A Generalization of a fixed point theorem of HONG-KUN XU

Sujata Goyal
Assistant Professor, Department of Mathematics, A.S. College
Khanna-141401, India

Abstract: *Xu. H. [1] introduced weakly asymptotic contraction and proved that if* $T:X\to X$ *is a continuous map where (X,d) is a complete metric space and* $\phi:R^+\to R^+$ *a map ,which is continuous and* $\phi(s)<s$ *for all* $s>0$ *,* $\phi(0)=0$ *such that given* $\in>0$ *, there exists* $n_\in>0$ *such that* $d(T^{n_\in}x, T^{n_\in}y)\le \phi(d(x,y))+\in$ *, for all x , y in X . It is also assumed that some orbit of T i.e.* $\{T^n x : n\in N\}$ *for some* $x\in X$ *is bounded . Then T has a unique fixed point y in X . Also* $T^n x\to y$ *as* $n\to\infty$*. In this paper ,it has been shown that result is still true if the function* ϕ *is assumed to be upper semicontinuous.*

Keywords: complete metric space, Cauchy sequence, fixed point, continuous map, upper semicontinuous map, limit superior

Introduction

Let (X, d) be a complete metric space. A map $T: X\to X$ is said to be contraction map if there exists a constant c ,$0<c<1$ such that $d(Tx, Ty)\le c\, d(x, y)$ for all x , y in X . Banach proved that in this case, T has a unique fixed point. Due to its wide applications, the theorem has been extended in a number of ways; see, for example, [3],[4] and [5].

In this direction, Kirk [2] introduced the notion of asymptotic contraction which is an asymptotic version of fixed point theorem by Boyd and Wong [7] and proved the fixed point theorem for this class of mappings.

Definition : Let (M,d) be a metric space. A mapping $T: M\to M$ is said to be an asymptotic contraction if

$d(T^n x, T^n y)\le \phi_n(d(x,y))$ for all $x,y\in M$, where $\phi_n:[0,\infty)\to[0,\infty)$ and $\phi_n\to\phi$ uniformly on the range of d.

where $\phi:R^+\to R^+$ is continuous such that $\phi(s)<s$ for all $s>0$, $\phi(0)=0$.

Kirk [2] proved the following theorem:

Theorem: Suppose (M,d) is a complete metric space and suppose $T:M\to M$ is an asymptotic contraction for which the mappings ϕ_n are also continuous. Assume also that some orbit of T is bounded.Then T has a unique fixed point $z\in M$, and moreover the Picard sequence $(T^n(x))$ converges to z for each $x\in M$.

The proof given by Kirk is nonconstructive , it uses ultrapower techniques and thus depends on the axiom of choice . Simple proofs of Kirk theorem has been given in [6] and further generalizations of the theorem have been given in [8] , [9] and [10].

Xu. H.[1] generalized the result of kirk by introducing weakly asymptotic contractions.

Definition : A continuous mapping T from a complete metric space to itself is said to be weakly asymptotic contraction if given $\in>0$, there exists $n_\in>0$ such that $d(T^{n_\in}x, T^{n_\in}y)\le\phi(d(x,y))+\in$, for all x, y in X, where $\phi:R^+\to R^+$ is a map ,which is continuous and $\phi(s)<s$ for all $s>0$, $\phi(0)=0$.

Xu. H. proved that if $T : X \rightarrow X$ is a weakly asymptotic contraction mapping , where (X,d) is a complete matric space .Also assume that some orbit of T i.e. $\{ T^n x : n \in N \}$ for some $x \in X$ is bounded. Then T has a unique fixed point y in X . Also

$T^n x \rightarrow y$ as $n \rightarrow \infty$

Main purpose of this paper is to show that result of Xu. H. [1] is still valid if the function ϕ is assumed to be upper semicontinuous only.

Main Theorem : *Let (X,d) be a complete metric space . $T: X \rightarrow X$ be a continuous map. Suppose there exists a map*

$\phi : R^+ \rightarrow R^+$ which is upper semicontinuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.

If T satisfies the following condition:

Given $\in > 0$, there exists $n_\in > 0$ such that

$d(T^{n_\in} x , T^{n_\in} y) \leq \phi(d(x,y)) + \in$

Assume that some orbit of T i.e. $\{ T^n x : n \in N \}$ for some $x \in X$ is bounded . Then T has a unique fixed point y in X . Also

$T^n x \rightarrow y$ as $n \rightarrow \infty$

To prove the theorem we need the following lemma :

Lemma : *If $\phi : R^+ \rightarrow R^+$ is upper semicontinuous and $\phi(s) < s$ for all $s > 0$, $\phi(0) = 0$.*

Define the function $\psi(t) = \max \{ \phi(\tau) : \tau \in [0 , t] \}$ then

1) *ψ is increasing*
2) *$\psi(s) < s$ for all $s > 0$*
3) *ψ is upper semicontinuous.*
(Note here that every upper semicontinuous map on a compact set assumes its maximum)

Proof : clearly ψ is increasing . Now if $s > 0$ then

$\psi(s) = \phi(\tau^\circ)$ for some $\tau^\circ \in [0, s]$, so that $\psi(s) < s$.

Now we show that ψ is upper semicontinuous :

Let $\in > 0$ be given :

Case 1 : $t > t_0 > 0$

$$\psi(t) - \psi(t_0) = \max \{ \phi(\tau) : \tau \in [0, t] \} - \max \{ \phi(\tau) : \tau \in [0, t_0] \}$$

$$\leq \max \{ \phi(\tau) : \tau \in [t_0 , t] \} - \phi(t_0)$$

$$= \max \{ \phi(\tau) - \phi(t_0) : \tau \in [t_0 , t] \}$$

Now since ϕ is upper semicontinuous , therefore there exists $\delta > 0$ such that

$\phi(t) < \phi(t_0) + \in$ whenever $|t-t_0| < \delta$. Thus $\psi(t) - \psi(t_0) < \in$ whenever $|t-t_0| < \delta$

Case 2 : $0 < t < t_0$

$\psi(t) - \psi(t_0) = \max\{\phi(\tau) : \tau \in [0, t]\} - \max\{\phi(\tau) : \tau \in [0, t_0]\}$

$\leq \max\{\phi(\tau) : \tau \in [t, t_0]\} - \phi(t_0)$

$= \max\{\phi(\tau) - \phi(t_0) : \tau \in [t, t_0]\}$

Now since ϕ is upper semicontinuous , therefore there exists $\delta > 0$ such that

$\phi(t) < \phi(t_0) + \in$ whenever $|t-t_0| < \delta$. Thus $\psi(t) - \psi(t_0) < \in$ whenever $|t-t_0| < \delta$

Case 3 : $t_0 = 0$,$t > t_0$

$\psi(t) - \psi(t_0) = \max\{\phi(\tau) : \tau \in [0, t]\} - \phi(t_0) \leq \max\{\phi(\tau) - \phi(t_0) : \tau \in [0, t]\}$

Now since ϕ is upper semicontinuous at $t_0 = 0$, therefore there exists $\delta > 0$ such that

$\phi(t) < \phi(t_0) + \in$ whenever $t-t_0 < \delta$. Thus $\psi(t) - \psi(t_0) < \in$ whenever $t-t_0 < \delta$

Proof of main theorem : Put $d_{n,m} = d(T^n x, T^m y)$

Let $d_\infty = \overrightarrow{\lim_{n.m\to\infty}} d_{n,m} = \lim_{k\to\infty} \sup\{d_{n,m} : n, m \geq k\} < \infty$

Let $\in > 0$ be given :

Now $d_{n,m} = d(T^{n_\in}(T^{n-n_\in} x), T^{n_\in}(T^{m-n_\in} y)) \leq \phi(d(T^{n-n_\in} x), (T^{m-n_\in} y)) + \in$

$\leq \psi(d(T^{n-n_\in} x), (T^{m-n_\in} y)) + \in$

Taking limit superior we get :

$$\overrightarrow{\lim_{n,m\to\infty}} d_{n,m} \leq \overrightarrow{\lim_{n,m\to\infty}} [\psi(d(T^{n-n_\in} x), (T^{m-n_\in} y)) + \in]$$

$$= \overrightarrow{\lim_{n,m\to\infty}} \psi(d(T^{n-n_\in} x), (T^{m-n_\in} y)) + \in$$

$$= \overrightarrow{\lim_{n,m\to\infty}} \psi(d_{n-n_\in, m-n_\in}) + \in \qquad ...(1)$$

Claim : = $\overrightarrow{\lim_{n,m\to\infty}} \psi(d_{n-n_\in, m-n_\in}) \leq \psi(\overrightarrow{\lim_{n,m\to\infty}} d_{n-n_\in, m-n_\in})$

Let $\overrightarrow{\lim_{n,m\to\infty}} d_{n-n_\in, m-n_\in} = L$

$\Rightarrow \lim_{k\to\infty} \sup\{d_{n-n_\in, m-n_\in} : n, m \geq k\} = L$

Now as ψ is upper semicontinuous , this implies that

$\lim_{k\to\infty} \psi(\sup\{d_{n-n_\in, m-n_\in} : n, m \geq k\}) \leq \psi(L)$

Clearly $\sup\{\psi(d_{n-n_\in, m-n_\in}) : n, m \geq k\} \leq \psi(\sup\{d_{n-n_\in, m-n_\in} : n, m \geq k\})$

Thus we have

$\lim_{k\to\infty} \sup\{\psi(d_{n-n_\in, m-n_\in}) : n, m \geq k\} \leq \psi(L)$

$\Rightarrow \overrightarrow{\lim_{n,m\to\infty}} \psi(d_{n-n_\in, m-n_\in}) \leq \psi(\overrightarrow{\lim_{n,m\to\infty}} d_{n-n_\in, m-n_\in})$, this proves the claim.

Thus from eq. (1) , we get :

$$\overrightarrow{\lim_{n.m\to\infty}} d_{n,m} = \psi\left(\overrightarrow{\lim_{n,m\to\infty}} d_{n-n_\in, m-n_\in} \right) + \in$$

$\Rightarrow d_\infty \leq \psi(d_\infty) + \in$

Since $\in > 0$ is arbitrary , therefore $d_\infty \leq \psi(d_\infty)$

$\Rightarrow d_\infty = 0$

$\Rightarrow$ (T^n (x)) is Cauchy sequence. Since X is complete, (T^n (x)) will converge to unique fixed point y of X

References:

[1] Xu, H. K. (2005), "Asymptotic and weakly asymptotic contractions", Indian Journal of Pure and Applied Mathematics, Vol. 36, No. 3, pp. 145-150.

[2] Kirk, W.A. (2004), "Fixed points of asymptotic contractions", Journal of Mathematical Analysis and Applications, Vol. 277, No. 2, pp. 645-650.

[3] Edelstein, M. (1961), "An Extension of Banach's Contraction Principle", Proceedings of the American Mathematical Society, Vol. 12, No. 1, pp. 7-10. 2

[4] Meir, A. and Keeler, E. (1969), "A theorem on contraction mappings", Journal of Mathematical Analysis and Applications, Vol. 28, No. 2, pp. 326-329.

[5] Ćirić, L.B. (1974), "A generalization of Banach's contraction principle", Proceedings of the American Mathematical Society, Vol. 45, No. 2, pp. 267-273.

[6] Arandelovic, I.D. (2005), "On a fixed point theorem of Kirk", Journal of Mathematical Analysis and Applications, Vol. 301, No. 2, pp. 384-385.

[7] Boyd, D.W. and Wong, J.S.W. (1969), "On Nonlinear Contractions", Proceedings of the American Mathematical Society, Vol. 20, No. 2,pp 458-464.

[8] Chen, Y.Z. (2005), "Asymptotic fixed points for nonlinear contractions" Fixed Point Theory and Applications, Vol. 2, pp. 213–217.

[9] Jachymski J. and Jóźwik I. (2004), "On Kirk's asymptotic contractions" Journal of Mathematical Analysis and Applications, Vol. 300, No. 1, pp. 147-159.

[10] Suzuki, T. (2007), "A definitive result on asymptotic contractions" Journal of Mathematical Analysis and Applications, Vol. 335, No. 1, pp. 707-715.