

Lattice Points of an Infinite Cone $x^2 + y^2 = (\alpha^{2n} + \beta^{2n})z^2$

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Abstract - The ternary homogeneous equation representing an infinite cone given by $x^2 + y^2 = (\alpha^{2n} + \beta^{2n})z^2, n \in \mathbf{N}, \alpha, \beta \in \mathbf{Z} - \{0\}$ is analyzed for its non-zero distinct integer points. Few different patterns of integer points satisfying the infinite cone under consideration are obtained.

Keyword- Diophantine equations, integer solutions, infinite cone, homogeneous cone, lattice points.

I. INTRODUCTION

Number theory is, as the name suggests, is devoted to the study of numbers, first and foremost integers. The classical problems in Number Theory are often easy to formulate knowing some basic mathematics and this makes the area attractive. It is often only the formulation are simple, proofs are very long, complicated and requires excessive numerical work.

A great deal of number theory also arises from the study of the solutions in integers of a polynomial equation $f(x_1, x_2, \dots, x_n) = 0$, called the Diophantine equation. They have fewer equations than unknown variables and involve integers that work correctly for all equations. Quadratic Diophantine equations are extremely important in Number theory. There are several Diophantine equations that have no solutions, trivial solutions, finitely or infinitely many solutions. This communication concerns with interesting ternary quadratic equation $x^2 + y^2 = (\alpha^{2n} + \beta^{2n})z^2, n \in \mathbf{N}, \alpha, \beta \in \mathbf{Z} - \{0\}$ representing an infinite cone for determining its infinitely many non-zero lattice points.

II. METHOD OF ANALYSIS

The ternary quadratic equation studied for its non-zero distinct integer solutions is given by

$$x^2 + y^2 = (\alpha^{2n} + \beta^{2n})z^2 \quad (1)$$

To start with, note that (1) is satisfied by $(\pm\alpha^n, \pm\beta^n, \pm 1)$.

We will discuss certain cases which can be related to some classic theorems in Number theory.

When $\alpha = \beta$, (1) reduces to a Congruum Problem [refer to IJPAMS, Volume 9, Number 2 (2016), Pp. 123-131]. If α, β are the generators of a Pythagorean triangle, then (1) takes the form of a Pythagorean equation. For the various different choices of α and β , (1) can be written as a Pellian equation. Also, the equation can be used to generate second order Ramanujan numbers.

However, we have other patterns of solutions which are illustrated as follows:

A. Pattern 1:

$$\text{Let } x = \alpha^n u + \beta^n v \text{ and } y = \beta^n u - \alpha^n v.$$

$$\text{Then } x^2 + y^2 = (\alpha^{2n} + \beta^{2n})(u^2 + v^2)$$

$$\text{and the equation (1) becomes } u^2 + v^2 = z^2.$$

We obtain

$$u = k(m^2 - n^2), v = 2kmn, z = k(m^2 + n^2),$$

for some integers k, m and n , hence the solution

$$x = k(\alpha^n m^2 + 2\beta^n mn - \alpha^n n^2)$$

$$y = k(\beta^n m^2 - 2\alpha^n mn - \beta^n n^2)$$

$$z = k(m^2 + n^2)$$

B. Pattern 2:

$$\text{Let } x = \alpha^n u - \beta^n v \text{ and } y = \beta^n u + \alpha^n v.$$

$$\text{Then } x^2 + y^2 = (\alpha^{2n} + \beta^{2n})(u^2 + v^2)$$

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$$x = k(\alpha^n m^2 - 2\beta^n mn - \alpha^n n^2)$$

$$y = k(\beta^n m^2 + 2\alpha^n mn - \beta^n n^2)$$

$$z = k(m^2 + n^2)$$

C. Pattern 3:

Assume $z = z(a, b) = a^{2n} + b^{2n}$, where $a, b > 0$ (2)

and write $\alpha^{2n} + \beta^{2n}$ as

$$\alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n i)(\alpha^n - \beta^n i) \quad (3)$$

Substituting (2) & (3) in (1) and employing the method of factorization,

Write $x + iy = (\alpha^n + \beta^n i)(a^n + ib^n)^2$

Equating the real and imaginary parts in the above equation, we get

$$x = x(a, b, \alpha, \beta, n) = \alpha^n(a^{2n} - b^{2n}) - 2\beta^n \alpha^n b^n$$

$$y = y(a, b, \alpha, \beta, n) = \beta^n(a^{2n} - b^{2n}) + 2\alpha^n \alpha^n b^n$$

$$z = z(a, b, \alpha, \beta, n) = a^{2n} + b^{2n}$$

which represents the distinct integer points on the cone (1).

D. Pattern 4:

We can also write $\alpha^{2n} + \beta^{2n}$ as

$$\alpha^{2n} + \beta^{2n} = (\beta^n + \alpha^n i)(\beta^n - \alpha^n i) \quad (4)$$

Substituting (2) & (4) in (1) and employing the method of factorization,

write $x + iy = (\beta^n + \alpha^n i)(a^n + ib^n)^2$

Equating the real and imaginary parts in the above equation, we get

$$x = x(a, b, \alpha, \beta, n) = \beta^n(a^{2n} - b^{2n}) - 2\alpha^n \alpha^n b^n$$

$$y = y(a, b, \alpha, \beta, n) = \alpha^n(a^{2n} - b^{2n}) + 2\beta^n \alpha^n b^n$$

$$z = z(a, b, \alpha, \beta, n) = a^{2n} + b^{2n}$$

which represents the distinct integer points on the cone (1).

E. Pattern 5:

Equation (3) can also be written in the following way:

$$\alpha^{2n} + \beta^{2n} = (-\alpha^n + \beta^n i)(-\alpha^n - \beta^n i)$$

Proceeding as above, we obtain

$$x = x(a, b, \alpha, \beta, n) = -[\alpha^n(a^{2n} - b^{2n}) + 2\beta^n \alpha^n b^n]$$

$$y = y(a, b, \alpha, \beta, n) = \beta^n(a^{2n} - b^{2n}) - 2\alpha^n \alpha^n b^n$$

$$z = z(a, b, \alpha, \beta, n) = a^{2n} + b^{2n}$$

F. Pattern 6:

Equation (3) can also be written in the following way:

$$\alpha^{2n} + \beta^{2n} = (-\beta^n + \alpha^n i)(-\beta^n - \alpha^n i)$$

Proceeding as above, we obtain

$$x = x(a, b, \alpha, \beta, n) = -[\beta^n(a^{2n} - b^{2n}) + 2\alpha^n \alpha^n b^n]$$

$$y = y(a, b, \alpha, \beta, n) = \alpha^n(a^{2n} - b^{2n}) - 2\beta^n \alpha^n b^n$$

$$z = z(a, b, \alpha, \beta, n) = a^{2n} + b^{2n}$$

G. Pattern 7:

Equation (1) can be written as

$$\frac{x + \alpha^n z}{\beta^n z + y} = \frac{\beta^n z - y}{x - \alpha^n z} = \frac{p}{q}, \text{ (say)}, q \neq 0 \quad (5)$$

This equation is equivalent to the following two equations:

$$qx - py + (\alpha^n q - \beta^n p)z = 0,$$

$$px + qy - (\alpha^n p + \beta^n q)z = 0$$

By the method of cross multiplication, we get the integral solutions of (1) to be

$$x = x(p, q, \alpha, \beta, n) = p(\alpha^n p + \beta^n q) - q(\alpha^n q - \beta^n p)$$

$$y = y(p, q, \alpha, \beta, n) = p(\alpha^n q - \beta^n p) + q(\alpha^n p + \beta^n q)$$

$$z = z(p, q, \alpha, \beta, n) = p^2 + q^2$$

H. Pattern 8:

Equation (1) can be written as

$$\frac{\beta^n z + y}{x + \alpha^n z} = \frac{x - \alpha^n z}{\beta^n z - y} = \frac{p}{q}, \text{ (say)}, q \neq 0$$

This equation is equivalent to the following two equations:

$$px - qy + (\alpha^n p - \beta^n q)z = 0,$$

$$qx + py - (\alpha^n q + \beta^n p)z = 0,$$

By the method of cross multiplication, we get the integral solutions of (1) to be

$$x = x(p, q, \alpha, \beta, n) = q(\alpha^n q + \beta^n p) - p(\alpha^n p - \beta^n q)$$

$$y = y(p, q, \alpha, \beta, n) = p(\alpha^n q + \beta^n p) + q(\alpha^n p - \beta^n q)$$

$$z = z(p, q, \alpha, \beta, n) = p^2 + q^2$$

Note 1:

Equation (1) can be written as

$$\frac{x - \alpha^n z}{\beta^n z + y} = \frac{\beta^n z - y}{x + \alpha^n z} = \frac{p}{q}, \text{ (say), } q \neq 0$$

Proceeding as above, we obtain

$$\begin{aligned} x &= x(p, q, \alpha, \beta, n) = -p(\alpha^n p - \beta^n q) + q(\alpha^n q + \beta^n p) \\ y &= y(p, q, \alpha, \beta, n) = -p(\alpha^n q + \beta^n p) - q(\alpha^n p - \beta^n q) \\ z &= z(p, q, \alpha, \beta, n) = q^2 + p^2 \end{aligned}$$

Note 2:

Applying the above method to the case,

$$\frac{x - \alpha^n z}{\beta^n z - y} = \frac{\beta^n z + y}{x + \alpha^n z} = \frac{p}{q}, \text{ (say), } q \neq 0$$

we obtain the solution of (1) as

$$\begin{aligned} x &= x(p, q, \alpha, \beta, n) = p(\alpha^n p - \beta^n q) - q(\alpha^n q + \beta^n p) \\ y &= y(p, q, \alpha, \beta, n) = -p(\alpha^n q + \beta^n p) - q(\alpha^n p - \beta^n q) \\ z &= z(p, q, \alpha, \beta, n) = -q^2 - p^2 \end{aligned}$$

I. Pattern 9:

Equation (1) can be written as

$$(\alpha^{2n} + \beta^{2n})z^2 - y^2 = x^2 * 1 \tag{6}$$

Assume $x = x(a, b, \alpha, \beta, n) = (\alpha^{2n} + \beta^{2n})a^{2n} - b^{2n}$,
where $a, b > 0$ (7)

Write 1 as

$$1 = \frac{(\sqrt{(\alpha^{2n} + \beta^{2n}) + \alpha^n})(\sqrt{(\alpha^{2n} + \beta^{2n}) - \alpha^n})}{\beta^{2n}} \tag{8}$$

Using (7) & (8) in (6) and applying the method of factorization,

$$\sqrt{(\alpha^{2n} + \beta^{2n})} z + y = \frac{(\sqrt{(\alpha^{2n} + \beta^{2n})} a^{n+b^n})^2 (\sqrt{(\alpha^{2n} + \beta^{2n}) + \alpha^n})}{\beta^n}$$

Equating the rational and irrational factors, we get

$$\begin{aligned} x &= (\alpha^{2n} + \beta^{2n})a^{2n} - b^{2n} \\ y &= \frac{1}{\beta^n} [\alpha^n ((\alpha^{2n} + \beta^{2n})a^{2n} + b^{2n}) + 2(\alpha^{2n} + \beta^{2n})a^n b^n] \end{aligned}$$

$$z = \frac{1}{\beta^n} [(\alpha^{2n} + \beta^{2n})a^{2n} + b^{2n} + 2\alpha^n b^n \alpha^n]$$

Since our interest centers on finding integral solutions, replace a by βA and b by βB in the above equations. Thus the corresponding solutions to (1) are given by

$$\begin{aligned} x &= \beta^{2n} [(\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}] \\ y &= \beta^n [\alpha^n ((\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n}) + 2(\alpha^{2n} + \beta^{2n})A^n B^n] \\ z &= \beta^n [(\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n} + 2\alpha^n A^n B^n] \end{aligned}$$

Note 3:

Equation (8) can be written as

$$1 = \frac{(-\sqrt{(\alpha^{2n} + \beta^{2n}) + \alpha^n})(-\sqrt{(\alpha^{2n} + \beta^{2n}) - \alpha^n})}{\beta^{2n}}$$

Proceeding as above, we obtain

$$\begin{aligned} x &= \beta^{2n} [(\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}] \\ y &= \beta^n [\alpha^n ((\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n}) - 2(\alpha^{2n} + \beta^{2n})A^n B^n] \\ z &= \beta^n [2\alpha^n A^n B^n - (\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}] \end{aligned}$$

J. Pattern 10:

Instead of (7) we can also write 1 as

$$1 = \frac{(\sqrt{(\alpha^{2n} + \beta^{2n}) + \beta^n})(\sqrt{(\alpha^{2n} + \beta^{2n}) - \beta^n})}{\alpha^{2n}}$$

Thus the corresponding solutions to (1) are given for the choice $a = \beta A$, $b = \beta B$, by

$$\begin{aligned} x &= \alpha^{2n} [(\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}] \\ y &= \alpha^n [\beta^n ((\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n}) + 2(\alpha^{2n} + \beta^{2n})A^n B^n] \\ z &= \alpha^n [(\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n} + 2A^n B^n \beta^n] \end{aligned}$$

Note 4:

Equation (8) can be written as

$$1 = \frac{(-\sqrt{(\alpha^{2n} + \beta^{2n}) + \beta^n})(-\sqrt{(\alpha^{2n} + \beta^{2n}) - \beta^n})}{\alpha^{2n}}$$

Proceeding as above, we obtain

$$x = \alpha^{2n} [(\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}]$$

$$y = \alpha^n [\beta^n ((\alpha^{2n} + \beta^{2n})A^{2n} + B^{2n}) - 2(\alpha^{2n} + \beta^{2n})A^n B^n]$$

$$z = \alpha^n [2A^n B^n \beta^n - (\alpha^{2n} + \beta^{2n})A^{2n} - B^{2n}]$$

III. CONCLUSION

The ternary quadratic Diophantine equations are rich in variety. One may search for other choices of Diophantine equations to find their corresponding integer solutions.

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