distinct vertices of $V(G) \setminus T$ namely u and x which is a contradiction.

Therefore $y \neq v$.

Similarly $y \neq u$.

Thus y is a unique vertex of $(V(G \setminus e) \setminus S)$in $G \setminus e$ which is adjacent to x.

Thus S is an RE set of $G \setminus e$ with $v \in S$ and $u \in S$.

Conversely Suppose i) and ii) conditions are satisfied.

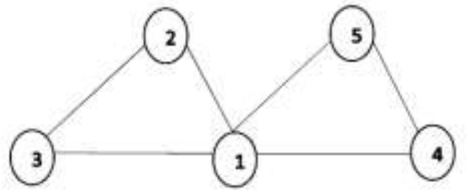
Let S be an RE set of $G \setminus e$ such that $v \in S$ and $u \in S$.Observe that S is maximal subset of Galso $v \in S$ and $u \in S$.

Therefore$|S| > RE(G)$.

$\Longrightarrow RE(G \setminus e) > RE(G)$. ■

Example 2.12.

Consider the graph G as follows:



G

Figure 2

Here $S = \{1\}$ is an RE set of G and therefore $RE(G) = 1$.

Now consider theedge $e = 45$. Here $T = \{1,2,3\}$ is an RE set of $G \setminus e$ and therefore RE $(G \setminus e) = 3$.

Therefore $RE(G) < RE(G \setminus e)$. ■

3. CONCLUDING REMARKS

For the Path GraphP_5, removal of every edge increases the RE number of a graph. The RE number of the Cycle Graph C_5 is 3 and removal of any edge decreases the RE number to 1.

It may be interesting to characterize the graphs for which

i) $RE(G) > RE(G \setminus e)$for every edge e.
ii) $RE(G) < RE(G \setminus e)$for every edge e.

4. REFERENCES

[1] D.K.Thakkar and B.M.Kakrech, Maximal Restrained Sets In Graphs, International Journal of Mathematics and Soft Computing, Volume 4 Issue 2, 2014, 207-211.

[2] G.S.Domke, J.H.Hettingh, S.T.Hedetniemi, R.C.Laskar and L.R.Markus, Restrained domination in graphs, Discrete Math. 203(1999), 61-69.

[3] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, FUNDAMENTAL OF DOMINATION IN GRAPHS, Marcel Dekker, Inc., New York, (1998).

[4] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, DOMINATION IN GRAPHS ADVANCED TOPICS, Marcel Dekker, Inc., New York, (1998).