E- Bayesian Estimation of the Parameter of Truncated Geometric Distribution

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October 15, 2016

Abstract

Dr. Han Ming (2005) is the first person who define E- Bayesian estimation of the estate probability. He also gave formulas of E- Bayesian estimation, forecast model and its applications in security investment. This paper aims at providing the E- Bayesian estimation procedure for the probability parameter of the truncated Geometric distribution. Simulation study has been done at the end of the paper to study the performance of the estimators.

Keywords: Failure Probability, Truncation, Geometric Distribution, E-Bayesian, estate probability, Reliability.

1 Introduction

Truncated data sets have to be oftenly dealt with in the area of reliability of industrial products, when concerned with the life testing of products with high reliability. Difficulty may arise when we estimate the parameter related to the lifetime of the product by using classical statistical techniques. E- Bayesian Estimation procedure has found to be more simplest than Hierarchical Bayesian estimation. We, in this paper, have developed the E- Bayesian estimation procedure for the parameter of the truncated Geometric distribution.

The Geometric distribution arises in many real life situations. Most commonly, we use this distribution to model the probability of getting first success when we perform independent Bernoulli trials. Also, it can be seen as the distribution of the total size of a population in a pure birth process and we can also find it as a limiting distribution of the size of the queue in a $M|M|1$ model. One of the important property of this distribution is the lack of memory property which plays an important part in the branch of applied probability. Many characterization of Geometric distribution can be found in the literature, mostly based on the independence of functions of order Statistics or on record values, see e.g Govindarajulu (1980), Arnold (1980), EL- Neweihi and Govindarajulu (1979), Srivastava (1979) and Galambos (1975). Xie and Goh (1997) proved that control charts based on Geometric distribution have shown to be useful when this is a better approximation of the underlying distribution than Poisson distribution. They also discussed the use of Geometric distribution for process control of high yield processes. Shanbhag (1970) gave a characteristic property of the Geometric distribution based on the means of the conditional distribution. We have given the E-Bayesian estimators of the parameter of truncated Geometric distribution and simulation study has been done to prove the property followed by E- Bayesian estimators.

2 Expected Bayesian Estimation of the Zero Truncated Geometric Distribution

Han(2007) introduced the concept of expected Bayesian estimator to estimate the failure probability. According to Han(2009), the following prior distributions for α and β are considered :

$$h_1(\alpha, \beta) = \frac{2(c-\beta)}{c^2}, 0 < \alpha < 1, 0 < \beta < c \tag{2.1}$$

$$h_2(\alpha, \beta) = \frac{1}{c}, 0 < \alpha < 1, 0 < \beta < c \tag{2.2}$$

$$h_3(\alpha, \beta) = \frac{2\beta}{c^2}, 0 < \alpha < 1, 0 < \beta < c \tag{2.3}$$

which are decreasing , constant and increasing functions of β , respectively.

In the following theorem, we have found the E-Bayesian estimators of the parameter θ of the truncated Geometric distribution. Simulation part has been done after the proof of the theorem where we have proved the hierarchy of the E-Bayesian Estimators.

Theorem 4.3 For prior distributions $(2.1), (2.2), (2.3)$ the corresponding Bayes estimators of θ based on SELF are, respectively :

$$
\begin{aligned}
\hat{\theta}_{ES1} &= \frac{2}{c^2}[-\frac{1}{6}ln(-1+n+z+c)+\frac{1}{6}ln(-1+n+z)+ln(n+z+c)cz \\
&- ln(-1+n+z+c)cz\frac{1}{2}ln(n+z)z^2+\frac{1}{2}ln(n+z+c)n^2+\frac{1}{2}ln(-1 \\
&+ n+z)n^2z-\frac{1}{2}ln(n+z)n^2+\frac{1}{2}ln(n+z+c)z^2+\frac{1}{2}ln(n+z+c)c^2 \\
&- \frac{1}{2}ln(-1+n+z)cz^2+\frac{1}{2}ln(-1+n+z)z^2-\frac{1}{2}ln(-1+n+z)z+ \\
& \frac{1}{2}ln(-1+n+z+c)n^2-\frac{1}{2}ln(-1+n+z)n^2-\frac{1}{2}ln(-1+n+z+c)z^2 \\
&+ \frac{1}{2}ln(-1+n+z+c)z-\frac{1}{2}ln(-1+n+z+c)c^2+\frac{1}{2}ln(-1+n+z+c) \\
&. \quad c-\frac{1}{6}c+\frac{1}{6}c^2-(2/3)cn-(\frac{1}{3})ln(n+z)n^3-\frac{1}{6}ln(n+z+c)z^3-\frac{1}{6}ln(n \\
&+ z+c)c^3+\frac{1}{6}ln(n+z)z^3+(\frac{1}{3})ln(n+z+c)n^3-(\frac{1}{3})ln(-1+n+z+c) \\
&. \quad n^3+(\frac{1}{3})ln(-1+n+z)n^3-\frac{1}{2}ln(-1+n+z)c+\frac{1}{6}ln(-1+n+z+c)
\end{aligned}
$$

$$
\begin{aligned}
. \quad & c^3 - \frac{1}{6}ln(-1+n+z)z^3 - \frac{1}{6}ln(-1+n+z)z^3 + \frac{1}{6}ln(-1+n+z+c)z^3 \\
- \quad & \frac{1}{6}zc - ln(n+z)cz - ln(n+z)nz + ln(n+z+c)cn + ln(n+z+c)nz - \\
& \frac{1}{2}ln(n+z+c)cz^2 - \frac{1}{2}ln(n+z+c)c^2z + \frac{1}{2}ln(n+z+c)n^2z + \frac{1}{2}ln(n+z \\
+ \quad & c)cn^2 + \frac{1}{2}ln(n+z)cz^2 - \frac{1}{2}ln(n+z)n^2z - \frac{1}{2}ln(n+z)cn^2 - ln(n+z)cn + \\
& \frac{1}{2}ln(-1+n+z)cn^2 + ln(-1+n+z)cz + \frac{1}{2}ln(-1+n+z+c)cz^2 + \frac{1}{2}ln(\\
- \quad & 1+n+z+c)c^2z - \frac{1}{2}ln(-1+n+z+c)n^2z - \frac{1}{2}ln(-1+n+z+c)cn^2]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\theta}_{ES2} \quad = \quad & \frac{1}{c}[-ln(n+z+c)cz + ln(-1+n+z+c)cz - \frac{1}{2}ln(-1+n+z) \\
+ \quad & \frac{1}{2}ln(n+z)z^2 - ln(n+z)z + ln(n+z+c)n - ln(n+z)n + \frac{1}{2}c \\
- \quad & \frac{1}{2}ln(n+z)n^2 - \frac{1}{2}ln(n+z+c)z^2 + ln(n+z+c)z + \frac{1}{2}ln(-1+ \\
& n+z+c) - \frac{1}{2}ln(n+z+c)c^2 + ln(n+z+c)c - \frac{1}{2}ln(-1+n+ \\
& z)z^2 + \frac{1}{2}ln(n+z+c)n^2 + ln(-1+n+z)z - \frac{1}{2}ln(-1+n+z+
\end{aligned}
$$

$$
\begin{aligned}
. \quad & c)n^2+\frac{1}{2}ln(-1+n+z)n^2+\frac{1}{2}ln(-1+n+z+c)z^2-ln(-1+n \\
+ \quad & z+c)z+\frac{1}{2}ln(-1+n+z+c)c^2-ln(-1+n+z+c)c]
\end{aligned}
$$

$$
\begin{aligned}
\hat{\theta}_{ES3} \quad = \quad & \frac{2}{c^2}[\frac{1}{6}ln(-1+n+z+c)-\frac{1}{6}ln(-1+n+z)+\frac{1}{2}ln(n+z)z^2+\frac{1}{2}ln(n \\
+ \quad & z)n^2-\frac{1}{2}ln(n+z+c)z^2+\frac{1}{2}ln(n+z+c)c^2-\frac{1}{2}ln(-1+n+z)z^2+\frac{1}{2} \\
& ln(-1+n+z)z-\frac{1}{2}ln(-1+n+z+c)n^2+\frac{1}{2}+\frac{1}{2}ln(-1+n+z^2)n^2 \\
+ \quad & \frac{1}{2}ln(-1+n+z+c)z^2-\frac{1}{2}ln(-1+n+z+cz)-\frac{1}{2}ln(n+z+c)n^2- \\
& \frac{1}{2}ln(-1+n+z+c)c^2+\frac{1}{6}c+(\frac{1}{3})c^2+(2/3)cn-\frac{1}{2}ln(-1+n+z)n^2z \\
+ \quad & \frac{1}{3}ln(n+z)n^3+\frac{1}{6}ln(n+z+c)z^3-\frac{1}{3}ln(n+z+c)c^3-\frac{1}{6}ln(n+z)z^3 \\
- \quad & (\frac{1}{3})ln(n+z+c)n^3+\frac{1}{3}ln(-1+n+z+c)n^3-(\frac{1}{3})ln(-1+n+z)n^3 \\
+ \quad & \frac{1}{3}ln(-1+n+z+c)c^3+\frac{1}{6}ln(-1+n+z)z^3-\frac{1}{6}ln(-1+n+z+c)z^3 \\
+ \quad & \frac{1}{6}zc+ln(n+z)nz-ln(n+z+c)nz-\frac{1}{2}ln(n+z+c)c^2z-\frac{1}{2}ln(n+
\end{aligned}
$$

$$z + c)n^2z + \frac{1}{2}ln(n + z)n^2z + \frac{1}{2}ln(-1 + n + z + c)c^2z + \frac{1}{2}ln(-1 + n +$$

$$z + c)n^2z]$$

Proof: For the prior distribution (2.1) ,

$$\begin{aligned}
\hat{\theta}_{ES1} &= \int_{\alpha}\int_{\beta}\hat{\theta}_{sh_1}(\alpha,\beta)d\beta d\alpha \\
&= \frac{2}{c^2}\int_0^1 \frac{n+\alpha}{n+\alpha+\beta+z-1}\int_0^c \frac{c-\beta}{n+\alpha+\beta+z-1}d\beta d\alpha \\
&= \frac{2}{c^2}[-\frac{1}{6}ln(-1+n+z+c) + \frac{1}{6}ln(-1+n+z) + ln(n+z+c)cz \\
&- \quad ln(-1+n+z+c)cz\frac{1}{2}ln(n+z)z^2 + \frac{1}{2}ln(n+z+c)n^2 - \frac{1}{2}ln(n \\
&+ \quad z)n^2 + \frac{1}{2}ln(n+z+c)z^2 + \frac{1}{2}ln(n+z+c)c^2 + \frac{1}{2} + ln(-1+n+z) \\
&. \quad z^2 - \frac{1}{2}ln(-1+n+z)z + \frac{1}{2}ln(-1+n+z+c)n^2 - ln(n+z)cz - \\
&\quad \frac{1}{2}ln(-1+n+z)n^2 - \frac{1}{2}ln(-1+n+z+c)z^2 + \frac{1}{2}ln(-1+n+z+c)z \\
&- \quad \frac{1}{2}ln(-1+n+z+c)c^2 + \frac{1}{2}ln(-1+n+z+c)c - \frac{1}{6}c + \frac{1}{6}c^2 - (2/3)cn \\
&- \quad (\frac{1}{3})ln(n+z)n^3 - \frac{1}{6}ln(n+z+c)z^3 - \frac{1}{6}ln(n+z+c)c^3 + \frac{1}{6}ln(n+z)z^3
\end{aligned}$$

$$
\begin{aligned}
+ & \quad (\frac{1}{3})ln(n+z+c)n^3 - (\frac{1}{3})ln(-1+n+z+c)n^3 + (\frac{1}{3})ln(-1+n+z)n^3 \\
- & \quad \frac{1}{2}ln(-1+n+z)c + \frac{1}{6}ln(-1+n+z+c)c^3 - \frac{1}{6}ln(-1+n+z)z^3 - \frac{1}{6}z \\
. & \quad c - \frac{1}{6}ln(-1+n+z)z^3 + \frac{1}{6}ln(-1+n+z+c)z^3 + \frac{1}{2}ln(-1+n+z)n^2z \\
- & \quad ln(n+z)nz + ln(n+z+c)cn + ln(n+z+c)nz - \frac{1}{2}ln(n+z+c)cz^2 - \\
& \quad \frac{1}{2}ln(n+z+c)c^2z + \frac{1}{2}ln(n+z+c)n^2z + \frac{1}{2}ln(n+z+c)cn^2 + \frac{1}{2}ln(n+z) \\
. & \quad cz^2 - \frac{1}{2}ln(n+z)n^2z - \frac{1}{2}ln(n+z)cn^2 - ln(n+z)cn + \frac{1}{2}ln(-1+n+z)cn^2 \\
+ & \quad ln(-1+n+z)cz + \frac{1}{2}ln(-1+n+z+c)cz^2 + \frac{1}{2}ln(-1+n+z+c)c^2z - \\
. & \quad \frac{1}{2}ln(-1+n+z+c)n^2z - \frac{1}{2}ln(-1+n+z+c)cn^2 - \frac{1}{2}ln(-1+n+z)cz^2]
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\hat{\theta}_{ES2} &= \int_\alpha \int_\beta \hat{\theta}_{sh_2}(\alpha,\beta)d\beta d\alpha \\
&= \frac{1}{c}[-ln(n+z+c)cz + ln(-1+n+z+c)cz - \frac{1}{2}ln(-1+n+z) \\
&+ \frac{1}{2}ln(n+z)z^2 - ln(n+z)z + ln(n+z+c)n - ln(n+z)n - \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
. & \quad ln(n+z)n^2 - \frac{1}{2}ln(n+z+c)z^2 + ln(n+z+c)z + \frac{1}{2}ln(-1+ \\
. & \quad n+z+c) - \frac{1}{2}ln(n+z+c)c^2 + ln(n+z+c)c - \frac{1}{2}ln(-1+n \\
+ & \quad ln(-1+n+z)z - \frac{1}{2}ln(-1+n+z+c)n^2 + \frac{1}{2}ln(-1+n+z) \\
. & \quad n^2 + \frac{1}{2}ln(-1+n+z+c)z^2 - ln(-1+n+z+c)z + \frac{1}{2}ln(-1 \\
+ & \quad n+z+c)c^2 - ln(-1+n+z+c)c + \frac{1}{2}c+z)z^2 + \frac{1}{2}ln(n+z+ \\
. & \quad c)n^2]
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\theta}_{ES3} &= \int_\alpha \int_\beta \hat{\theta}_{sh_3}(\alpha,\beta) d\beta d\alpha \\
&= \frac{2}{c^2}[\frac{1}{6}ln(-1+n+z+c) - \frac{1}{6}ln(-1+n+z) + \frac{1}{2}ln(n+z)z^2 + \frac{1}{2} \\
. & \quad ln(n+z)n^2 - \frac{1}{2}ln(n+z+c)z^2 + \frac{1}{2}ln(n+z+c)c^2 - \frac{1}{2}ln(-1+n \\
+ & \quad z)z^2 + \frac{1}{2}ln(-1+n+z)z - \frac{1}{2}ln(-1+n+z+c)n^2 + \frac{1}{2} + \frac{1}{2}ln(-1 \\
+ & \quad n+z^2)n^2 + \frac{1}{2}ln(-1+n+z+c)z^2 - \frac{1}{2}ln(-1+n+z+cz) - \frac{1}{2}ln
\end{aligned}
$$

$$. \quad (n+z+c)n^2 - \frac{1}{2}ln(-1+n+z+c)c^2 + \frac{1}{6}c + (\frac{1}{3})c^2 + (2/3)cn - \frac{1}{2}$$

$$. \quad ln(-1+n+z)n^2z + \frac{1}{3}ln(n+z)n^3 + \frac{1}{6}ln(n+z+c)z^3$$

$$- \quad (\frac{1}{3})ln(n+z+c)n^3 + \frac{1}{3}ln(-1+n+z+c)n^3 - (\frac{1}{3})ln(-1+n+z)n^3$$

$$+ \quad \frac{1}{3}ln(-1+n+z+c)c^3 + \frac{1}{6}ln(-1+n+z)z^3 - \frac{1}{6}ln(-1+n+z+c)z^3$$

$$+ \quad \frac{1}{6}zc + ln(n+z)nz - ln(n+z+c)nz - \frac{1}{2}ln(n+z+c)c^2z - \frac{1}{2}ln(n+$$

$$z+c)n^2z + \frac{1}{2}ln(n+z)n^2z + \frac{1}{2}ln(-1+n+z+c)c^2z - \frac{1}{3}ln(n+z+$$

$$c)c^3 - \frac{1}{6}ln(n+z)z^3 + \frac{1}{2}ln(-1+n+z+c)n^2z]$$

Result : The Expected Bayes estimator of parameter θ satisfy :

$$\hat{\theta}_{ES3} < \hat{\theta}_{ES2} < \hat{\theta}_{ES1}$$

As these estimators are in the form of long numerical expressions, so for the sake of simplicity we have compared them with the help of simulation study. Table 1 and Table 2 proves the above result.

Table 1: **Expected Bayes Estimators for** $n = 10$ **and** $c = 2$

$\hat{\theta}_{ES3}$	$\hat{\theta}_{ES2}$	$\hat{\theta}_{ES1}$	z
.47	.92	.94	1
.32	.78	.84	3
.2	.68	.70	5
.11	.6	.61	7
.03	.28	.55	9

Table 2: **Expected Bayes Estimators for** $n = 15$ **and** $c = 4$

$\hat{\theta}_{ES3}$	$\hat{\theta}_{ES2}$	$\hat{\theta}_{ES1}$	z
.75	.91	.94	1
.68	.84	.86	3
.62	.78	.80	5
.56	.72	.74	7
.51	.67	.69	9

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