Normal-Uniform Distributed Stochastic Production Frontier Model

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Abstract— *Stochastic Production Frontier Model (SFM) plays a vital role in measuring technical efficiency in the field of production. In this paper, the technical efficiency of normal uniform stochastic frontier model was derived. The parameters were evaluated using Maximum likelihood Estimates. In the model* $v_i \sim N(0,\sigma_v^2)$, *is a two sided error term representing the statistical noise and* $u_i \geq 0$ *is a one sided error term, representing inefficiency.*

Keywords- *Normal-Uniform distribution, stochastic normal-uniform production frontier model, Technical Efficiency, Maximum likelihood estimation.*

I. INTRODUCTION

Stochastic Production Frontier analysis (SPFA) is a method of economic modelling, and it is widely used to estimate individual efficiency scores. The model can be expressed as $y = f(x,\beta)\, e^{v-u}$,where y is scalar output,x is a vector of inputs, β is vector of technology parameter .This model contains two error components. The first error component u represents the unmeasured variables such as weather, walkout, epidemic, and other variables which undefined in the production function and also u is intended to capture the technical inefficiency. The second component v is the random shock variable which is identically and independently normal distributed with mean 0 and variance ${\sigma_v}^2$ (ie)v always follows normal distribution $N(0,\sigma^2)$.

The technical inefficiency term u follows any one of the continuous distribution. Distributional assumption plays a major role to estimate the technical efficiency of each producer. Meeusen and van den Broeck(1977) assigned a exponential distribution to u. Battese and Corra(1977) assigned a half normal distribution to u. Aigner, Lovell and Schmidt (1977) assigned both exponential and half normal to u. Greene (1990) proposed a Gamma distribution and Stevenson (1980) proposed Gamma and truncated normal distributions. In this paper, uniform distribution is assigned for u.

A formal definition of technical efficiency is achieving maximum output from a given input vector (Koopmans, 1951). The output-oriented technical efficiency of producer is $TE_i = \dfrac{y_i}{f(x_i,\beta)\exp\{v_i\}}$ which defines technical efficiency as the ratio of observed output to the maximum feasible output, conditional on $\exp\{v_i\}$}. Producer i attain its maximum feasible output of $f(x_i,\beta)\exp\{v_i\}$ } if and only if TEi = 1. Otherwise, 0 < TEi < 1 provides a measure of the shortfall of observed output from the maximum feasible output in an environment characterized by $\exp\{v_i\}$ (Kumbhakar and Lovell (2003)

In general, Technical efficiency, TEi, can be obtained as the exponential conditional expectation of u given the composed error term ε , which is given by $TE_i = \exp\left(-E\left(u_i / \varepsilon_i\right)\right)$, This estimation was proposed by Jondrow et al,(1982)..

Aigner and Chu (1968) were the first researchers to use the maximum likelihood estimate (MLE) method to estimate point estimators. The MLE method can be used to maximize the log likelihood function corresponding to the marginal density function f(ε). Once the maximum point estimates of parameters are obtained, we substitute these values into the estimate of technical efficiency of each producer.

II.NORMAL-UNIFORM STOCHASTIC PRODUCTION FRONTIER MODEL (NUSPFM)

The Stochastic Production Frontier Model can be expressed as

$$y = f(x,\beta)\, e^{v-u}$$

Considering the stochastic production frontier model, the following assumptions in the distribution were made.

1) $v \sim iid\ N(0, \sigma_v^2)$

2) $u \sim iid$ Uniform distribution in the interval $(0, \theta)$

3) u and v are distributed independently of each other and of the regressions.

The probability density function of $u \geq 0$ is given by

$$f(u) = \frac{1}{\theta - 0}, \quad 0 < u < \theta$$

$$= \frac{1}{\theta}, \quad 0 < u < \theta \qquad \text{\_\_\_\_\_\_\_\_(1)}$$

The probability density function of v is given by

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}}, -\infty < v < \infty \qquad \text{\_\_\_\_(2)}$$

Since u and v are independent

$$f(u,v) = f(u).f(v)$$

$$= \frac{1}{\sigma\theta\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} \qquad \text{\_\_\_\_\_(3)}$$

Making the transformation , $\varepsilon = v - u \Rightarrow v = u + \varepsilon$

$$f(u,\varepsilon) = \frac{1}{\sigma\theta\sqrt{2\pi}} e^{-\frac{(u+\varepsilon)^2}{2\sigma^2}} \qquad \text{\_\_(4)}$$

The marginal density function of ε is obtained by integrating $f(u, \varepsilon)$ with respect to u.

$$f(\varepsilon) = \int_0^\theta f(u,\varepsilon)\, du$$

$$= \int_0^\theta \frac{1}{\sigma\theta\sqrt{2\pi}} e^{-\frac{(u+\varepsilon)^2}{2\sigma^2}} du$$

$$put\ t = \frac{u+\varepsilon}{\sigma},$$

$$dt = \frac{du}{\sigma} \Rightarrow \sigma\, dt = du$$

$$when u = 0, t = \frac{\varepsilon}{\sigma,}$$

$$when u = \theta, t = \frac{\theta+\varepsilon}{\sigma}$$

$$f(\varepsilon) = \frac{1}{\theta} \int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\theta}\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right) - \Phi\left(\frac{\varepsilon}{\sigma}\right)\right] \qquad \text{\_\_(5)}$$

where $\phi(.), \Phi(.)$ are and density function and the standard normal cumulative distribution respectively. The marginal density function $f(\varepsilon)$ is asymmetrically distributed, with mean and variance as below

$$E(\varepsilon) = E(v-u) \quad = E(v) - E(u)$$

$$= -E(u) \qquad \text{\_\_\_\_\_\_\_\_\_\_(6)}$$

$$E(\varepsilon) = -\int_0^\theta u\ f(u) du$$

$$= -\int_0^\theta u \frac{1}{\theta} du = \frac{1}{\theta}\left[\frac{u^2}{2}\right]_0^\theta = -\frac{\theta}{2} \qquad \text{\_\_\_(7)}$$

Variance of ε

Var(ε) =Var(v-u)

=Var (v)+var (u)

$$= \sigma^2 + Var(u)$$

$$Var(u) = E(u^2) - (E(u))^2$$

$$E(u^2) = \int_0^\theta u^2\ f(u) du$$

$$= \int_0^\theta u^2 \frac{1}{\theta} du = \frac{\theta^2}{3}$$

$$Var(u) = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

$$\therefore Var(\varepsilon) = \sigma^2 + \frac{\theta^2}{12} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_(8)}$$

The likelihood function of the sample is the product of the density function of the individual observations, which is given as,

$$L(sample) = \prod_{i=1}^{i=N} f(\varepsilon_i) \qquad \text{\_\_\_\_\_\_\_\_(9)}$$

The Log likelihood function for a sample of I producers is

$$InL = -I\; In\theta + \sum_{i=1}^{I} In\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right] \quad \_(10)$$

Technical efficiency can be measured once the parameters are calculated using log – likelihood function.

III.ESTIMATION OF PARAMETERS OF NUSPFM

Parameters θ, σ^2, β can be estimated using the first order conditions of the maximization of log-likelihood function.

$$\frac{\partial\, InL}{\partial\theta} = K_\theta^* = -\frac{I}{\theta} + \sum_{i=1}^{I} \frac{\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right]\frac{1}{\sigma}}{\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]} \quad \_(11)$$

$$\frac{\partial\; InL}{\partial\sigma^2} = K_{\sigma^2}^*$$

$$= \sum_{i=1}^{I} \frac{\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right]\frac{-1}{2\sigma^3}(\theta+\varepsilon_i) - \left[\phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]\frac{-1}{2\sigma^3}(\varepsilon_i)}{\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]}$$

$$\frac{\partial\; InL}{\partial\sigma^2} = K_{\sigma^2}^*$$

$$= \frac{1}{2\sigma^3}\left[\sum_{i=1}^{I} \frac{-\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right](\theta+\varepsilon_i) + \left[\phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]\varepsilon_i}{\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]}\right] \quad \_\_\_\_\_\_\_\_\_(12)$$

The Cobb-Douglas production function is

$$In(y_i) = \beta_0 + \sum_{n=1}^{N} \beta_n\; In\,(x_{ni}) + \varepsilon_i \quad \_\_\_\_\_\_\_\_\_(13)$$

From this

$$In(y_i) = \beta_0 + \sum_{n=i}^{N} \beta_n\; In\,(x_{ni}) + \varepsilon_i \quad \_\_\_\_\_\_\_\_\_(14)$$

$$\frac{\partial InL}{\partial\beta_0} = \frac{\partial InL}{\partial\varepsilon_i} * \frac{\partial\varepsilon_i}{\partial\beta_0}$$

$$\frac{\partial InL}{\partial\varepsilon_i} = \frac{\frac{1}{\sigma}\sum_{i=1}^{N}\left[\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right] - \left[\phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]\right]}{\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]} \quad \_\_\_\_\_\_\_\_\_(15)$$

$$\frac{\partial\varepsilon_i}{\partial\beta_0} = -1 \quad \_\_\_\_\_\_\_\_\_(16)$$

$$\frac{\partial InL}{\partial\beta_0} = \frac{-\frac{1}{\sigma}\sum_{i=1}^{N}\left[\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right] - \left[\phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]\right]}{\left[\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]} \quad \_\_\_\_\_\_\_\_\_(17)$$

$$\frac{\partial InL}{\partial\beta_n} = \frac{\partial InL}{\partial\varepsilon_i} * \frac{\partial\varepsilon_i}{\partial\beta_n} \quad \_\_\_\_\_\_\_\_\_(18)$$

$$\frac{\partial\varepsilon_i}{\partial\beta_n} = -\sum_{n=1}^{N} In\,(x_{ni}) \quad \_\_\_\_\_\_\_\_\_(19)$$

$$\frac{\partial InL}{\partial\beta_n}$$

$$= \frac{-\frac{1}{\sigma}\sum_{n=1}^{N} In\,(x_{ni})\left[\left[\phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right)\right] - \left[\phi\left(\frac{\varepsilon_i}{\sigma}\right)\right]\right]}{\Phi\left(\frac{\theta+\varepsilon_i}{\sigma}\right) - \Phi\left(\frac{\varepsilon_i}{\sigma}\right)} \quad \_\_\_\_\_\_\_\_\_(20)$$

Equating the equations (11),(12),(17),(20) to zero and then solving, we get the maximum likelihood estimation of all parameters.

IV. MEASURE OFTECHNICAL EFFICIENCY OF NORMAL-UNIFORM STOCHASTIC PRODUCTION FRONTIER MODEL

Technical Efficiency,

$$TE_i = \exp(-\hat{u}_i) = \exp\left(-E\left(u_i/\varepsilon_i\right)\right) \quad \_\_\_\_ (21)$$

Consider,

$$E(u/\varepsilon) = \int_0^\theta u \quad \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u+\varepsilon)^2}{2\sigma^2}}}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right) - \Phi\left(\frac{\varepsilon}{\sigma}\right)\right]} \; du \text{ -(22)}$$

$$=\frac{\frac{1}{\sigma\sqrt{2\pi}}}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\int_0^\theta u\ e^{-\frac{(u+\varepsilon)^2}{2\sigma^2}}\ du$$

$$put\ t=\frac{u+\varepsilon}{\sigma},\Rightarrow u=t\sigma-\varepsilon$$

$$dt=\frac{du}{\sigma}\Rightarrow\sigma\,dt=du$$

$$when\ u=0,t=\frac{\varepsilon}{\sigma,}$$

$$when\ u=\theta,t=\frac{\theta+\varepsilon}{\sigma}$$

$$E(u/\varepsilon)=\frac{\frac{1}{\sigma\sqrt{2\pi}}}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\times\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}(t\sigma-\varepsilon)\,e^{-\frac{t^2}{2}}\ \sigma dt \quad \_(23)$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\left[\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}(t\sigma)e^{-\frac{t^2}{2}}\ dt-\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}(\varepsilon)e^{-\frac{t^2}{2}}\ dt\right]$$

$$=\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\left[\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}t\,e^{-\frac{t^2}{2}}\ dt-\frac{\varepsilon}{\sqrt{2\pi}}\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}\varepsilon\,e^{-\frac{t^2}{2}}\ dt\right]$$

$$=\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\left[\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{\varepsilon}{\sigma,}}^{\frac{\theta+\varepsilon}{\sigma}}t\,e^{-\frac{t^2}{2}}\ dt-\varepsilon\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]\right]$$

$$put\ u=\frac{t^2}{2},du=t\ dt$$

$$when\ t=\frac{\varepsilon}{\sigma},\ u=\frac{\left(\frac{\varepsilon}{\sigma}\right)^2}{2}$$

$$when\ t=\frac{\theta+\varepsilon}{\sigma},\quad u=\frac{\left(\frac{\theta+\varepsilon}{\sigma}\right)^2}{2}$$

$$E(u/\varepsilon)=\left[\frac{\sigma}{\sqrt{2\pi}}\int_{\frac{\left(\frac{\varepsilon}{\sigma,}\right)^2}{2}}^{\frac{\left(\frac{\theta+\varepsilon}{\sigma}\right)^2}{2}}e^{-u}\ du-\varepsilon\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]\right]\_\_\_(24)$$

$$=\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\times\left[\frac{-\sigma}{\sqrt{2\pi}}\left(e^{-\frac{\left(\frac{\theta+\varepsilon}{\sigma}\right)^2}{2}}-e^{-\frac{\left(\frac{\varepsilon}{\sigma,}\right)^2}{2}}\right)-\varepsilon\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]\right]$$

$$=\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\times\left[\left(\sigma\,\phi\left(\frac{\varepsilon}{\sigma}\right)-\sigma\,\phi\left(\frac{\theta+\varepsilon}{\sigma}\right)\right)-\varepsilon\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]\right]$$

$$=\frac{1}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\times\left[-\sigma\left(\phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\phi\left(\frac{\varepsilon}{\sigma}\right)\right)-\varepsilon\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]\right]$$

The expected value of inefficiency term u given ε in the normal-uniform model is

$$E(u/\varepsilon)=\left[-\sigma\left(\frac{\phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\phi\left(\frac{\varepsilon}{\sigma}\right)}{\left[\Phi\left(\frac{\theta+\varepsilon}{\sigma}\right)-\Phi\left(\frac{\varepsilon}{\sigma}\right)\right]}\right)-\varepsilon\right]\_\_\_(25)$$

V. CONCLUSION:

Econometricians have been developing stochastic production frontier models since 1977. As a result, the models have taken on a rich variety of forms. The models applied to a number of areas, such as banks, agriculture, hospitals, schools, industries, and so on. In this paper, normal distribution for v and uniform distribution for u are assigned. Technical

Efficiency for NUSPFM was derived. For Technical efficiency, the expectation of u_i given ε_i was obtained from the joint density function of u_i and ε_i, and marginal density function of f(ε). The joint density function of u_i and ε, is the product of individual density functions u and ε . The marginal density function f(ε) is obtained by integrating u out of f (u, ε).Parameters also estimated using Log likelihood function. In a similar way we can derive the technical efficiency and estimate the parameters for other distributions also.

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