A Bilevel Quadratic–Quadratic Fractional Programming through Fuzzy Goal Programming approach

Dr. Madhuchanda Rakshit [1], Mr. Suchet Kumar[2]

(Assistant Professor in Department of Mathematics, Guru Kashi University, Talwandi Sabo, India)[1]
(Research Scholar in Department of Mathematics, Guru Kashi University, Talwandi Sabo, India)[2]

ABSTRACT

This paper proposed a new model of quadratic fractional programming problem where our purpose is to study the quadratic fractional programming problem through fuzzy goal programming procedure by utilizing the bilevel linear programming. The bilevel is a class of multilevel optimization hierarchy with two decision levels and each objective function in both decision maker levels has fractional form with quadratic function in numerator as well as in denominator. In this paper we construct two bilevel quadratic programming problems from one bilevel quadratic fractional programming problem by separating the numerator and denominator in fractional objective function of each decision maker. Next our purpose isto solve both bilevel quadratic programming problems separately and thus to form a solution procedure for our proposed model which named as Bilevel Quadratic –Quadratic Fractional Programming Problem.

Keywords*: Quadratic Programming, Quadratic Fractional programming, Fuzzy Goal Programming, Bilevel Quadratic fractional programming problem, Bilevel Quadratic – Quadratic Fractional Programming Problem*

1. INTRODUCTION

Bi-level programming is a powerful and robust technique for solving hierarchical decision- making problem. The bi-level programming problem (BLPP) is a nested optimization problem, which has two levels in hierarchy. The first level is called leader and the second level is called follower which has their own objective functions and constraints. The first level or upper level or first level decision maker (FLDM), sets his goals or decisions, which is referred as the leader and the second level decision maker (SLDM) decides and submits their decisions, which is referred as follower. Bi-level programming is also a class of multi-level programming which is computationally more complex and expensive that conventional mathematical programming. BLPP is a practical tool for solving various decision making problems. The BLPP is used in areas such as economic systems, traffic, finance, engineering, banking, transportation, network design, management planning and so on. As an example in a congestion pricing problem which provides an optimal price for vehicles entering the bridges or in a specified areas, the BLPP model is the best known model in which, at first level the income of the leader that in this case usually the municipal is maximized whenever in the second level the users or drivers are trying to minimize their route from origin to the destination. Bilevel programming problems are very complex because of their non-continuity and non-convexity nature, especially the nonlinear bilevel programming problems. Therefore, most researchers in this field have been focused to especially in this type of programming. The idea of bilevel programming problems was firstly introduced by a Candler and Townsley [2] as well as by Fortuni-Amat and McCarl [4]. Many researchers have designed algorithms for the solution of the BLPP [10, [11], [17], [20], [34]. Amat and McCarl [4] presented the formal formulation of BLPP. Anandalingam [1] proposed Stackelberg solution concept to multi-level programming problem (MLPP) as well as bi-level decentralized programming problem (BLDPP). Lai [16] applied the concept of fuzzy set theory at first to MLPP by using tolerance membership functions. Sinha [29], [30], [31] presented alternative multi-level programming based on fuzzy mathematical programming.

Quadratic Fractional Programming is a particular type of nonlinear programming problem in which the target function may be a ratio of two quadratic objective functions subject to a set of linear constraints. Such problems also arise naturally in decision-making when several rates have to be compelled for the optimization at the same time. In the fields like financial and corporate planning, production planning, hospital and healthcare planning the quadratic fractional programming is widely used. The concept of decision-making takes place in an environment in which the objectives and constraints are not appropriate exactly, and in such cases the concept of quadratic fractional programming has been used extensively. Initially, Ibaraki, T. et al [8] developed an algorithm for quadratic fractional programming problem and later on Terlaky [33] also gives an algorithm to solve QFPP. Also Tantawy [32] used feasible direction method to solve QFPP. Nejmaddin [21] solved QFPP by Wolfe's and modified simplex method and Khurana [14] studied such types of QFPP with linear homogeneous constraints.

The use of fuzzy set theory for decision problems was first introduced by a Zimmermann[5]. After that the various approaches were introduced and investigated in the literature of Bilevel Programming Problems. Lai [16] applied the concepts of membership function of optimality and degree of decision powers to solve the technical inefficiency problems in Kuhn–Tucker condition or penalty functions based Multilevel Programming approaches . Lai's concept has been extended further by Shih et al [28] but that approach was quite lengthy for solution procedure. To overcome this type of problem, the fuzzy goal programming approach (FGP) was proposed by Mohamed [19] which is extended by Pramanik and Roy [26] to solve the multilevel linear programming problems. Lachhwani and Poonia[13] proposed FGP approach for multi-level linear fractional programming problem. Baky [7] used fuzzy goal programming approach to solve decentralized bilevel multiobjective programming problems. Chang [3] suggested goal programming approach for fuzzy multiobjective fractional programming problems. Also, Pal and Gupta [22] studied a genetic algorithm to fuzzy goal programming formulation of fractional multiobjective decision-making problems. Recently, Moitra and Pal [24] has been introduced approach to solve bilevel linear programming problems. Mishra and Ghosh [18] studied interactive fuzzy programming approach to bi-level quadratic fractional programming problems by updating the satisfactory level of the DM at the first level with consideration of overall satisfactory balance between the levels. In fuzzy environment, Pal and Moitra [23] proposed fuzzy goal programming (FGP) procedure for solving quadratic bilevel programming problem in 2003. In which they formulated quadratic bilevel programming problem in two phases by using the notion of distance function. At the first phase of the solution process, Pal and Moitra transform quadratic bilevel programming problem model into nonlinear goal programming model in order to maximize the membership value of each of the fuzzy objective goals based on their priorities in the decision context. However, each level DM has only one objective function, therefore their concept of priority is not appropriate.

In this paper we focus on solving the bilevel Quadratic Fractional Programming Problem by using the fuzzy goal programming approach which is denoted as BQFPP. BQFPP are the special case of BLPP but in this paper we proposed that the leader and the follower each has fractional objective function and also the numerator as well as denominator of fraction has quadratic function. Our purpose is to organize this BQFPP into two Bilevel Quadratic Programming Problems (BLQPP) by converting the numerator as well as denominator of fractional objective function of each level into separate functions. Thus, we get two BLQPPs from one BQFPP. Therefore our proposed approach we denoted as the bilevel quadratic-quadratic fractional programming problem (BQQFPP). After converting into two BLQPPs, we solve both of them separately by using fuzzy goal programming approach and then finally combined the solutions of both BQPPs to achieve the solution procedure of BQFPP.

2. *PROBLEM FORMULATION*

Let both the leader and follower have a motivation to cooperate with each other and try to maximize their own benefit paying serious attention to the preferences of the other. Therefore the vectors of decision variables x_1 and x_2 are under the control of the leader and follower respectively and let f_1 and f_2 be the leader and the follower respective differentiable quadratic fractional objective functions which we denoted as $f_1 = \frac{f_{11}}{f_{12}}$ and $f_2 = \frac{f_{21}}{f_{22}}$. Thus a Bilevel Quadratic-Quadratic Fractional Programming Problem can be defined as:-

$[1^{st}\ Level]$

$$\max_{x_1} f_1(x_1,x_2) = \frac{f_{11}(x_1,x_2)}{f_{12}(x_1,x_2)} = \frac{C_{11}x + \frac{1}{2}x^T D_{11} x}{C_{12}x + \frac{1}{2}x^T D_{12} x}$$

where x_1 solves and x_1 is vector of decision variable.

$[2^{nd}\ Level]$

$$\max_{x_2} f_2(x_1,x_2) = \frac{f_{21}(x_1,x_2)}{f_{22}(x_1,x_2)} = \frac{C_{21}x + \frac{1}{2}x^T D_{21} x}{C_{22}x + \frac{1}{2}x^T D_{22} x}$$

where x_2 solves and x_2 is vector of decision variable.

subject to $(x_1, x_2) \in S$

$$A_1x_1 + A_2x_2 \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$$

$$x_1, x_2 \ge 0$$

where f_1 *and* f_2 are objective functions of the first level decision maker (FLDM), and second level decision maker (SLDM) and C_{ij} are (1×2) matrices and D_{ij} are 2×2 real matrices for $i = 1,2$ *and* $j = 1,2$. A_1, A_2 are matrices of coefficients.. The first-level decision maker has control over x_1, and second-level decision maker has control over the x_2. The objective functions $f_1$and $f_2$are assumed to be concave, differentiable and bounded.

Now, our intention is to solve BQQFPP by converting it into two BLQPPs as following:-

BLQPP-I

$[1^{st}\ Level]$

$$\max_{x_1} f_1(x_1, x_2) = f_{11}(x_1, x_2) = C_{11}x + \frac{1}{2}x^T D_{11}x$$

where x_1 solves and x_1 is vector of decision variable

$[2^{nd}\ Level]$

$$\max_{x_2} f_2(x_1, x_2) = f_{21}(x_1, x_2) = C_{21}x + \frac{1}{2}x^T D_{21}x$$

where x_2 solves and x_2 is vector of decision variable

subject to $(x_1, x_2) \in S$

$$A_1x_1 + A_2x_2 \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$$

$$x_1, x_2 \ge 0$$

BLQPP-II

$[1^{st}\ Level]$

$$\max_{x_1} f_1(x_1, x_2) = -f_{12}(x_1, x_2) = -\left(C_{12}x + \frac{1}{2}x^T D_{12}x\right)$$

where x_1 solves and x_1 is vector of decision variable

$[2^{nd}\ Level]$

$$\max_{x_2} f_2(x_1, x_2) = -f_{22}(x_1, x_2) = -\left(C_{22}x + \frac{1}{2}x^T D_{22}x\right)$$

where x_2 solves and x_2 is vector of decision variable

subject to $(x_1, x_2) \in S$

$$A_1x_1 + A_2x_2 \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$$

$$x_1, x_2 \ge 0$$

3. *FORMULATION OF THE FUZZY PROGRAMMING PROBLEM APPROACH TO THE BQQFPP*

In BLPP, the fuzzy objectives are termed as fuzzy goals if an imprecise aspiration level is assigned to each objective function and they are characterized by their associated membership function by defining the tolerance limit for the achievement of their aspired levels.

3.1. CONSTRUCTION OF MEMBERSHIP FUNCTIONS

Let $\left({x^{11}}_1, {x^{11}}_2, {f^{11}}_{11}\right)$ and $\left({x^{21}}_1, {x^{21}}_2, {f^{21}}_{11}\right)$ be the optimal solutions of the leader and the follower for BLQPP-I over the region S and Let $\left({x^{12}}_1, {x^{12}}_2, {f^{12}}_{12}\right)$ and $\left({x^{22}}_1, {x^{22}}_2, {f^{22}}_{22}\right)$ be the optimal solutions of the leader and the follower for BLQPP-II over the region S.

Then, the fuzzy goals appears as :-

$f_{11} \ge {f^{11}}_{11}$ *and* $f_{21} \ge {f^{21}}_{21}$ for BLQPP-I and $f_{12} \ge {f^{12}}_{12}$ *and* $f_{22} \ge {f^{22}}_{22}$ for BLQPP-II

Now, it can be assumed that

${f^{21}}_{11}\left(= f_{11}({x^{21}}_1, {x^{21}}_2\ )\right)$, ${f^{11}}_{21}\left(= f_{21}({x^{11}}_1, {x^{11}}_2\ )\right)$ be the tolerance limits of respective fuzzy objective goals for BLQPP-I.

Similarly, ${f^{22}}_{12}\left(= f_{12}({x^{22}}_1, {x^{22}}_2\ )\right)$, ${f^{12}}_{22}\left(= f_{22}({x^{12}}_1, {x^{12}}_2\ )\right)$ be the tolerence limits of respective fuzzy objective goals for BLQPP-II.

Thus membership functions $\mu_{f_{11}}$ and $\mu_{f_{21}}$ for objective functions of upper level and lower level respectively for BLQPP-I can be defined as following:-

$$\mu_{f_{11}}(f_{11}(x_1,x_2)) = \begin{cases} 1, & if\ f_{11}(x_1,x_2) \geq f_{11}{}^{11} \\ \dfrac{f_{11} - f_{11}^{21}}{f_{11}^{11} - f_{11}{}^{21}}, & if\ \ f_{11}{}^{21} \leq f_{11}(x_1,x_2) \leq f_{11}{}^{11} \\ 0, & if \quad f_{11}(x_1,x_2) \leq f_{11}{}^{21} \end{cases}$$

$$\mu_{f_{21}}(f_{21}(x_1,x_2)) = \begin{cases} 1, & if\ f_{21}(x_1,x_2) \geq f_{21}{}^{21} \\ \dfrac{f_{21} - f_{21}^{11}}{f_{21}^{21} - f_{21}{}^{11}}, & if\ \ f_{21}{}^{11} \leq f_{21}(x_1,x_2) \leq f_{21}{}^{21} \\ 0, & if \quad f_{21}(x_1,x_2) \leq f_{21}{}^{11} \end{cases}$$

Similarly, membership functions $\mu_{f_{12}}$ and $\mu_{f_{22}}$ for objective functions of upper level and lower level respectively for BLQPP-II can be defines as following:-

$$\mu_{f_{12}}(f_{12}(x_1,x_2)) = \begin{cases} 1, & if\ f_{12}(x_1,x_2) \geq f_{12}{}^{12} \\ \dfrac{f_{12} - f_{12}^{22}}{f_{12}^{12} - f_{12}{}^{22}}, & if\ \ f_{12}{}^{22} \leq f_{12}(x_1,x_2) \leq f_{12}{}^{12} \\ 0, & if \quad f_{12}(x_1,x_2) \leq f_{12}{}^{22} \end{cases}$$

$$\mu_{f_{22}}(f_{22}(x_1,x_2)) = \begin{cases} 1, & if\ f_{22}(x_1,x_2) \geq f_{22}{}^{22} \\ \dfrac{f_{22} - f_{22}^{12}}{f_{22}^{22} - f_{22}{}^{12}}, & if\ \ f_{22}{}^{12} \leq f_{22}(x_1,x_2) \leq f_{22}{}^{22} \\ 0, & if \quad f_{22}(x_1,x_2) \leq f_{22}{}^{12} \end{cases}$$

4. FORMULATION OF FUZZY GOAL PROGRAMMING PROBLEM APPROACH TO THE BQQFPP

In the Fuzzy Goal Programming problem formulation, the defined membership functions are converted into the membership goals with assigning the highest membership value as the aspiration level to each of them .The fuzzy goal programming approach to BQQFPP can be formulate through quadratic goal programming model for BLQPP-I , can be represented as the following:-

Find X so as to

$$Min\ Z = W^-{}_{11}\ d^-{}_{11} + W^-{}_{21}\ \ d^-{}_{21}$$

and satisfy

$$\frac{f_{11}-f_{11}^{21}}{f_{11}^{11}-f_{11}{}^{21}} + d^-{}_{11} - \ d^+{}_{11} = 1$$

$$\frac{f_{21} - f_{21}^{11}}{f_{21}^{21} - f_{21}{}^{11}} + d^-{}_{21} - \ d^+{}_{21} = 1$$

$x^{L_1}{}_1 \leq x_1 \leq x^{U_1}{}_1$ and $x^{L_2}{}_2 \leq x_2 \leq x^{U_2}{}_2$

$d^-{}_{i1}\ , d^+{}_{i1} \geq 0$ with $d^-{}_{i1}\ , d^+{}_{i1} = 0,\ i = 1{,}2$. where $d^-{}_{i1}\ and\ d^+{}_{i1}$ represents the under and over deviational variables and $W^-{}_{11} = \frac{1}{f_{11}^{11}-f_{11}{}^{21}}$, $W^-{}_{21} = \frac{1}{f_{21}^{21}-f_{21}{}^{11}}$

Similarly, The fuzzy goal programming approach BQQFPP can be formulate through quadratic goal programming model for BLQPP-II, can be represented as the following:-

Find X' so as to

$$Min\ Z' = W^-{}_{12}\ d^-{}_{12} + W^-{}_{22}\ \ d^-{}_{22}$$

and satisfy

$$\frac{f_{12}-f_{12}^{22}}{f_{12}^{12}-f_{12}{}^{22}} + d^-{}_{12} - \ d^+{}_{12} = 1$$

$$\frac{f_{22} - f_{22}^{12}}{f_{22}^{22} - f_{22}{}^{12}} + d^-{}_{22} - \ d^+{}_{22} = 1$$

$x^{L_3}{}_1 \leq x_1 \leq x^{U_3}{}_1$ and $x^{L_4}{}_2 \leq x_2 \leq x^{U_4}{}_2$

$d^-{}_{i2}\ , d^+{}_{i2} \geq 0$ with $d^-{}_{i2}\ , d^+{}_{i2} = 0,\ i = 1{,}2$. where $d^-{}_{i1}\ and\ d^+{}_{i1}$ represents the upper and over deviational variables and $W^-{}_{12} = \frac{1}{f_{12}^{12}-f_{12}{}^{22}}$, $W^-{}_{22} = \frac{1}{f_{22}^{22}-f_{22}{}^{12}}$

5. LINEAR APPROXIMATION PROCEDURE TO QADRATIC FUZZY GOAL PROGRAMMING

The linearization techniques for general fuzzy programming problems with non linear membership function have been studied by [7], [9], [35]. Here the concept of non linear goal programming approach of [27] is further extended and the solving procedure for BQQFPP is presented. The approximate solution $x_p{}^0(x_1{}^{0p}, x_2{}^{0p})$ is determined by initial inspection, where the solution lies in the tolerance ranges specified for decisions $x_1$and x_2. Then, for getting satisfactory decisions in the neighbourhood of $x_p{}^0(x_1{}^{0p}, x_2{}^{0p})$. The linear approximate techniques for goals can be used as

$$G_{ip}: \mu_{ip}(x_p{}^0) + [\nabla\mu_{ip}(x_p{}^0)]^T(V_p - W_p) + d^-{}_{ip} - d^+{}_{ip} = I, \quad where\ i,p = 1{,}2$$

where $\nabla\mu_{ip}(x_p{}^0)$ is the gradient of $\mu_{ip}(x_p{}^0)$

Now, the use of linear approximation to $\mu_{ip}(x_p{}^0)$ in the neighborhood of $(x_{jp}{}^0)$, it can be taken as $x_j{}^p$ must not take any value lower than the corresponding lower tolerance limit $x_j{}^{Lp}$. Thus tolerance distance that $x_j{}^p$ may move can be taken as $t_j^p = x_{jp}^0 - x_j^p$ and tolerance range can be calculated as $-t_j^p \le x_{jp}^0 - x_j^p \le t_j^p$

also $-t_j^p \le v_j^p - w^p \le t_j^p$ and $v_j^p, w^p \le t_j^p$ for $j = 1,2,3, \ldots\ldots, n$

and maximum value of $x^p{}_j$ can be taken as $t_j^p = x_{jp}^0 - x_j^p$

consequently, the resultant upper-bound restrictions are obtained as

$0 \le v_j^p \le \min\,(x_j^{U_p} - x_{jp}^0, t_j^p)$ and $0 \le w_j^p \le \min\,(x_{jp}^0, t_j^p)$

and $a_j = \min\,(x_j^{U_p} - x_{jp}^0, t_j^p)$ and $b_j = \min\,(x_{jp}^0, t_j^p)$

5.1. FORMULATION OF THE LINEAR FUZZY GOAL PROGRAMMING PROBLEM

The achievements of the goals depends on the vales of v_j and w_j with their upper-bound restrictions and also these can be optimized by minimizing their over deviational variables.

Thus the linear fuzzy goal programming model for BLQPP-I can be described as:-

$$Min\ Z_1 = \left[P_1\left(\sum_{j=i}^{n} p_{j1}^{\ +} + q_{j1}^{\ +}\right), P_2(W^-{}_{11}\ \ d^-{}_{11} + W^-{}_{21}\ \ d^-{}_{21})\right]$$

and satisfy

$$v_{j1} + p_{j1}^{\ -} - p_{j1}^{\ +} = a_{j1}$$

$$w_{j1} + q_{j1}^{\ -} - q_{j1}^{\ +} = b_{j1}$$

$$\mu_{i1}(x_1^{\ 0}) + [\nabla\mu_{i1}(x_1^{\ 0})]^T(V_1 - W_1) + d^-{}_{i1} - d^+{}_{i1} = I;\quad for\ i = 1,2$$

$d^-{}_{i1}\ , d^+{}_{i1} \ge 0$ with $d^-{}_{i1}\ , d^+{}_{i1} = 0,\ i = 1,2.$

$p_{j1}^{\ -},\ p_{j1}^{\ +},\ q_{j1}^{\ -},\ q_{j1}^{\ +} \ge 0$ with $q_{j1}^{\ -}.\ q_{j1}^{\ +} = 0$, $p_{j1}^{\ -}.\ p_{j1}^{\ +} = 0$

where $\left(p_{j1}^{\ -},\ p_{j1}^{\ +}\right)$ and $\left(q_{j1}^{\ -}.\ q_{j1}^{\ +}\right)$ represents the under and overdeviational variables of respective goal.

Consider that its solution is $x_1^{\ N}, x_2^{\ N}, f_{11}^{\ N}, f_{21}^{\ N}, f_{12}^{\ N}, f_{22}^{\ N}$

Thus the linear fuzzy goal programming model for BLQPP-II can be described as:-

$$Min\ Z_2 = \left[P_1\left(\sum_{j=1}^{n} p_{j2}^{\ +} + q_{j2}^{\ +}\right), P_2(W^-{}_{12}\ \ d^-{}_{12} + W^-{}_{22}\ \ d^-{}_{22})\right]$$

and satisfy

$$v_{j2} + p_{j2}^{\ -} - p_{j2}^{\ +} = a_{j2}$$

$$w_{j2} + q_{j2}^{\ -} - q_{j2}^{\ +} = b_{j2}$$

$$\mu_{i2}(x_2^{\ 0}) + [\nabla\mu_{i2}(x_2^{\ 0})]^T(V_2 - W_2) + d^-{}_{i2} - d^+{}_{i2} = I;\quad for\ i = 1,2$$

$d^-{}_{i2}\ , d^+{}_{i2} \ge 0$ with $d^-{}_{i2}\ , d^+{}_{i2} = 0,\ i = 1,2.$

$p_{j2}^{\ -},\ p_{j2}^{\ +},\ q_{j2}^{\ -},\ q_{j2}^{\ +} \ge 0$ with $q_{j2}^{\ -}.\ q_{j2}^{\ +} = 0$, $p_{j2}^{\ -}.\ p_{j2}^{\ +} = 0$

where $\left(p_{j2}^{\ -},\ p_{j2}^{\ +}\right)$ and $\left(q_{j2}^{\ -}.\ q_{j2}^{\ +}\right)$ represents the under and overdeviational variables of respective goal.

Consider that its solution is $x_1^{\ D}, x_2^{\ D}, f_{11}^{\ D}, f_{21}^{\ D}, f_{12}^{\ D}, f_{22}^{\ D}$

6. SOLUTION PROCEDURE OF BQQFPP

Now, we can take the best solution values of respective objective function for above models (i.e. from $f_{11}^{\ N}, f_{21}^{\ N}, f_{12}^{\ N}, f_{22}^{\ N}$ and $f_{11}^{\ D}, f_{21}^{\ D}, f_{12}^{\ D}, f_{22}^{\ D}$) as the new fuzzy objective goals and their worst values of as the lower tolerance limits. Then, we transform quadratic membership functions into linear membership functions by using the Taylor's polynomial theorem at the points $(x_1^{\ N}, x_2^{\ N})$ and $(x_1^{\ D}, x_2^{\ D})$ which can be defined as the following:-

$$\mu'_{f_{11}} = \mu_{f_{11}}(x_1^{\ N}, x_2^{\ N}) + (x_1 - x_1^{\ N})\,\frac{\partial\mu_{f_{11}}}{\partial x_1}[(x_1^{\ N}, x_2^{\ N})] + (x_2 - x_2^{\ N})\,\frac{\partial\mu_{f_{11}}}{\partial x_2}[(x_1^{\ N}, x_2^{\ N})]$$

$$\mu'_{f_{21}} = \mu_{f_{21}}(x_1^{\ N}, x_2^{\ N}) + (x_1 - x_1^{\ N})\,\frac{\partial\mu_{f_{21}}}{\partial x_1}[(x_1^{\ N}, x_2^{\ N})] + (x_2 - x_2^{\ N})\,\frac{\partial\mu_{f_{21}}}{\partial x_2}[(x_1^{\ N}, x_2^{\ N})]$$

$$\mu'_{f_{12}} = \mu_{f_{12}}(x_1^{\ D}, x_2^{\ D}) + (x_1 - x_1^{\ D})\,\frac{\partial\mu_{f_{12}}}{\partial x_1}[(x_1^{\ D}, x_2^{\ D})] + (x_2 - x_2^{\ D})\,\frac{\partial\mu_{f_{12}}}{\partial x_2}[(x_1^{\ D}, x_2^{\ D})]$$

$$\mu'_{f_{22}} = \mu_{f_{22}}(x_1{}^D, x_2{}^D) + (x_1 - x_1{}^D)\frac{\partial \mu_{f_{22}}}{\partial x_1}[(x_1{}^D, x_2{}^D)] + (x_2 - x_2{}^D)\frac{\partial \mu_{f_{22}}}{\partial x_2}[(x_1{}^D, x_2{}^D)]$$

Hence the BQQFPP can be formulated as Linear Programming Problem can be defined as

$$Max\ Z^* = \frac{1}{4}\left[\mu'_{f_{11}} + \mu'_{f_{21}} + \mu'_{f_{12}} + \mu'_{f_{22}}\right]$$

subject to

$$A_1x_1 + A_2x_2 \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} b$$

$$x_1, x_2 \ge 0$$

Thus, we get the required solution of BQQFPP.

7. NUMERICAL EXAMPLE

The following example is illustrated to show the proposed methodology for BQQFPP. The following BQQFPP is formulated as

$[1^{st}\ Level]$

$$Max_{x_1} f_1 = \frac{f_{11}}{f_{12}} = \frac{4x_1{}^2 + x_2{}^2 + 1}{2x_1{}^2 + 5x_2{}^2 + 1}$$

$[2^{nd}\ Level]$

$$Max_{x_2} f_2 = \frac{f_{21}}{f_{22}} = \frac{3x_1{}^2 + 5x_2{}^2 + 1}{4x_1{}^2 + 3x_2{}^2 + 2}$$

$$-5x_1 + 3x_2 \le 15, \quad 4x_1 + 3x_2 \le 45, \quad x_1, x_2 \ge 0$$

Firstly, we are separating above problem into two Bilevel quadratic programming problems as following:-

<u>BQPP-I</u>

$[1^{st}\ Level]$

$$Max_{x_1} f_{11} = 4x_1{}^2 + x_2{}^2 + 1$$

$[2^{nd}\ Level]$

$$Max_{x_2} f_{21} = 3x_1{}^2 + 5x_2{}^2 + 1$$

$$-5x_1 + 3x_2 \le 15, \quad 4x_1 + 3x_2 \le 45, \quad x_1, x_2 \ge 0$$

<u>BQPP-II</u>

$[1^{st}\ Level]$

$$Max_{x_1} f_{12} = -(2x_1{}^2 + 5x_2{}^2 + 1)$$

$[2^{nd}\ Level]$

$$Max_{x_2} f_{22} = -(4x_1{}^2 + 3x_2{}^2 + 2)$$

$$-5x_1 + 3x_2 \le 15, \quad 4x_1 + 3x_2 \le 45, \quad x_1, x_2 \ge 0$$

BQPP-I

The individual optimal solutions of leader and follower are $(x_1{}^1, x_2{}^1) = (11.25,0)$ with $f_{11}{}^1 = 507.25$ and $(x_1{}^2, x_2{}^2) = (3.3,10.5)$ with $f_{21}{}^1 = 591.43$. Then, fuzzy objective goals appear as $f_{11} \ge 507.25, f_{21} \ge 591.43$. The lower tolerance limits of goals are defined as $f_{11}{}^2 = 1, f_{21}{}^1 \ge 1$ and the lower tolerance limits of decision variables is $3.3 \le x_1 \le 11.25, \quad 0 \le x_2 \le 10.5$

Fuzzy Quadratic Goal Programming model for this problem is:

$$Min\ Z_1 = 0.001975d^-{}_1 + 0.00169d^-{}_2$$

$$\frac{4x_1{}^2 + x_2{}^2}{506.25} + d^-{}_1 - d^+{}_1 = 1$$

$$\frac{3x_1{}^2 + 5x_2{}^2}{590.43} + d^-{}_2 - d^+{}_2 = 1$$

$$3.3 \le x_1 \le 11.25$$

$$0 \le x_2 \le 10.5$$

To approximate the goals in above model, consider the initial solution as $(x_1{}^0, x_2{}^0) = (4,2)$. Then, resulting goal programming model is as following

$$Min\ Z_2 = [(P_1\{d^+{}_3 + d^+{}_4 + d^+{}_5 + d^+{}_6\}), (P_2\{0.001975d^-{}_1 + 0.00169d^-{}_2\})]$$

$$0.06321(v_1 - w_1) + 0.0079(v_2 - w_2) + d^-{}_1 - d^+{}_1 = 0.8657$$

$$0.04065(v_1 - w_1) + 0.03387(v_2 - w_2) + d^-{}_2 - d^+{}_2 = 0.8848$$

$$v_1 + d^-{}_3 - d^+{}_3 = 0.7$$

$$v_2 + d^-{}_4 - d^+{}_4 = 2.0$$

$$w_1 + d^-{}_5 - d^+{}_5 = 0.7$$

$$w_2 + d^-{}_6 - d^+{}_6 = 2.0$$

Software LINGO 15 is used to solve the above mentioned model and solution is obtained as :

$$v_1 = 0.7, \quad v_2 = 2.0, \quad w_1 = 0, \quad w_2 = 0$$

Thus, solution to original problem is $x_1 = 4.7, \quad x_2 = 4$ and

$f_{11} = 105.36, \quad f_{12} = -125.18, \quad f_{21} = 147.27, \quad f_{22} = -138.36$

BQPP-II

The individual optimal solutions of leader and follower are $(x_1{}^1, x_2{}^1) = (0.000035, 0.000022)$ with $f_{12}{}^1 = -1$ and $(x_1{}^2, x_2{}^2) = (0.000024, 0.000027)$ with $f_{22}{}^1 = -2$. Then, fuzzy objective goals appear as $f_{12} \geq -1, f_{22} \geq -2$. The lower tolerance limits of goals are defined as $f_{12}{}^2 = -1, f_{22}{}^1 \geq -3$ and the lower tolerance limits of the decision variables is

$0.000024 \leq x_1 \leq 0.000035, \quad 0.000022 \leq x_2 \leq 0.000027$

Fuzzy Quadratic Goal Programming model for this problem is:

$Min\ Z_1 = 1d^-{}_1 + 1d^-{}_2$

$-2x_1{}^2 - 5x_2{}^2 + 1 + d^-{}_1 - d^+{}_1 = 1$

$-4x_1{}^2 - 3x_2{}^2 + 1 + d^-{}_2 - d^+{}_2 = 1$

$0.000024 \leq x_1 \leq 0.000035 \ , \quad 0.000022 \leq x_2 \leq 0.000027$

To approximate the goals in above model, consider the initial solution as $(x_1{}^0, x_2{}^0) = (0.000030, 0.000025)$. Then, resulting goal programming model is as following

$$Min\ Z_2 = [(P_1\{d^+{}_3 + d^+{}_4 + d^+{}_5 + d^+{}_6\}), (P_2\{1d^-{}_1 + 1d^-{}_2\})]$$

$$-0.00012(v_1 - w_1) - 0.00025(v_2 - w_2) + d^-{}_1 - d^+{}_1 = 0$$

$$-0.00024(v_1 - w_1) - 0.00015(v_2 - w_2) + d^-{}_2 - d^+{}_2 = 0$$

$$v_1 + d^-{}_3 - d^+{}_3 = 0.000005, \quad v_2 + d^-{}_4 - d^+{}_4 = 0.000002$$

$$w_1 + d^-{}_5 - d^+{}_5 = 0.000006, \quad w_2 + d^-{}_6 - d^+{}_6 = 0.000003$$

Software LINGO 15 is used to solve the above mentioned model and solution is obtained as :

$v_1 = 0, \quad v_2 = 0, \quad w_1 = 0.000006, \quad w_2 = 0.000003$

Thus, solution to original problem is $x_1 = 0.000024, \ x_2 = 0.000022$ and

$f_{11} = 1.0000000278, \quad f_{12} = -1.000000035, \quad f_{21} = 1.000000004, \quad f_{22} = -2.0000003$

Solution procedure for BQQFPP:-

Now we can consider the fuzzy objective goals appear as $f_{11} \geq 105.36, \ f_{21} \geq 147.27, \ f_{12} \geq -1.000000035, \ f_{22} \geq -2.0000003$. The lower tolerance limits of goals are defined as

$f_{11}{}^2 = 1, f_{21}{}^1 \geq 1 , \ f_{12}{}^2 = -125.18, \ , \ f_{22}{}^1 = -138.36$

Thus, membership functions about the neighborhood of the points (4.7,4) and $(0.000024, 0.000022)$ are

$$\mu_{f_{11}}{}' = 0.3603x_1 + 0.0767x_2 - 1, \quad \mu_{f_{21}}{}' = 0.1928x_1 + 0.2737x_2 - 1$$

$$\mu_{f_{12}}{}' = 0.0000007x_1 + 0.0000018x_2 + 1, \quad \mu_{f_{22}}{}' = -(3.3E-9)x_1 - (2.1E-9)x_2 + 1$$

Hence, the LPP model is formulated as

$$Max_{x_2} Z_3 = \frac{1}{4}(0.5531x_1 + 0.3504x_2) = 0.138275x_1 + 0.0876x_2$$

$$-5x_1 + 3x_2 \leq 15, \quad 4x_1 + 3x_2 \leq 45, \quad x_1, x_2 \geq 0$$

Software LINGO 15 is used to solve the above mentioned model and solution is obtained as:

$x_1 = 3.33, \quad x_2 = 10.56, \quad f_1 = 0.27, \quad f_2 = 1.56,$

8. CONCLUSION

We solve the BQQFPP by converting it into two BQPPs with goal approximation procedure and this study can be extended to solve nonlinear multilevel programming problems and it is hoped that the approach presented here can contribute to future study of hierarchical decision-making problems.

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