Fixed Point Theorem in Fuzzy Metric Space

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Abstract *- The intent of this paper is to initiate the concept of weak-compatibility and semi-compatibility in the context of fuzzy metric spaces. The follow-up investigations by many other mathematicians in due course established a lot of interesting results. Picked up some ideas from these results we established some common fixed point theorem on fuzzy metric space for eight mappings which is the generalization of results of Som [2] and Mukherjee [1].*

Key Words *- complete fuzzy Metric Space, semi-compatible Mapping*

Main result

Theorem 1. Let A be a self mapping of a complete metric space (X, d) and S, T to be two continuous self mappings of X satisfying:

(a) The pair{A, S} and {A, T}are compatible with $A(X) \subset S(X) \cap T(X)$ and there exist an upper semi continuous function $\varphi:(R^+)^5 \rightarrow R^+$ nondecreasing in each coordinate variable such that for all $x, y \in X$.

(b) $$a\,d(Ax, Ay) - b\,d(Sx, Ty) \leq \phi\{d(Sx, Ty), d(Sx, Ax), d(Sx, Ay), d(Ty, Ax), d(Ty, Ay)\},$$

where $a > b+1$ and for any $t > 0$, $\phi(t, t, a_1 t, a_2 t) < t$, $a_1 + a_2 = 3$.

Then A, S and T have a unique common fixed point in X.

Now, we are proving the following theorems in fuzzy metric space for eight mappings which are semi-compatible and weak-compatible.

Theorem 2. Let A, B, C, D, R, S, T and Y be mappings from a complete fuzzy metric space (X, M, *) into itself satisfying following conditions:

(a) $A(X) \subset U(X)$, $B(X) \subset T(X)$, $C(X) \subset S(X)$ and $D(X) \subset R(X)$

(b) (A, U), (B, T), (C, S) and (D, R) are semi-compatible.

(c) A, B, C, D are continuous.

(d) For all $x, y \in X$ and $t > o$

$$M(Ax, By, t) \geq r[M(Ux, Ty, t)] \quad \text{(i)}$$

$$M(Bx, Cy, t) \geq r[M(Tx, Sy, t)] \quad \text{(ii)}$$

$$M(Cx, Dy, t) \geq r[M(Sx, Ry, t)] \quad \text{(iii)}$$

$$M(Dx, Ay, t) \geq r[M(Rx, Uy, t)] \quad \text{(iv)}$$

where $r : [0, 1] \rightarrow [0, 1]$ is some continuous function such that $r(t) > t$, for each

$0 < t < 1$. Then A, B, C, D, R, S, T and U have a unique common fixed point.

Proof. Let $x_o \in X$ be an arbitrary point. Since $A(X) \subset U(X)$, there exist a $x_1 \in X$ such that $Ax_o = Ux_1$. Since $B(X) \subset T(X)$, therefore for this point $x \in X$, we can choose a point $x_2 \in X$ such that $Bx_1 = Tx_2$. Also since $\subset (X) \subset S(X)$, so for this point x_2, we can choose a point $x_3 \in X$ such that $Cx_2 = Sx_3$ and since $D(X) \subset R(X)$, therefore, for this point x_3. We can choose a point $x_4 \in X$ such that $Dx_3 = Rx_4$ and so on. Inductively, construct two sequences $\{y_n\}$ and $\{x_n\}$ in X such that –

$$y_{4n} = Ax_{4n} = Ux_{4n+1},$$

$$y_{4n+1} = Bx_{4n+1} = Tx_{4n+2},$$

$$y_{4n+2} = Cx_{2n+2} = Sx_{4n+3},$$

$$y_{4n+3} = Dx_{4n+3} = Rx_{4n+4}, \qquad \text{for } n = 0, 1, 3, \ldots$$

In order to show that $\{y_n\}$ is a canchy sequence, it is sufficient to prove that $\{y_{4n}\}$, $\{y_{4n+1}\}$, $\{y_{4n+2}\}$ and $\{y_{4n+3}\}$ are cauchy sequence in X

Now, $M(y_{4n+1}, y_{4n+2}, t) = M(Bx_{4n+1}, Cx_{2n+2}, t)$

$\geq r\,[M(Tx_{4n+1}, Sy_{2n+2}, t)]$ [form (ii)]

$M(y_{4n+1}, y_{4n+2}, t) \geq r\,[M(y_{4n}, y_{4n+1}, t)]$

$M(y_{4n+1}, y_{4n+2}, t) > M(y_{4n}, y_{4n+1+1}, t).$

Now, $M(y_{4n+2}, y_{4n+3}, t) = M(Cx_{4n+2}, Dx_{2n+3}, t)$

$\geq r\,[M(Sx_{4n+2}, Rx_{2n+3}, t)]$ [form (iii)]

$= r\,[M(y_{4n+1}, y_{4n+2}, t)]$

$> M(y_{4n+1}, y_{4n+2}, t).$

Now, $M(y_{4n+3}, y_{4n+4}, t) = M(Dx_{4n+3}, Ax_{4n+4}, t)$

$\geq r\,[M(Rx_{4n+3}, Ux_{4n+4}, t)]$ [form (iv)]

$= r\,[M(y_{4n+2}, y_{4n+3}, t)]$

$> M(y_{4n+2}, y_{4n+3}, t).$

Now, $M(y_{4n+4}, y_{4n+5}, t) = M(Ax_{4n+4}, Bx_{4n+5}, t)$

$\geq r\,[M(Ux_{4n+4}, Tx_{4n+5}, t)$ [form (i)]

$= r\,[M(y_{4n+3}, y_{4n+4}, t)]$

$> M(y_{4n+3}, y_{4n+4}, t)$

Thus in general, we obtain

$$M(y_{n+1}, y_{n+2}, t) > M(y_n, y_{n+1}, t), \text{ for all } n = 0, 1, 2, 3, \ldots$$

Thus $\{M(y_{n+1}, y_n, t), n > 0\}$ is an increasing sequence of positive real numbers in $[0, 1]$ and therefore tends to a limit $L \leq 1$. If $L < 1$ then

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) = L > r(L) > L,$$

which is a contradiction.

Hence $L = 1$. Thus $\lim_{n\to\infty} M(y_{n+1}, y_n, t) = 1, \; \forall\; t \geq 0.$

Now, for any positive integer, P, we have

$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p).$

By taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \ldots\ldots * 1 = 1,$$

$$\therefore \lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1; \quad \forall\; t > 0.$$

Hence $\{y_n\}$ is a cauchy sequence in X and by completeness of X,

$$\lim_{n\to\infty} \{y_n\} = z \text{ in } X.$$

also the subsequences

$$\left.\begin{array}{ll} \{Ax_{4n}\} \to z, & \{Bx_{4n+1}\} \to z, \\ \{Cx_{4n+2}\} \to z, & \{Dx_{4n+3}\} \to z, \\ \{Rx_{4n}\} \to z, & \{Tx_{4n+2}\} \to z, \\ \{Sx_{4n+3}\} \to z, & \{Ux_{4n+1}\} \to z. \end{array}\right\} \quad (1)$$

Case 1.

Since A is continuous and (A, U) is semi-compatible, we have

$$AUx_{4n+1} \to Az, \quad (2)$$

$$AUx_{4n+1} \to Uz. \quad (3)$$

Since the limit of a sequence in fuzzy metric space is unique,

we obtain that

$$Az = Uz \quad (4)$$

Now, we prove Az = z, Suppose on the contradiction Az $\neq$ z, we get

$M(Az, Bx_{4n+1}, t) \geq r\,[M(Uz, Tx_{4n+1}, t)] > M(Uz, Tx_{4n+1}, t)$,

taking limit as $n \to \infty$, using (1) and (2)

$$M(Az, z, t) > M\ (Az, z, t)$$

which is a contradiction.

Hence, $z = Az = Uz$. (5)

Case 2. Since B is continuous and (B, T) is semi-compatible, we obtain

$$B\,Tx_{4n+1} \to Bz,$$

$$B\,Tx_{4n+1} \to Tz.$$

Since limit is unique in fuzzy metric space, then $Bz = Tz$. (6)

Now, we prove Bz = z.

For this suppose on the contrary that Bz $\neq$ z and putting x = z and

$y = x_{4n+1}$ in (ii) we get.

$$M(BZ, Cx_{4n+1}, t) \geq r\ [M(Tz, Sx_{4n+1}, t)]$$

$$> M(Tz, Sx_{4n+1}, t)$$

taking $n \to \infty$ and from (6) and (3)

$$M(Bz, z, t) > M(Bz, z, t)$$

which is a contradiction.

Hence $z = Bz = Tz$. (7)

Case 3. Since C is continuous and (C, S) is semi–compatible, we get

$$CSx_{4x+3} \to Cz,$$

$$CSx_{4x+3} \to Sz.$$

Since limit is unique in fuzzy metric space, then

$$Cz = Sz \quad (8)$$

Now, we prove $Cz = z$.

For this, suppose on the contrary that Bz $\neq$ z.

Putting $x = z$, $y = x_{4x+3}$ (ii), we get

$$M(Cz, Dx_{4x+3}, t) \geq r\,[M(Sz, Dx_{4x+3}, t)] > M(Sz, Dx_{4x+3}, t).$$

On taking $n \to \infty$ and using (8), we obtain

$$M(Cz, z, t) > M(Cz, z, t)$$

Which is a contradiction.

Hence $z = Cz = Sz$. (9)

Case 4. Since D is continuous and (D, R) is semi-compatible, then $DRx_{2n} \to Dz$, $DRx_{2n} \to Rz$.

Since limit is unique in fuzzy metric space . Hence $Dz = Rz$ (10)

Now, we have to prove that $Dz = z$, for this suppose on the contrary $Dz \neq z$.

And putting $x = z$, $y = x_{2n}$ in (iv), we get

$$M(Dz, Ax_{2n}, t) \geq r\,[M(Rz, Ux_{2n+1}, t)] > M(Rz, Ux_{2n+}, t).$$

Taking $n \to \infty$ and using (10), we get

$$M(Dz, z, t) > M(Dz, z, t)$$

Which is a contradiction.

Hence $z = Dz = Rz$ (11)

form (5), (7), (9) and (11)

$$z = Az = Uz = Bz = Tz = Cz = Sz = Dz = Rz.$$

Hence z is a fixed point of A, B, C, D, S, T, U and R.

Uniqueness

Let w be the another fixed point of A, B, C, D, R, S, T and U

i.e. $w \neq z$ and $w = Aw = Bw = Cw = Dw = Rw = Sw = Tw = Uw$.

Putting $x = z$ and $y = w$ in (i)

$$M(Az, Bw, t) \geq r\,[M(Uz, Tw, t)]$$

$$M(z, w, t) > M(z, w, t)$$

which is a contradiction. Hence $z = w$.

Thus z is the unique common fixed point of A, B, C, D, R, S, T and U.

References

[1] Mukherjee, R.N. : *Indian J. pure Appl. Math.* 12(8) (1981), 930.
[2] Som, T. : Few common fixed points for comparative mappings, *Bull. Cal. Math. Soc*., 95(4) (2003), 307-312.
[3] Som, T. and Mukherjee, R.N.: A fixed point theorems for two non-self mapping, *Proc. Acad. Sci. India,* 56(111), (1986).