A Normal-Half Normal Distributed Stochastic Cost Frontier Model

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Abstract -- *Stochastic Cost Frontier Model (SCFM) plays a major role in measuring Cost Efficiency Scores (CES) in the field of production. The present study is an attempt to derive the cost efficiency of normal half normal stochastic cost frontier model. The parameters were evaluated using Maximum likelihood Estimates. In the model* $v_i \sim N(0, \sigma_v^2)$, *is a two sided error term representing the statistical noise and* $u_i \geq 0$ *is one sided error term representing cost efficiency.*

Key words*: Normal-Half normal distribution, Stochastic Cost Frontier Model, Cost Efficiency scores.*

I.INTRODUCTION

Starting with the pioneering work of Farrell (1957) on the calculations of cost efficiency, stochastic frontier models have been used successfully in many field. Aigner and Lovell (1976) worked on the model $\boldsymbol{y_i = f(x_i, \beta) + \varepsilon_i}$ by taking $\boldsymbol{\varepsilon_i = v_i + u_i}$. To estimate the cost efficiency of each producer, distribution assumptions are required. In 1977 Aigner,Lovell and Schmidt published a paper in which $\boldsymbol{v_i}$ assumed to follow normal distribution and $\boldsymbol{u_i}$ half normal distribution and exponential distribution.Battes and Corra (1977) assumed half normal distribution for $\boldsymbol{u_i}$ for the production frontier. Steven B Coudill (2003) considered normal-half normal distribution for $\boldsymbol{v_i}$ ***and*** $\boldsymbol{w_i}$ for the model $\boldsymbol{y_i = x_i\beta + w_i + v_i}$,the stochastic frontier (cost) regression model. In this paper normal- half normal distribution is used to calculate the joint density function of u and v. Once the marginal density function of $\boldsymbol{\varepsilon_i}$ is calculated, using log-likelihood functions Parameters like $\boldsymbol{\beta, \sigma^2, \lambda}$ are estimated. Measures of cost efficiency for NHSCFM are obtained once the conditional probability of $\boldsymbol{u_i}$ given $\boldsymbol{\varepsilon_i}$ is derived.

II. THE NORMAL-HALF NORMAL STOCHASTIC COST FRONTIER MODEL (NHSCFM)

Considering the stochastic cost frontier model, the following assumptions in the distribution were made.

1) $v_i \sim iid\ N(0, \sigma_v^2)$

2) $u_i \sim iid\ N^+(0, \sigma_u^2)$ that is non negative half normal.

3) v_i and u_i are distributed independently of each other and of the regressors.

Probability density function of u is given by

$$f(u) = \frac{2}{\sigma_u\sqrt{2\pi}} e^{\frac{-u^2}{2\sigma_u^2}} \quad (1)$$

Probability density function of v is given by

$$f(v) = \frac{1}{\sigma_v\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma_v^2}} \quad (2)$$

The joint density function of u and v is the product of their individual density functions,

Joint distribution of u and v is ,f (u, v) = f (u).f (v)

$$f(u, v) = \frac{1}{\pi\sigma_u\sigma_v} e^{\frac{-v^2}{2\sigma_v^2} - \frac{u^2}{2\sigma_u^2}} (3)$$

Making the transformations $\varepsilon = v + u$, the joint density function of u and ε is

$$f(u, \varepsilon) = \frac{1}{\pi\sigma_u\sigma_v} e^{\frac{-(\varepsilon-u)^2}{2\sigma_v^2} - \frac{u^2}{2\sigma_u^2}} \quad (4)$$

The marginal density function of ε is obtained by integrating $f(u, \varepsilon)$ with respect to u.

$$f(\varepsilon) = \int_0^\infty f(u, \varepsilon) du \quad (5)$$

$$f(\varepsilon) = \int_0^\infty \frac{1}{\pi\sigma_u\sigma_v} e^{\frac{-(\varepsilon^2 - 2\varepsilon u + u^2)}{2\sigma_v^2} - \frac{u^2}{2\sigma_u^2}} du (6)$$

Let $\sigma^2 = \sigma_u^2 + \sigma_v^2$; $\lambda = \frac{\sigma_u}{\sigma_v}$

Thus,

$$f(\varepsilon) = \frac{1}{\pi\sigma_u\sigma_v} \int_0^\infty e^{\frac{-\frac{1}{2}(\sigma_u^2\varepsilon^2 - 2\varepsilon u\sigma_u^2 + u^2\sigma_u^2 + u^2\sigma_v^2)}{\sigma_u^2\sigma_v^2}} du$$

(7)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{\frac{-\frac{1}{2}(u^2(\sigma_u^2+\sigma_v^2)+\sigma_u^2\varepsilon^2-2\varepsilon u\sigma_u^2)}{\sigma_u^2\sigma_v^2}}du$$

(8)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{\frac{-\frac{1}{2}(u^2\sigma^2+\sigma_u^2\varepsilon^2-2\varepsilon u\sigma_u^2)}{\sigma_u^2\sigma_v^2}}du(9)$$

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{\sigma^2}{2\sigma_u^2\sigma_v^2}\left(u^2+\frac{\sigma_u^2\varepsilon^2-2\varepsilon u\sigma_u^2}{\sigma^2}\right)}du(10)$$

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{\sigma^2}{2\sigma_u^2\sigma_v^2}\left[\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2\sigma_u^2}{\sigma^2}-\frac{\sigma_u^4\varepsilon^2}{\sigma^4}\right]}du$$

(11)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\sigma^2\varepsilon^2\sigma_u^2}{\sigma_u^2\sigma_v^2\sigma^2}-\frac{\sigma^2\sigma_u^4\varepsilon^2}{\sigma_u^2\sigma_v^2\sigma^4}\right]}du$$

(12)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2}{\sigma_v^2}-\frac{\sigma_u^2\varepsilon^2}{\sigma_v^2\sigma^2}\right]}du$$

(13)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2}{\sigma_v^2}\left(1-\frac{\sigma_u^2}{\sigma^2}\right)\right]}du$$

(14)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2}{\sigma_v^2}\left(\frac{\sigma_u^2+\sigma_v^2-\sigma_u^2}{\sigma^2}\right)\right]}du$$

(15)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2}{\sigma_v^2}\left(\frac{\sigma_v^2}{\sigma^2}\right)\right]}du$$

(16)

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}e^{-\frac{\varepsilon^2}{2\sigma^2}}\int_0^\infty e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2\right]}du$$

(17)

Define:

$$\frac{\sigma}{\sigma_u\sigma_v}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)=s$$

$$\frac{\sigma}{\sigma_u\sigma_v}du=ds\Rightarrow du=\frac{\sigma_u\sigma_v}{\sigma}ds$$

When $u=o,\ s=-\frac{\sigma\varepsilon\sigma_u^2}{\sigma_u\sigma_v\sigma^2}=-\frac{\varepsilon\lambda}{\sigma}$

When $u\to\infty,\ s\to\infty$

$$f(\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}e^{-\frac{\varepsilon^2}{2\sigma^2}}\int_{-\frac{\varepsilon\lambda}{\sigma}}^\infty e^{-\frac{1}{2}s^2}\left(\frac{\sigma_u\sigma_v}{\sigma}\right)ds$$

(18)

$$f(\varepsilon)=\frac{1}{\pi\sigma}e^{-\frac{\varepsilon^2}{2\sigma^2}}\int_{-\frac{\varepsilon\lambda}{\sigma}}^\infty e^{-\frac{1}{2}s^2}ds(19)$$

$$f(\varepsilon)=\frac{2}{2\pi\sigma}e^{-\frac{\varepsilon^2}{2\sigma^2}}\int_{-\frac{\varepsilon\lambda}{\sigma}}^\infty e^{-\frac{1}{2}s^2}ds(20)$$

$$f(\varepsilon)=\frac{2}{\sqrt{2\pi}\sigma}e^{-\frac{\varepsilon^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}}\int_{-\frac{\varepsilon\lambda}{\sigma}}^\infty e^{-\frac{1}{2}s^2}ds(21)$$

$$f(\varepsilon)=\frac{2}{\sigma}\frac{1}{\sqrt{2\pi}}e^{-\frac{\varepsilon^2}{2\sigma^2}}\left[1-\Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right](22)$$

$$f(\varepsilon)=\frac{2}{\sigma}\phi\left(\frac{\varepsilon}{\sigma}\right)\Phi\left(\frac{\varepsilon\lambda}{\sigma}\right)(23)$$

where $\phi(.),\Phi(.)$ are the density function and standard normal cumulative distribution respectively. The marginal density function $f(\varepsilon)$ is asymmetrically distributed, with mean and variance as below.

$$E(\varepsilon)=E(v+u)=E(v)+E(u)=0+E(u)=E(u)$$

$$E(\varepsilon)=\int_0^\infty u\,f(u)du \qquad (24)$$

$$E(\varepsilon)=\frac{2}{\sigma_u\sqrt{2\pi}}\int_0^\infty u\,e^{-\frac{u^2}{2\sigma_u^2}}du(25)$$

Substituting $s=\frac{u^2}{2\sigma_u^2}$; $udu=\sigma_u^2\,ds$

When $u=o,s=0\,;u\to\infty,s\to\infty$

$$E(\varepsilon)=\frac{2\sigma_u^2}{\sigma_u\sqrt{2\pi}}\int_0^\infty e^{-s}ds \qquad (26)$$

$$E(\varepsilon)=\frac{2\sigma_u}{\sqrt{2\pi}}[-e^{-s}]_0^\infty \qquad (27)$$

Therefore,

$$E(\varepsilon)=E(u)=\sigma_u\sqrt{\frac{2}{\pi}}$$

Similarly,

$$V(\varepsilon)=V(u)+V(v).$$

$$V(u)=E(u^2)-(E(u))^2.$$

$$E(u^2)=\frac{2}{\sigma_u\sqrt{2\pi}}\int_0^\infty u^2e^{-\frac{u^2}{2\sigma_u^2}}du(28)$$

Define $s = \frac{u^2}{2\sigma_u^2}$

$2udu = 2\sigma_u^2 ds$

$udu = \sigma_u^2 ds$

$du = \frac{\sigma_u}{\sqrt{2s}} ds$

When $u = o, s = 0\,; u \to \infty, s \to \infty$

$E(u^2) = \frac{2}{\sigma_u\sqrt{2\pi}} \int_0^\infty 2s\ \sigma_u^2 \left(\frac{\sigma_u}{\sqrt{2s}}\right) e^{-s}\ ds (29)$

$E(u^2) = \frac{2\sigma_u^2}{\sqrt{\pi}} \int_0^\infty s^{\frac{1}{2}} e^{-s}\ ds (30)$

$E(u^2) = \frac{2\sigma_u^2}{\sqrt{\pi}} \int_0^\infty s^{\frac{3}{2}-1} e^{-s}\ ds (31)$

$E(u^2) = \frac{2\sigma_u^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right) (32)$

$E(u^2) = \frac{2\sigma_u^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) (33)$

$E(u^2) = \frac{2\sigma_u^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} (34)$

$E(u^2) = \sigma_u^2 (35)$

Therefore,

$$V(\varepsilon) = \sigma_u^2 - \frac{2\sigma_u^2}{\pi} + \sigma_v^2$$

$V(\varepsilon) = \frac{\pi-2}{\pi}\sigma_u^2 + \sigma_v^2 (36)$

The likelihood function of the sample is the product of the density function of the individual observations, which is given as,

$$L(sample) = \prod_{i=1}^{i=N} f(\varepsilon_i)$$

Christopher F. Parmeter and Subal C.Kumbhakar (2014), got the corresponding log likelihood function for $\varepsilon_i = y_i - m(x_i; \beta)$ as

$$In\ L = -n\ In\ \sigma + \sum_{i=1}^{n} In\ \Phi\left(\frac{-\varepsilon_i \lambda}{\sigma}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \varepsilon_i^2$$

The log likelihood equation for a sample of N producers for equation 22 is:

$$In\ L = -\left(\frac{N}{2}\right)(In\ 2\pi + In\sigma^2) + \sum_{i=1}^{N}\left\{In\Phi\left(\frac{\varepsilon\lambda}{\sigma}\right) - \frac{1}{2}\left(\frac{\varepsilon_i^2}{\sigma^2}\right)\right\}$$

(37)

Cost efficiency can be measured,once the parameters are calculated using log – likelihood function.

III .ESTIMATION OF THE PARAMETERS OF NHSCFM

Parameters β, σ^2, λ can be estimated using the first order conditions of the maximization of log-likelihood function.

Consider,

$L[\beta, \sigma^2, \lambda] = In\ L$

$= -\left(\frac{N}{2}\right)(In\ 2\pi + In\sigma^2) + \sum_{i=1}^{N}\left\{In\Phi\left(\frac{\varepsilon\lambda}{\sigma}\right) - \frac{1}{2}\left(\frac{\varepsilon_i^2}{\sigma^2}\right)\right\} (38)$

$$L[\beta, \sigma^2, \lambda] = In\ L$$

$-\left(\frac{N}{2}\right) In 2\pi - \frac{N}{2} In\sigma^2 - \frac{1}{2}\sum_{i=1}^{N}\left(\frac{y_i - x_i'\beta}{\sigma}\right)^2 + \sum_{i=1}^{N} In\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right) (39)$

The first order partial derivatives of (39) with respect to β, σ^2, λ are

$$\frac{\partial InL}{\partial \beta} = F_\beta^* =$$

$$\frac{1}{\sigma^2}\sum_{i=1}^{N}(y_i - x_\beta')x_i + \sum_{i=1}^{N}\frac{\phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}{\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}\left(-\frac{x_i'\lambda}{\sigma}\right)$$

(40)

$\frac{\partial InL}{\partial \lambda} = F_\lambda^* = \sum_{i=1}^{N}\frac{\phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}{\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}\left(\frac{y_i - x_i'\beta}{\sigma}\right) (41)$

$$\frac{\partial In\ L}{\partial \sigma^2} = F_{\sigma^2}^*$$

$= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(y_i - x_i'\beta)^2 + \frac{\lambda}{2\sigma^3}\sum_{i=1}^{N}\frac{\phi\left(\frac{(y_i - x_i'\beta)}{\sigma}\right)}{\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}(y_i - x_i'\beta) (42)$

Where x_i is a (m × 1) vector consisting of elements in the i the row of x, and ϕ *and* Φ are the standard normal density and distribution function respectively. These functions are evaluated at $\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)$.

Equating equation (41) to Zero

$$\frac{\partial inL}{\partial \lambda} = 0$$

$$\sum_{i=1}^{N} \frac{\phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}{\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)} \left(\frac{y_i - x_i'\beta}{\sigma}\right) = 0$$, at the optimum (43)

Equating (42) to Zero and substituting (43) following equation is obtained

$$-\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{N} (y_i - x_i'\beta)^2 = 0 \qquad (44)$$

Simplifying further,

$$\frac{1}{\sigma^2} \sum_{i=1}^{N} (y_i - x_i'\beta)^2 = N \qquad (45)$$

The likelihood estimator of σ^2 is given by

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - x_i'\beta)^2 (46)$$

To evaluate the likelihood estimator of β following is defined

X be an N x m data matrix

Y be an N x1 data vector

εbe an N x 1 vector $(\varepsilon_1 \varepsilon_2 \dots \dots . \varepsilon_N)$

Let γ be $\frac{\phi\left(\frac{(y_i - x_i'\beta)}{\sigma}\right)}{\Phi\left(\frac{(y_i - x_i'\beta)\lambda}{\sigma}\right)}$

Using above definitions equations (40), (41) and (42) changes in to

$$F_\beta^* = \frac{1}{\sigma^2}[X'y - X'X\beta] - \frac{\lambda X'\gamma}{\sigma} = 0 (47)$$

$$F_\lambda^* = \frac{1}{\sigma} \varepsilon'\gamma = 0 (48)$$

$$F_{\sigma^2}^* = -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^4} \varepsilon\varepsilon' + \frac{\lambda\varepsilon'\gamma}{2\sigma^3} = 0 (49)$$

Multiplying (47) by $\sigma^2 (XX')^{-1}$

$$(XX')^{-1}[X'y - X'X\beta] - \sigma(XX')^{-1}\lambda X'\gamma = 0$$

(50)

$$(XX')^{-1}X'y - \beta - \sigma(XX')^{-1}\lambda X'\gamma = 0 (51)$$

Gives,

$$\beta = (XX')^{-1}X'[y - \sigma\lambda\gamma] (52)$$

Define, $a = (XX')^{-1}X'y$ and $b = \sigma(XX')^{-1}\lambda X'\gamma$

(53) Then the likelihood estimator of β is,

$$\hat{\beta} = a - b (54)$$

Where a is the slope vector of the OLS estimator and b is the OLS estimate.

Equation (47) can be re written as

$$\frac{1}{\sigma^2} X'\varepsilon - \frac{\lambda X'\gamma}{\sigma} = 0 (55)$$

Multiplying (55) by $\sigma\beta'$

$$\frac{\beta' X'\varepsilon}{\sigma} - \beta'\lambda X'\gamma = 0 (56)$$

The likelihood estimator of λ is,

$$\lambda = \frac{-\beta' X'\varepsilon}{\sigma\beta' X'\gamma} (57)$$

Thus the maximum likelihood estimator can be evaluated using

(46), (54) and (57). The inefficiency, u_i , can be obtained once the parameters are estimated. If$\varepsilon_i < 0$, then the possibility is that u_i is not large, which in turn means that producer is relatively cost efficient .If $\varepsilon_i > 0$ then u_i is large means the producer is cost inefficient.(S.Kumbhakar and C.A Knox Lovell 2000).

A solution to the problem is obtained from the conditional distribution of u_i given ε_i , which contains whatever information ε_i contains concerning u_i . If $u_i \sim N^+(0, \sigma_u^2)$,the conditional distribution of u_i given ε_i is

$$f(u/\varepsilon) = \frac{f(\varepsilon, u)}{f(\varepsilon)} \qquad (58)$$

$$f(u/\varepsilon) = \frac{\frac{1}{\pi\sigma_u\sigma_v} e^{\left[-\frac{u^2}{2\sigma_u^2} - \frac{(\varepsilon-u)^2}{2\sigma_v^2}\right]}}{\frac{2}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2}}\left[1 - \Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]} (59)$$

$$f(u/\varepsilon) = \frac{\sigma\left[1 - \Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left(\frac{u^2}{\sigma_u^2} + \frac{\varepsilon^2 - 2\varepsilon u + u^2}{\sigma_v^2} - \frac{\varepsilon^2}{\sigma^2}\right)} (60)$$

$$f(u/\varepsilon) = \frac{\sigma\left[1 - \Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left(\frac{u^2(\sigma_u^2 + \sigma_v^2) + \varepsilon^2\sigma_u^2 - 2\varepsilon u\sigma_u^2}{\sigma_u^2\sigma_v^2} - \frac{\varepsilon^2}{\sigma^2}\right)} (61)$$

$$f(u/\varepsilon) = \frac{\sigma\left[1 - \Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left(\frac{u^2\sigma^2 + \varepsilon^2\sigma_u^2 - 2\varepsilon u\sigma_u^2}{\sigma_u^2\sigma_v^2} - \frac{\varepsilon^2}{\sigma^2}\right)}$$

(62)

$$f(u/\varepsilon) = \frac{\sigma\left[1 - \Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u - \frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2 - \frac{\varepsilon^2\sigma_u^2}{\sigma^2\sigma_v^2} + \frac{\varepsilon^2}{\sigma_v^2} - \frac{\varepsilon^2}{\sigma^2}\right]} (63)$$

$$f(u/\varepsilon) = \frac{\sigma\left[1-\Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2+\frac{\varepsilon^2}{\sigma^2}-\frac{\varepsilon^2}{\sigma^2}\right]}$$

(64)

$$f(u/\varepsilon) = \frac{\sigma\left[1-\Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left[\frac{\sigma^2}{\sigma_u^2\sigma_v^2}\left(u-\frac{\varepsilon\sigma_u^2}{\sigma^2}\right)^2\right]}$$

(65)

$$f(u/\varepsilon) = \frac{\sigma\left[1-\Phi\left(-\frac{\varepsilon\lambda}{\sigma}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_u\sigma_v} e^{-\frac{1}{2}\left(\frac{u-\frac{\varepsilon\sigma_u^2}{\sigma^2}}{\frac{\sigma_u\sigma_v}{\sigma}}\right)^2} \quad (66)$$

John Drow et al. (1982)considered the two point disturbance with $v_i \sim N(0,\sigma_v^2), u_i \sim N(0,\sigma_u^2)$. They defined $\sigma^2 = \sigma_u^2 + \sigma_v^2$, $u_* = -\frac{\sigma_u^2\varepsilon}{\sigma^2}$ &$\sigma_*^2 = \frac{\sigma_u^2\sigma_v^2}{\sigma^2}$,for production function.

In this case we define $\mu_* = \frac{\varepsilon\sigma_u^2}{\sigma^2}$ and $\sigma_* = \frac{\sigma_u\sigma_v}{\sigma}$

$$f(u/\varepsilon) = \frac{\left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}}{\sqrt{2\pi}\sigma_*} e^{-\frac{1}{2}\left(\frac{u-\mu_*}{\sigma_*}\right)^2} (67)$$

IV. *Measure of cost efficiency ofNHSCFM*

Since $f(u/\varepsilon)$ is distributed as$N^+(\mu_*,\sigma_*^2)$, the mean of this distribution can serve as a point estimator of $u_i$which is given by

$$E(u/\varepsilon) = \int_0^\infty u\, f(u/\varepsilon)\, du (68)$$

$$E(u/\varepsilon) = \int_0^\infty u \frac{1}{\sigma_*\sqrt{2\pi}}\left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1} e^{-\frac{1}{2}\left(\frac{u-\mu_*}{\sigma_*}\right)^2} du$$

(69)

Define, $s = \frac{u-\mu_*}{\sigma_*} \sigma_* ds = du$

When $u = o\,, s = -\frac{\mu_*}{\sigma_*}\,; u \to \infty, s \to \infty$

$$E(u/\varepsilon) = \frac{\left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}}{\sigma_*\sqrt{2\pi}} \int_{-\frac{\mu_*}{\sigma_*}}^{\infty} (\sigma_* s + \mu_*) e^{-\frac{1}{2}(s)^2} \sigma_*\, ds \quad (70)$$

$$= \left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}\left[\frac{\sigma_*^2}{\sigma_*\sqrt{2\pi}}\int_{-\frac{\mu_*}{\sigma_*}}^{\infty} s\, e^{-\frac{1}{2}(s)^2} ds + \frac{\mu_*\sigma_*}{\sigma_*\sqrt{2\pi}}\int_{-\frac{\mu_*}{\sigma_*}}^{\infty} e^{-\frac{1}{2}(s)^2} ds\right] (71)$$

Let $t = \frac{s^2}{2},\ dt = sds$

When $s = -\frac{\mu_*}{\sigma_*},\ t = \frac{\mu_*^2}{2\sigma_*^2}\,; s \to \infty, t \to \infty$

$$E(u/\varepsilon) = \left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}\left\{\frac{\sigma_*}{\sqrt{2\pi}}\int_{\frac{\mu_*^2}{2\sigma_*^2}}^{\infty} e^{-t}\, dt + \mu_*\left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]\right\} \quad (72)$$

$$E(u/\varepsilon) = \left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}\left(\frac{\sigma_*}{\sqrt{2\pi}}\left[\frac{e^{-t}}{-1}\right]_{\frac{\mu_*^2}{2\sigma_*^2}}^{\infty}\right) + \mu_*$$

(73)

$$E(u/\varepsilon) = \left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}\left(\frac{\sigma_*}{\sqrt{2\pi}} e^{-\frac{\mu_*^2}{2\sigma_*^2}}\right) + \mu_*$$

(74)

$$E(u/\varepsilon) = \left[1-\Phi\left(-\frac{\mu_*}{\sigma_*}\right)\right]^{-1}\sigma_*\phi\left(-\frac{\mu_*}{\sigma_*}\right) + \mu_* (75)$$

$$=\left[1-\Phi\left(-\frac{\varepsilon\sigma_u^2}{\sigma^2}\frac{\sigma}{\sigma_u\sigma_v}\right)\right]^{-1}\sigma_*\phi\left(-\frac{\varepsilon\sigma_u^2}{\sigma^2}\frac{\sigma}{\sigma_u\sigma_v}\right) + \mu_*$$

(76)

$$E(u/\varepsilon) = \sigma_*\left[1-\Phi\left(-\frac{\varepsilon_i\lambda}{\sigma}\right)\right]^{-1}\phi\left(-\frac{\varepsilon\lambda}{\sigma}\right) + \frac{\varepsilon\sigma_u^2}{\sigma^2}$$

(77)

$$E(u_i/\varepsilon_i) = \sigma_*\left[\frac{\phi\left(-\frac{\varepsilon_i\lambda}{\sigma}\right)}{1-\Phi\left(-\frac{\varepsilon_i\lambda}{\sigma}\right)} + \frac{\varepsilon_i\lambda}{\sigma}\right] \quad (78)$$

Estimates of u_i can be obtained from

$$CE_i = E\left(e^{\{-u_i\}/\varepsilon_i}\right)$$

V.Conclusion

Oncetheestimates of u_i are can be obtained cost efficiency of each producer can be obtained from $CE_i = e^{-\widehat{u_i}}$ where $u_i = E(u_i/\varepsilon_i) or\ M(u_i/\varepsilon_i)$ .The point estimator of $CE_i = E\left(e^{\{-u_i\}/\varepsilon_i}\right)$and $CE_i = e^{-\widehat{u_i}}$ can give different results since $e^{-E(u_i/\varepsilon_i)} \neq E\left(e^{-u_i/\varepsilon_i}\right)$.The estimator $CE_i = E\left(e^{\{-u_i\}/\varepsilon_i}\right)$is used when u_i is not close to zero. $E(e^{-u})$is consistent with the definition of cost efficiency given in $V(\varepsilon) = \frac{\pi-2}{\pi}\sigma_u^2 + \sigma_v^2$. Battese and Coelli (1988) proposed alternative point estimator $E\left(e^{\{-u_i\}/\varepsilon_i}\right)$for technical efficiency, which can be adopted for cost efficiency.

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