Bulk arrival, fixed batch service queue with unreliable server, Bernoulli vacation, Two stages of service and with Delay time

Kalyanaraman.R[1], Nagarajan.P[2]

[1] *Departmemt of mathematics, Annamalai University, Annamalai nagar-608002, India*
2 Department of mathematics, SCSVMV University, Enathur-631561, Kanchipuram,

Abstract- *A Poisson queue with batch arrival and two stages of batch service has been considered. In addition the server take Bernoulli vacation, the server may breakdown and the services are given in two stages. For this model, using supplementary variable technique, the probability generating function of numbers of customers in the queue at various server states have been obtained. Some operating characteristics have been derived and numerical examples are given.*

Keywords- *queuing; vacation; unreliable server; probability generating function and operating characteristics.*

AMS Subject classification number- 90B22, 60K25 and 60K30.

1. INTRODUCTION

The congestion situatins encountered in computer, communication, manufacturing, production system, etc, can be modelled as queueing system with vacation. Several researchers have contributed significantly on vacation models [Takagi [18], Lee etal. [15], Bacot etal [1] and Choudhury [2], Ke [12]]. Doshi [5]and Takagi [18] are the two excellent survey works on vacation queues. In many real life situations, the server may break downs, so that a more realistic queueing model is that which incorporates the assumption of unreliable server. Many researchers have contributed on queue with unreliable servers [Li etal [16], Wang [20], Wang [21]]. Some notable works on queuing with break down are Wang [20,21] Wang etal [22] and Ke [11]. Many researchers have studied the queueing model with unreliable server in different frameworks and suggested ways and means to tackles related situations. Grey etal [7] incorporated the server breakdown on vacation queueing model. Haridass and Arumuganathan [8] studied M $^{[X]}$ /G/1 queuing system with an unreliable server and with single vacation. Choudhury and Deka [2] investigated an M/G/1 unreliable server Bernoulli vacation queue with two phases of service . In 2013, the same authors studied a batch arrival unreliable server Bernoulli vacation queue with two phases of service and delayed repair [3]. Ke etal [13], analyzed an M $^{[X]}$ /G/1 queuing system with an unreliable server and repair, in which the server operates with a randomized vacation policy with multiple available vacation. The motivation of queueing model with two phases of services mainly comes from communication networks, in which messages are processed in two stages by a single server. This type of models has been first studied by Krishna and Lee [14]. Some notable works are Doshi [6], Selvam and Sivasankaran [17], Kalyanaraman and Ayyappan [9], Kalyanaraman and Nagarajan [10], Choudhoury etal [4] and Thangaraj and vanitha [19].

In this article we consider an M$^{[X]}$/G^{K}/1 queue with unreliable server and with Bernoulli vacation. In addition each service has two stages, In addition the server provides two stages of service. This type of queuing system exists in manufacturing industries, Transportation system etc. In manufacturing industries, after products are approved for transportation to customer shops, they are transported to the shops in bulks by truck. After transporting the products, if no batch is available for transportation, the truck will be used for other work or the truck is sent for maintenance (vacation period). During the service period (transportation period), the trucks may break down. During the production stage, the product undergo serveral stages of service like compiling, quality testing ect., The above situation can be modeled as an M$^{[X]}$/G^{K}/1 queue with unreliable server and Bernoulli vacation and with two stages of service.

The remainder of this article is organized as follows: Section 2 provides the model description and mathematical analysis. In section 3, we obtain some queuing characteristics of the model discussed in this paper. In section 4, we present some particular models. In section 5, we illustrate the model by some numerical examples. Finally, In section 6 we present a conclusion.

2.THE MODEL AND ANALYSIS

We condisder and $M^{[X]}/G^K/1$ queueing system, where the number of customers arrives to the system at time instant follows a compound Poisson process with arrival rate λ. The size of the successive arriving batches is a random vabirable with probability P{X=j}=C $_j$,whose probability generating function is defined by $C(z)=\sum_{j=1}^{\infty} C_j z^j$.

The services are given in batchs of fixed size 'K'. Each batch undergoes two stages of heterogenous service provided by a single server on a first come first served basis. The service time random periods of the two stages follow different gernerally distributed random variables with distribution function $G_i(x)$ and the density function $g_i(x)$ for i=0,1.

After completion of second stage of service, the server takes a Bernoulli vacation of random duration. The vacation period is also generally distributed with distribution function $B(x)$.

In addition, the server may breakdown during a service and the breakdowns are assumed to occur according to a Poisson process with rate 'α'. Once the server breakdown, the customer whose service is interrupted goes to the head of the queue and the repair to server starts immediately. The duration of the repair period is generally distributed with distribution function $H(x)$. Immediatly after the broken server is repaired, the server is ready to start its service. Further, we assume that the input process, server life time, server repair time, sevice time and vacation times are independent of each other.

The analysis of this model is based on supplementary variable technique and the supplementary variable is elapsed service time / elapsed vacation time / elapsed repair time.

We define the following probabilities and conditional probabilities:

$\mu_i(x)=\dfrac{g_i(x)}{1-G_i(x)}$ for i=1,2 is the conditional probability that the completion of i^{th} phase service during the interval $(x, x+dx)$, given that the elapsed service time is 'x'.

$\beta(x)=\dfrac{b(x)}{1-B(x)}$ is the conditional probability that the completion of vacation during the interval $(x, x+dx)$, given that the elapsed vacation time is 'x'. $\varepsilon(x)=\dfrac{w(x)}{1-W(x)}$ is the conditional probability that the completion of repair during the interval $(x, x+dx)$, given that the elapsed repair time is' x'. $\gamma(x)=\dfrac{h(x)}{1-H(x)}$ is the conditional probability that the completion of repair during the interval $(x, x+dx)$, given that the elapsed repair time is' x'.

The Markov process related to this model is $\{(N(t), S(t)): t \geq 0\}$ where $N(t)$ be the number of customer in the queue and $S(t)$ be the supplementary variable at time t. and

$S(t)=S_1(t)$, elapsed 1^{th} stage service time

$=S_2(t)$, elapsed 2^{nd} stage service time.

$=S_3(t)$, the elapsed vacation time

$=S_4(t)$, the elapsed delay time

$=S_5(t)$, the elapsed repair time

$P_n^{(i)}(t,x)$=Probability that, at time 't', there are 'n' customers in the queue, the server provides the 'i' stage of service and (excluding the customer in service) the elapsed service time is 'x'. where i=1,2.

$V_n(t,x)$=Probability that, at time 't', there are 'n' customers in the queue and the elapsed vacation time is 'x'

$U_n(t,x)$=Probability that, at time 't', there are 'n' customers in the queue and the elapsed delay time is 'x'

$R_n(t,x)$=Probability that, at time 't', there are 'n' customers in the queue and the elapsed repair time is 'x'

$Q_n(t)$=Probability that, at time 't', there are n customers in the queue and the server is idle

The differential-difference equations for this model are

$$\frac{dP_0^{(1)}(x)}{dx}=-(\lambda+\mu_1(x)+\alpha)P_0^{(1)}(x) \quad (1)$$

$$\frac{dP_n^{(1)}(x)}{dx}=-(\lambda+\mu_1(x)+\alpha)P_n^{(1)}(x)$$

$$+\lambda\sum_{j=1}^{n} C_j P_{n-j}^{(1)}(x), \quad \text{for n=0,1,....} \quad (2)$$

$$\frac{dP_0^{(2)}(x)}{dx}=-(\lambda+\mu_2(x)+\alpha)P_0^{(2)}(x) \quad (3)$$

$$\frac{dP_n^{(2)}(x)}{dx} = -(\lambda+\mu_2(x)+\alpha)P_n^{(2)}(x)$$
$$+\lambda\sum_{j=1}^{n}C_jP_{n-j}^{(2)}(x), \quad \text{for n=0,1,......} \tag{4}$$
$$\frac{dV_0(x)}{dx} = -(\lambda+\beta(x))V_0(x) \tag{5}$$
$$\frac{dV_n(x)}{dx} = -(\lambda+\beta(x))V_n(x)$$
$$+\lambda\sum_{j=1}^{n}C_jV_{n-j}(x), \quad \text{for n=0,1,......} \tag{6}$$
$$\frac{dU_0(x)}{dx} = 0 \tag{7}$$
$$\frac{dU_n(x)}{dx} = -(\lambda+\varepsilon(x))U_n(x)$$
$$+\lambda\sum_{j=1}^{n}C_jU_{n-j}(x), \quad \text{for n=0,1,......} \tag{8}$$
$$\frac{dR_0(x)}{dx} = -(\lambda+\gamma(x))R_0(x) \tag{9}$$
$$\frac{dR_n(x)}{dx} = -(\lambda+\gamma(x))R_n(x)$$
$$+\lambda\sum_{j=1}^{n}C_jR_{n-j}(x), \quad \text{for n=1,......} \tag{10}$$
$$0 = -\lambda Q_n + \lambda(1-\delta_{n,K})\sum_{j=1}^{n}c_jQ_{n-j}$$
$$+\int_0^\infty R_n(x)\gamma(x)dx + \int_0^\infty V_n(x)\beta(x)dx$$
$$+(1-p)\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx \tag{11}$$

The boundary conditions are

$$P_n^{(1)}(0) = \int_0^\infty V_{n+K}(x)\beta(x)dx$$
$$+\int_0^\infty R_{n+K}(x)\gamma(x)dx + (1-p)\int_0^\infty P_{n+K}^{(2)}(x)\mu_2(x)$$
$$+\lambda\sum_{j=0}^{K-1}C_{n+K-j}Q_j, \quad \text{for n=0,1,......} \tag{12}$$
$$P_n^{(2)}(0) = \int_0^\infty P_n^{(1)}(x)\mu_1(x)dx, \quad \text{for n=0,1,.} \tag{13}$$
$$V_n(0) = p\int_0^\infty P_n^{(2)}(x)\mu_2(x)dx, \quad \text{for n=0,1,.....} \tag{14}$$
$$U_n(0) = \alpha\int_0^\infty (P_{n-K}^{(1)}(x) + P_{n-K}^{(2)}(x))dx, \quad \text{for n=K,K+1......} \tag{15}$$
$$U_n(0) = 0, \quad \text{for n=0,1,2,.....K-1} \tag{16}$$
$$R_n(0) = \int_0^\infty U_n(x)\varepsilon(x)dx, \quad \text{for,n=0,1,..} \tag{17}$$

and the normalization condition is

$$\sum_{n=0}^{K-1}Q_n + \int_0^\infty\sum_{n=0}^{\infty}[P_n^{(1)}(x)+P_n^{(2)}(x)+V_n(x)$$
$$+U_n(x)+R_n(x)]dx = 1 \tag{18}$$

For the analysis, We define the following probability generating functions

$$P_1(x,z) = \sum_{n=0}^{\infty}P_n^{(1)}(x)z^n, \quad P_2(x,z) = \sum_{n=0}^{\infty}P_n^{(2)}(x)z^n$$
$$R(x,z) = \sum_{n=0}^{\infty}R_n(x)z^n, \quad C(z) = \sum_{j=0}^{\infty}C_jz^j,$$
$$Q(z) = \sum_{n=0}^{K-1}Q_nz^n, \quad V(x,z) = \sum_{n=0}^{\infty}V_n(x)z^n.$$
$$U(x,z) = \sum_{n=0}^{\infty}U_n(x)z^n.$$

Multiplying equation (2) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial\sum_{n=1}^{\infty}P_n^{(1)}(x)z^n}{\partial x} = -(\lambda+\mu_1(x)+\alpha)\sum_{n=1}^{\infty}P_n^{(1)}(x)z^n$$
$$+\lambda\sum_{n=1}^{\infty}\sum_{j=1}^{n}C_jP_{n-j}^{(1)}(x)z^n$$

Adding the above equation with equation (1), we have

$$\frac{\partial P_1(x,z)}{\partial x} + (\lambda-\lambda C(z)+\mu_1(x)+\alpha)P_1(x,z) = 0 \tag{19}$$

Multiplying equation (4) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial\sum_{n=1}^{\infty}P_n^{(2)}(x)z^n}{\partial x} = -(\lambda+\mu_2(x)+\alpha)\sum_{n=1}^{\infty}P_n^{(2)}(x)z^n$$
$$+\lambda\sum_{n=1}^{\infty}\sum_{j=1}^{n}C_jP_{n-j}^{(2)}(x)z^n$$

Adding the above equation with equation (3), we have

$$\frac{\partial P_2(x,z)}{\partial x}+(\lambda-\lambda C(z)+\mu_2(x)+\alpha)P_2(x,z)=0 \quad (20)$$

Multiplying equation (6) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial \sum_{n=1}^{\infty} V_n(x)z^n}{\partial x} = -(\lambda+\beta(x))\sum_{n=1}^{\infty} V_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{j=1}^{n} C_j V_{n-j}(x)z^n$$

Adding the above equation with equation (5), we have

$$\frac{\partial V(x,z)}{\partial x}+(\lambda-\lambda C(z)+\beta(x))V(x,z)=0 \quad (21)$$

Multiplying equation (8) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial \sum_{n=1}^{\infty} U_n(x)z^n}{\partial x} = -(\lambda+\varepsilon(x))\sum_{n=1}^{\infty} U_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{j=1}^{n} C_j U_{n-j}(x)z^n$$

Adding the above equation with equation(7) , we have

$$\frac{\partial U(x,z)}{\partial x}+(\lambda-\lambda C(z)+\varepsilon(x))U(x,z)=0 \quad (22)$$

Multiplying equation (10) by z^n and applying $\sum_{n=1}^{\infty}$, we have

$$\frac{\partial \sum_{n=1}^{\infty} R_n(x)z^n}{\partial x} = -(\lambda+\gamma(x))\sum_{n=1}^{\infty} R_n(x)z^n + \lambda\sum_{n=1}^{\infty}\sum_{j=1}^{n} C_j R_{n-j}(x)z^n$$

Adding the above equation with equation(7) , we have

$$\frac{\partial R(x,z)}{\partial x}+(\lambda-\lambda C(z)+\gamma(x))R(x,z)=0 \quad (23)$$

Multiplying equation (12) by z^{n+K} and applying $\sum_{n=0}^{\infty}$, we have

$$\sum_{n=0}^{\infty} P_n^{(1)}(0)z^{n+K} = \lambda\sum_{n=0}^{\infty}\sum_{j=0}^{K-1} Q_j C_{n+K-j} z^{n+K} + \int_0^{\infty}\gamma(x)\sum_{n=0}^{\infty} R_{n+K}(x)z^{n+K}dx + \int_0^{\infty}\beta(x)\sum_{n=0}^{\infty} V_{n+K}(x)z^{n+K}dx + (1-p)\int_0^{\infty}\mu_2(x)\sum_{n=0}^{\infty} P_{n+K}^{(2)}(x)z^{n+K}dx$$

$$\Rightarrow z^K P_1(0,z) = \int_0^{\infty}\gamma(x)\sum_{n=K}^{\infty} R_n(x)z^n dx + (1-p)\int_0^{\infty}\mu_2(x)\sum_{n=K}^{\infty} P_n^{(2)}(x)z^n dx + K(z) + \int_0^{\infty}\beta(x)\sum_{n=K}^{\infty} V_n(x)z^n dx \quad (24)$$

where $K(z)=\lambda\sum_{n=0}^{\infty}\sum_{j=0}^{K-1} Q_j C_{n+K-j} z^{n+K}$

We multiplying equation (11) by z^n and applying $\sum_{n=0}^{K-1}$, we get

$$0 = -\lambda\sum_{n=0}^{K-1} Q_n z^n + \int_0^{\infty}\gamma(x)\sum_{n=0}^{K-1} R_n(x)z^n dx + \int_0^{\infty}\beta(x)\sum_{n=0}^{K-1} V_n(x)z^n dx + \lambda\sum_{n=0}^{K-1}(1-\delta_{n,K})\sum_{j=1}^{n} C_j Q_{n-j} z^n + (1-p)\int_0^{\infty}\mu_2(x)\sum_{n=0}^{K-1} P_n^{(2)}(x)z^n dx$$

$$0 = -\lambda Q(z) + \int_0^{\infty}\gamma(x)\sum_{n=0}^{K-1} R_n(x)z^n dx + \int_0^{\infty}\beta(x)\sum_{n=0}^{K-1} V_n(x)z^n dx + \lambda L(z) + (1-p)\int_0^{\infty}\mu_2(x)\sum_{n=0}^{K-1} P_n^{(2)}(x)z^n dx \quad (25)$$

where $L(z)=\sum_{n=0}^{K-1}(1-\delta_{n,K})\sum_{j=1}^{n} C_j Q_{n-j} z^n$

We add the equations (24) and (25), we get

$$z^K P_1(0,z) = \int_0^{\infty}\beta(x)V(x,z)dx + \int_0^{\infty}\gamma(x)R(x,z)dx - \lambda Q(z) + K(z) + \lambda L(z) + (1-p)\int_0^{\infty}\mu_2(x)P_2(x,z)dx$$

where $K(z)=\lambda[C(z)Q(z)-L(z)]$

$$z^K P_1(0,z) = \int_0^{\infty}\beta(x)V(x,z)dx + \int_0^{\infty}\gamma(x)R(x,z)dx$$

$$-\lambda Q(z)+[\lambda C(z)Q(z)-\lambda L(z)]+\lambda L(z)$$
$$+(1-p)\int_0^\infty \mu_2(x)P_2(x,z)dx$$
$$z^K P_1(0,z)=\int_0^\infty \beta(x)V(x,z)dx+\int_0^\infty \gamma(x)R(x,z)dx$$
$$+(1-p)\int_0^\infty \mu_2(x)P_2(x,z)dx$$
$$+\lambda[C(z)-1]Q(z) \qquad (26)$$

Multiplying equation (13) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$\sum_{n=0}^{\infty} P_n^{(2)}(0)z^n=\int_0^\infty \mu_1(x)\sum_{n=0}^{\infty} P_n^{(1)}(x)z^n dx$$
$$P_2(0,z)=\int_0^\infty \mu_1(x)P_1(x,z)dx \qquad (27)$$

Multiplying equation (14) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$V(0,z)=p\int_0^\infty \mu_2(x)P_2(x,z)dx \qquad (28)$$

Multiplying equation (15) by z^n and applying $\sum_{n=K}^{\infty}$, we have

$$\sum_{n=K}^{\infty} U_n(0)z^n=\alpha\int_0^\infty (\sum_{n=K}^{\infty} P_{n-K}^{(1)}(x)z^n$$
$$+\sum_{n=K}^{\infty} P_{n-K}^{(2)}(x)z^n)dx$$

Multiply equation (16) z^n and apply $\sum_{n=0}^{K-1}$ and Adding with the above equation, we have

$$U(0,z)=\alpha z^K\int_0^\infty (P_1(x,z)+P_2(x,z))dx$$
$$=\alpha z^K(P_1(z)+P_2(z)) \qquad (29)$$

Multiplying equation (17) by z^n and applying $\sum_{n=0}^{\infty}$, we have

$$R(0,z)=\int_0^\infty \gamma(x)U(x,z)dx \qquad (30)$$

Integrating equation (19) partially with respect to 'x' with the limits from '0' to 'x', we have

$$P_1(x,z)=P_1(0,z)e^{-ax-\int_0^x \mu_1(x)dx}dx \qquad (31)$$

where $a=\lambda-\lambda C(z)+\alpha$

Integrating equation (31) partially with respect to 'x' with the limits from '0' to '∞', we have

$$P_1(z)=\frac{P_1(0,z)[1-G^*{}_1(a)]}{a}, \qquad (32)$$

Multiplying equation (31) by $\mu_1(x)$ and integrating partially with respect to 'x', with the limits from '0' to '∞', we have

$$\int_0^\infty \mu_1(x)P_1(x,z)\partial x=P_1(0,z)G^*{}_1(a), \qquad (33)$$

Sub (33) in (,27), we have

$$P_2(z,0)=P_1(0,z)G^*{}_1(a), \qquad (35)$$

Integrating equation (20) partially with respect to 'x' with the limits from '0' to 'x', we have

$$P_2(x,z)=P_2(0,z)e^{-ax-\int_0^x \mu_2(x)dx}dx \qquad (36)$$

where $a=\lambda-\lambda C(z)+\alpha$

Integrating equation (36) partially with respect to 'x' with the limits from '0' to '∞', we have

$$P_2(z)=\frac{P_2(0,z)[1-G^*{}_2(a)]}{a}, \qquad (37)$$

Multiplying equation (36) by $\mu_2(x)$ and integrating partially with respect to 'x', from '0' to '∞',

$$\int_0^\infty \mu_2(x)P_2(x,z)\partial x=P_2(0,z)G^*{}_2(a), \qquad (38)$$

Substituting equation (38) in (28), we have

$$V(0,z)=pP_2(0,z)G^*{}_2(a) \qquad (39)$$

Sub (35) in (39) , we have

$$V(0,z)=pG_1^*(a)P_1(0,z)G^*{}_2(a) \qquad (40)$$

Integrating equation (21) partially with respect to 'x', with the limits from '0' to 'x', we have

$$V(x,z)=V(0,z)e^{-mx-\int_0^x \beta(x)dx} \qquad (41)$$

where $m=\lambda-\lambda C(z)$

Substituting equation (40) in equation (41), we get

$$V(x,z)=pP_1(0,z)G_1^*(a)G^*{}_2(a)e^{-mx-\int_0^x \beta(x)dx} \qquad (42)$$

Integrating equation (42) partially with respect to 'x', with the limits form '0' to '∞', we have

$$V(z)=\frac{pP_1(0,z)G_1^*(a)G^*{}_2(a)[1-B^*(m)]}{m} \qquad (43)$$

Multiplying equation (42) by $\beta(x)$ and integrating partially with respect to x, with limits from 0 to ∞.

$$\int_0^\infty V(x,z)\beta(x)dx = pP_1(0,z)G_1^*(a)G^*{}_2(a)B^*(m) \quad (44)$$

Integrating equation (22) partially with respect to 'x', with the limits from '0' to 'x', we have

$$U(x,z) = U(0,z)e^{-mx-\int_0^x \varepsilon(x)dx} \quad (45)$$

Substituting equation (29),(32), (37)and (35) in (45), we have

$$U(x,z) = \frac{\alpha z^K e^{-mx-\int_0^x \varepsilon(x)dx}}{a} \times P_1(0,z)\left[1-G_1^*(a)G^*{}_2(a)\right] \quad (46)$$

Integrating equation (46) partially with respect to 'x', with the limits from 0 to ∞, we have

$$U(z) = \frac{\alpha z^K[1-W^*(m)]}{am} \times P_1(0,z)[1-G^*{}_1(a)G^*{}_2(a)] \quad (47)$$

Multiplying equation (46) by $\varepsilon(x)$ and integrating partially with respect to 'x', from 0 to ∞, we have

$$\int_0^\infty U(x,z)\varepsilon(x)dx = \frac{\alpha z^K W^*(m)}{a} P_1(0,z) \times[1-G^*{}_1(a)G^*{}_2(a)] \quad (48)$$

Substitute equation (48) in (30), we have

$$R(0,z) = \frac{\alpha z^K W^*(m)}{a} P_1(0,z)[1-G^*{}_1(a)G^*{}_2(a)] \quad (49)$$

Integrating equation (23) partially with respect to 'x', with the limits from '0' to 'x', we have

$$R(x,z) = R(0,z)e^{-mx-\int_0^x \gamma(x)dx}$$

Substituting equation (49) in (50), we have

$$R(x,z) = \frac{\alpha z^K P_1(0,z)\left[1-G_1^*(a)G^*{}_2(a)\right]W^*(m)}{a} \times e^{-mx-\int_0^x \gamma(x)dx} \quad (51)$$

Integrating equation (51) partially with respect to 'x', with the limits from 0 to ∞, we have

$$R(z) = \frac{\alpha z^K[1-H^*(m)]P_1(0,z)}{am} \times[1-G^*{}_1(a)G^*{}_2(a)]W^*(m) \quad (52)$$

Multiplying equation (46) by $\gamma(x)$ and integrating partially with respect to 'x', from 0 to ∞, we have

$$\int_0^\infty R(x,z)\gamma(x)dx = \frac{\alpha z^K W^*(m)H^*(m)P_1(0,z)}{a} \times[1-G^*{}_1(a)G^*{}_2(a)] \quad (53)$$

Now using equation (53), (44), (38) and (35) in equation (26), we have

$$P_1(0,z) = \frac{aQ(z)m}{D} \quad (54)$$

where $D = \alpha z^K[1-G^*{}_1(a)G^*{}_2(a)]W^*(m)H^*(m) - a[z^K-(1-p+pB^*(m))G^*{}_1(a)G^*{}_2(a)]$

Substituting $P_1(0,z)$ in the equation (32), we have

$$P_1(z) = \frac{mQ(z)[1-G^*{}_1(a)]}{D} \quad (55)$$

Substituting (35)and (54) in the equation (37), we have

$$P_2(z) = \frac{mQ(z)[1-G^*{}_2(a)]G_1^*(a)}{D} \quad (56)$$

Substituting (54) in the equation (43), we have

$$V(z) = \frac{paG^*{}_1(a)G_2^*(a)[1-B^*(m)]Q(z)}{D} \quad (57)$$

Substituting (54) in the equation (47), we have

$$U(z) = \frac{\alpha z^K Q(z)[1-G^*{}_1(a)G_2^*(a)][1-W^*(m)]}{D} \quad (58)$$

Substituting (54) in the equation (52), we have

$$R(z) = \frac{\alpha z^K Q(z)[1-G^*{}_1(a)G_2^*(a)][1-H^*(m)]W^*(m)}{D} \quad (59)$$

$$S(z) = P_1(z)+P_2(z)+V(z)+R(z)^{(50)} \quad (60)$$

Here S(z) is the probability generating function of number of customer in the queue, independent of server state

$$S(z) = \frac{Q(z)N}{D} \quad (61)$$

$$N = \{apG^*{}_1(a)G_2^*(a)[1-B^*(m)] +[m+\alpha z^K(1-W^*(m)H^*(m))][1-G_1^*(a)G_2^*(a)]\}$$

where m= λ - λ C(z), a= λ - λ C(z)+ α

We know that S(z) is probability generating function, it has the property that it must converge inside the unit circle $|z|=1$. Here it can be seen that the expression in the denominator of S(z) has 'K' zero. By Rouches theorem, we notice that K-1 zero's of this expression lies inside the unit circle $|z|=1$, and must coinside with K-1 zero's of numerator of S(z), and one zero lies out side the unit circle $|z|=1$. Let z_0 be the zero which lies outside the circle $|z|=1$. As S(z) converges, K-1 zero's of numerator and denominator of S(z) will be cancelled. Therefore we have $S(z)=\frac{A}{z-z_0}$ By substituting z=1 ,we get

$$A=(1-z_0)S(1)$$

$$S(1)=\frac{QN_1}{D_1} \quad \text{since } Q=Q(1) \qquad (62)$$

$$N_1=Q\lambda E(X)\{\alpha pG^*{}_1(\alpha)G_2^*(\alpha)E(V)$$
$$+[1-G^*{}_1(\alpha)G_2^*(\alpha)][1+\alpha(E(U)+E(R))]\}$$

$$D_1=\alpha G^*{}_1(\alpha)G_2^*(\alpha)[K-\lambda pE(X)E(V)]$$
$$-\lambda E(X)[1-G^*{}_1(\alpha)G_2^*(\alpha)][1+\alpha(E(U)+E(R))]$$

Substituting the value of S(1) in the above equation, we get $A=\frac{(1-z_0)QN_1}{D_1}$ (63)

Substituting the value of A , we get

$$S(z)=\frac{(z_0-1)QN_1}{z_0D_1}\sum_{n=0}^{\infty}(\frac{z}{z_0})^n \qquad (64)$$

Which is probability generating function of number of customer in the queue.

3. SYSTEM PERFORMANCE MEASURES

In this section, the system performance measures, the mean number of customers in the queue and idle probability have been presented.

(i) The mean number of customers in the queue

$$E(N)=S'(1)=\frac{QN_1}{(z_0-1)D_1}$$

(ii) The idle probability

Since $Q+S(1)=1$, where $Q=\sum_{n=0}^{K-1}Q_n$, which lead to

$$Q=1-\lambda E(X)\{\frac{1}{\alpha KG^*{}_1(\alpha)G_2^*(\alpha)}-\frac{1}{\alpha K}$$
$$+\frac{(E(U)+E(R))}{KG^*{}_1(\alpha)G_2^*(\alpha)}-\frac{(EU)+E(R))}{K}+\frac{pE(V)}{K}\}$$
(66)

4. SOMR PARTICULAR MODELS

In this section, four particular models have been obtained by assigning particular forms to the parameters and to the distribution function.

PARTICULAR MODEL-01

In the above model, we assume that batch arrival size random variable X follows geometric distributin with probablilty $C_n=(1-s)^{n-1}s$ for n ≥ 1 and s=1-t, then E(X)=$\frac{1}{s}$. Also we assume that the service time random variables follows exponential distribution with mean $\frac{1}{\mu_i}$ then $G^*(\alpha)=\frac{\mu_i}{\alpha+\mu_i}$, for i=1,2 and the repair time random variable R follows exponential distribution with mean $\frac{1}{\gamma}$, In addition, we assume that vacation time random variable V follows exponential distribution with mean $\frac{1}{\beta}$, also we assume that delay time random variable U follows exponential distribution with mean $\frac{1}{\varepsilon}$ .Now equations (64), (65) and (66) becomes

The probability generating function of number of customers in the queue

$$S(z)=\frac{N}{D}\sum_{n=0}^{\infty}(\frac{z}{z_0})^n$$

$$N=(z_0-1)Q\lambda[\beta(\alpha(\gamma+\varepsilon)+\gamma\varepsilon)(\alpha+\mu_1+\mu_2)+p\mu_1\mu_2\gamma\varepsilon]$$
$$D=z_0\{\mu_1\mu_2\gamma\varepsilon(Ks\beta-p\lambda)-\lambda\beta(\alpha(\gamma+\varepsilon)+\gamma\varepsilon)(\alpha+\mu_1+\mu_2)\} \qquad (65)$$

The idle probability

$$Q=\frac{\begin{Bmatrix}\beta\mu_1\mu_2\gamma Ks\varepsilon+\lambda p\mu_1\mu_2\gamma\varepsilon\\-\lambda\beta(\alpha(\gamma+\varepsilon)+\gamma\varepsilon)(\alpha+\mu_1+\mu_2)\end{Bmatrix}}{\beta\mu_1\mu_2\gamma Ks\varepsilon}$$

The mean number of customers in the queue

$$E(N) = \frac{m1}{(z_0 - 1)m2}$$

$$m1 = Q\lambda[(\alpha(\gamma+\varepsilon)+\gamma\varepsilon)\beta(\alpha+\mu_1+\mu_2) + p\mu_1\mu_2\gamma\varepsilon]$$

$$m2 = \{\mu_1\mu_2\gamma\varepsilon[Ks\beta - p\lambda] - \beta\lambda(\alpha(\gamma+\varepsilon)+\gamma\varepsilon)(\alpha+\mu_1+\mu_2)\}$$

PARTICULAR MODEL -02

If we put p=0, we get the model without vacation.

The probability generating function of number of customers in the queue

$$S(z) = \frac{N}{D}$$

$$N = Q(z)[m + \alpha z^K(1 - W^*(m)H^*(m))][1 - G_1^*(a)G_2^*(a)]$$

$$D = \alpha z^K[1 - G_1^*(a)G_2^*(a)]W^*(m)H^*(m) - a[z^K - G_1^*(a)G_2^*(a)]$$

where m= λ - λ C(z),a= λ - λ C(z)+ α

The idle probability

$$Q = 1 - \lambda E(X)\{\frac{1}{\alpha K G_1^*(\alpha)G_2^*(\alpha)} - \frac{1}{\alpha K} + \frac{(E(U)+E(R))}{K G_1^*(\alpha)G_2^*(\alpha)} - \frac{(E(U)+E(R))}{K}\}$$

The mean number of customers in the queue

$$E(N) = \frac{m1}{(z_0 - 1)m2}$$

$$m1 = Q\lambda E(X)[1 - G_1^*(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R))]$$

$$m2 = \alpha G_1^*(\alpha)G_2^*(\alpha)K - \lambda E(X)[1 - G_1^*(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R)]$$

PARTICULAR MODEL -03

If we put K=1,we get a model with batch service of size one.

The probability generating function of number of customers in the queue,

$$S(z) = \frac{Q(z)J_1}{J_2}$$

$$J_1 = \{+apG^*{}_1(a)G_2^*(a)[1 - B^*(m)] + [m + \alpha z(1 - W^*(m)H^*(m))][1 - G^*{}_1(a)G_2^*(a)]\}$$

$$J_2 = \alpha z[1 - G^*{}_1(a)G_2^*(a)]W^*(m)H^*(m) - a[z - (1 - p + pB^*(m))G^*{}_1(a)G_2(a)]$$

where m= λ - λ C(z), a= λ - λ C(z)+ α

The Idle probability

$$Q = 1 - \lambda E(X)\{\frac{1}{\alpha G^*{}_1(\alpha)G_2^*(\alpha)} - \frac{1}{\alpha} - (E(U)+E(R)) + \frac{(E(U)+E(R))}{G^*{}_1(\alpha)G_2^*(\alpha)} + pE(V)\}$$

The mean number of customers in the queue

$$E(N) = \frac{Q\lambda E(X)L_1}{(z_0 - 1)L_2}$$

$$L_1 = \{[1 - G^*{}_1(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R))] + \alpha pG^*{}_1(\alpha)G_2^*(\alpha)E(V)\}$$

$$L_2 = \alpha G^*{}_1(\alpha)G_2^*(\alpha)[1 - \lambda pE(X)E(V)] - \lambda E(X)[1 - G^*{}_1(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R))]$$

PARTICULAR MODEL -04

If we put k=1,and X=1, we get a model with single arrival and batch size of one.

The probability generating function of number of customers in the queue

$$S(z) = \frac{Q(z)J_1}{J_2}$$

$$J_1 = \{[m + \alpha z(1 - W^*(m)H^*(m))][1 - G^*{}_1(a)G_2^*(a)] + apG^*{}_1(a)G_2^*(a)[1 - B^*(m)]\}$$

$$J_2 = \alpha z[1 - G^*{}_1(a)G_2^*(a)]W^*(m)H^*(m) - a[z - (1 - p + pB^*(m))G^*{}_1(a)G_2^*(a)]$$

where m= λ - λ z, a= λ - λ z+ α

The idle probability

$$Q = 1 - \lambda\{\frac{1}{\alpha G_1^*(\alpha)G_2^*(\alpha)} - \frac{1}{\alpha} + \frac{(E(U)+E(R))}{G_1^*(\alpha)G_2^*(\alpha)} - (E(U)+E(R)) + E(V)\}$$

The mean number of customers in the queue

$$E(N) = \frac{Q\lambda L_1}{(z_0 - 1)L_2}$$

$$L_1 = \{[1 - G_1^*(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R))] + \alpha pG_1^*(\alpha)G_2^*(a)E(V)\}$$

$$L_2 = \{\alpha G_1^*(\alpha)G_2^*(\alpha)[1 - \lambda pE(V)] - \lambda[1 - G^*{}_1(\alpha)G_2^*(\alpha)][1 + \alpha(E(U)+E(R))]\}$$

5. NUMERICAL EXAMPLE

In this section, we present some numerical examples related to the models in section 4. We fix the values of β, α, γ, μ_1, μ_2, ε, K, s, p and we vary the values of the arrival rate λ. For various values of z_0, we find the values of E(N). Also we find the vlaues of Q. The results are presented in tables 1 to 4, respectively for models 01 to 04. From the values, it is clear that, as the arrival rate increases, the idle probability decreases. Which is very much coincide with our expectations. Aslo the mean number of customers in the queue increases, for increasing values of arrival rate. Again, which is very much coincide with our expectation. Surprisingly in all the models, If the zero z_0 increases from 1.00001 to 15, the mean number of customers in the queue considerably decreases.

Table 1: Q and E(N) ($\beta = 2, \alpha = 1, \gamma = 1 \mu_1 = 8$
$\mu_2 = 9, K = 15, s = 0.7, p = 0.5, \varepsilon = 1$)

λ	Q	E(N)					
		Z_0 values					
		1.0000	1.0001	1.5	5	10	15
1	0.9639	3611	361	0.0722	0.0090	0.0040	0.0026
2	0.9278	7222	722	0.1444	0.0181	0.0080	0.0052
3	0.8917	10833	1083	0.2167	0.0271	0.0120	0.0077
4	0.8556	14444	1444	0.2889	0.0361	0.0160	0.0103
5	0.8194	18056	1805	0.3611	0.0451	0.0201	0.0129
6	0.7833	21667	2166	0.4333	0.0542	0.0241	0.0155
7	0.7472	25278	2527	0.5056	0.0632	0.0281	0.0181
8	0.7111	28889	2888	0.5778	0.0722	0.0321	0.0206
9	0.6750	32500	3250	0.6500	0.0813	0.0361	0.0232
10	0.6389	36111	3611	0.7222	0.0903	0.0401	0.0258

Table 02:Q andE(N) ($\beta = 20, \alpha = 10, \gamma = 10,$
$\mu_1 = 80, \mu_2 = 90, K = 10, s = 0.9, p = 0.5, \varepsilon = 1$)

Table 03:Q and E(N) ($\beta = 20, p = 0.5$
$\alpha = 10, \gamma = 10, \mu_1 = 80, \mu_2 = 90, K = 10, s = 0.7, \varepsilon = 1$)

λ	Q	E(N)					
		Z_0 values					
		1.0000	1.0001	1.5	5	10	15
1	0.9536	4642	464	0.929	0.0116	0.0052	0.0033
2	0.9071	9285	928	0.1857	0.0232	0.0103	0.0066
3	0.8607	13929	1392	0.2786	0.0348	0.0155	0.0099
4	0.8143	18571	1857	0.3714	0.0464	0.0206	0.0133
5	0.7679	23214	2321	0.4643	0.0580	0.0258	0.0166
6	0.7214	27857	2785	0.5571	0.0696	0.0310	0.0199
7	0.6750	32500	3250	0.6500	0.0813	0.0361	0.0232
8	0.6286	37143	3714	0.7429	0.0929	0.0413	0.0265
9	0.5821	41786	4178	0.8357	0.1045	0.0464	0.0298
10	0.5357	46429	4642	0.9286	0.1161	0.0516	0.0332

λ	Q	E(N)					
		Z_0 values					
		1.00001	1.0001	1.5	5	10	15
1	0.9048	9523	952	0.1905	0.0238	0.0106	0.0068
2	0.8095	19048	1904	0.3810	0.0476	0.0212	0.0136
3	0.7143	28571	2857	0.5714	0.0714	0.0317	0.0204
4	0.6190	38095	3809	0.7619	0.0952	0.0423	0.0272
5	0.5238	47619	4761	0.9524	0.1190	0.0529	0.0340
6	0.4286	57143	5714	1.1429	0.1429	0.0635	0.0408
7	0.3333	66667	6666	1.3333	0.1667	0.0741	0.0476
8	0.2381	76190	7619	1.5238	0.1905	0.0847	0.0544
9	0.1429	85714	8571	1.7143	0.2143	0.0952	0.0612
10	0.0476	95238	9523	1.9048	0.2381	0.1058	0.0680

Table 04:Q and E(N) $(\beta = 20, p = 0.5, \alpha = 10, \gamma = 10, \mu_1 = 80, \mu_2 = 90, K = 15, s = 0.9, \varepsilon = 1)$

λ	Q	E(N)					
		Z_0 values					
		1.00001	1.0001	1.5	5	10	15
1	0.9759	2407	240	0.0481	0.0060	0.0027	0.0017
2	0.9519	4814	481	0.0963	0.0120	0.0053	0.0034
3	0.9278	7222	722	0.1444	0.0181	0.0080	0.0052
4	0.9037	9629	962	0.1926	0.0241	0.0107	0.0069
5	0.8796	12037	1203	0.2407	0.0301	0.0134	0.0086
6	0.8556	14444	1444	0.2889	0.0361	0.0160	0.0103
7	0.8315	16852	1685	0.3370	0.0421	0.0187	0.0120
8	0.8074	19259	1925	0.3852	0.0481	0.0214	0.0138
9	0.7833	21667	2166	0.4333	0.0542	0.0241	0.0153
10	0.7593	24074	2407	0.4815	0.0602	0.0267	0.0172

6. CONCLUTION

In this article, a single server batch arrival, heterogeneous two stage batch service(fixed) queue with Bernoulli vacation and with unreliable server has been completely analysed. To illustrate the analytical compatability of the model we present some numerical examples by taking particular values to the parameters and particular form to the probability distribution. The model can be extended by taking the break down period as generally distributed.

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