ON IDEALS OF A CLASS OF SEMILATTICE ORDERED SEMIRINGS

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ABSTRACT. The class of lattice ordered semirings studied by P. Ranga Rao[5] in 1981 is a common abstraction of lattice ordered rings and Boolean rings. In 2007, we introduced the notion of a semilattice ordered semirings (sl-semirings)[3] as a generalization of lattice ordered semirings. In this paper, we introduce the notions of nilpotent, prime and irreducible ideals to a class of semilattice ordered semirings and obtain the characteristics of them.

1. INTRODUCTION

To obtain a common abstraction of Boolean algebras(rings) and lattice ordered groups, K. L. N. Swamy[7] in 1965, introduced the notion of a dually residuated lattice ordered semigroup (DRl-semigroup) and obtained many common properties of Boolean algebras and l-groups. Later P. Ranga Rao[5] in 1981, introduced the notion of a lattice ordered semiring considering DRl-semigroup with a binary multiplication and observed that this class provide a common abstraction of lattice ordered rings and Boolean rings. In 2007, we introduced the notion of a semilattice ordered semirings (sl-semirings)[3] as a generalization of lattice ordered semiring and obtained the ideal theory for a class of semilattice ordered semirings. In this paper, we introduce the notions of nilpotent and prime ideals to the class of semilattice ordered semirings and obtain characteristics of them. Also, we define an irreducible ideal to this class and observe that the set $\{S(I) \mid I \text{ is an ideal of } A\}$ is a topology on $spec(A)$, where $spec(A)$ is the set of all irreducible ideals and $S(I) = \{P \in spec(A) \mid I \nsubseteq P\}$.

2. PRELIMINARIES

In this section we collect important definitions and examples from the literature for our use in the next sections.

Definition 2.1. [7] A system $A = (A, +, \leq, -)$ is called a *Dually residuated lattice ordered semigroup* (or simply a *DRl-semigroup*) if and only if

$(D1)$ $A = (A, +, \leq)$ is a lattice ordered semigroup with identity 0. i.e., $(A, +)$ is a semigroup with identity 0 and $(A, \leq)$ is a lattice (where the lattice operations are denoted by $\vee, \wedge$) such that $x + (a \vee b) + y = (x + a + y) \vee (x + b + y)$ and $x + (a \wedge b) + y = (x + a + y) \wedge (x + b + y)$ for all x, y, a, b in A,

$(D2)$ to each a, b in A $\exists$ a least x in A such that $x + b \geq a$ and this x is denoted by $a - b$,

$(D3)$ $(a - b) \vee 0 + b \leq a \vee b$ for all a, b in A,

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$(D4)$ $a - a \geq 0$.
If $(A, +)$ is a commutative semigroup in the DRl-semigroup $A = (A, +, \leq, -)$ then A is called a *commutative DRl-semigroup*.

Definition 2.2. [4] A *semi Brouwerian algebra* is a system $A = (A, \leq, -, 0)$ where
$(B1)$ $(A, \leq)$ is a join semilattice with 0 as least,
$(B2)$ $a - b \leq c$ if and only if $a \leq b + c$ for all $a,\ b,\ c \in A$.

Definition 2.3. [5] A system $A = (A, +, \cdot, \leq, -)$ is called a *lattice ordered semiring* (or *l.o.semiring* or *l-semiring*) if and only if
$(L1)$ $A = (A, +, \leq, -)$ is a commutative DRl-semigroup,
$(L2)$ $(A, \cdot)$ is a semigroup,
$(L3)$ $a(b + c) = ab + ac$, $(a + b)c = ac + bc$,
$(L4)$ $a(b - c) = ab - ac$, $(a - b)c = ac - bc$,
$(L5)$ $a \geq 0$, $b \geq 0$ implies $ab \geq 0$ for all a, b, c in A.
If $(A, \cdot)$ is a commutative semigroup in the l-semiring $A = (A, +, \cdot, \leq, -)$ then A is called a *commutative l-semiring*.

Definition 2.4. [3] A system $A = (A, +, \leq, \cdot, -)$ is said to be a *semilattice ordered semiring* (in short *sl-semiring*) if and only if it satisfies the following:
$(S1)$ $(A, +)$ is a commutative semigroup with 0,
$(S2)$ $(A, \leq)$ is a join semilattice such that $a + (b \vee c) = (a + b) \vee (a + c)$,
$(S3)$ for $a, b \in A$ $\exists$ a least $x \in A \ni a \leq b + x$ and this x is denoted by $a - b$,
$(S4)$ $(a - b) \vee 0 + b \leq a \vee b$,
$(S5)$ $a - a \geq 0$,
$(S6)$ $(A, \cdot)$ is a semigroup,
$(S7)$ $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$,
$(S8)$ $a(b - c) = ab - ac$ and $(a - b)c = ac - bc$,
$(S9)$ if $a \geq 0$, $b \geq 0$ then $ab \geq 0$ for all a, b, c in A.

Every semi Brouwerian algebra can be made into an sl-semiring with 0 least and thus Boolean algebras, Brouwerian algebras are sl-semirings with 0 least.

The following is an example of sl-semiring with 0 which is not a lattice ordered semiring.

Example 2.5. [3] *Let* $A = N \bigcup \{a, b, c\}$ *where* N *is the set of all nonnegative integers and* $a,\ b,\ c$ *are elements which are not in* N. *Define* $\leq$ *in* A *as follows:*
For the elements in N, *let* $\leq$ *be the usual ordering and* $a > x$, $b > x$ *for all* x *in* N *and* $a < c$, $b < c$.
Define $+$ *on* A *as* $x + y = x \vee y$ *for every* $x, y \in A$.
Define $\cdot$ *on* A *as* $x \cdot y = 0$ *for every* $x, y \in A$.
Then $-$ *is defined as follows:*
For any $x, y \in N$, $x - y$ *is usual subtraction if* $y \leq x$, $x - y = 0$ *for* $x < y$, *and* $x - a = x - b = x - c = a - c = b - c = 0$, $a - x = a - b = c - b = a$, $b - x = b - a = c - a = b$ *and* $c - x = c$.
Then $A = (A, +, \cdot, \leq, -)$ *is an sl-semiring with 0 least. But it is not l-semiring, since* $a \wedge b$ *does not exist.*

Following is an example of sl-semiring with 0 least but not a semi Brouwerian algebra.

Example 2.6. [3] *Let* $A = \{0, a, b, 1\}$. *Define* $\leq$ *as* $0 < a < b < 1$ *and define the operations* $-$, $+$ *on* A *by the following tables:*

-	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	1	1	0

+	0	a	b	1
0	0	a	b	1
a	a	b	b	1
b	b	b	b	1
1	1	1	1	1

And define $\cdot$ *as* $x \cdot y = 0$ *for all* $x, y \in A$. *Then* $A = (A, +, \leq, \cdot, -)$ *is an sl-semiring with 0 least, but it is not a semi Brouwerian algebra since* $a + a = b \neq a$.

Definition 2.7. [3] A nonempty subset I of A is called an *ideal* if and only if the following conditions hold:
$(I1)$ $a \in I, b \in I$ implies $a + b \in I$,
$(I2)$ $a \in I, b \in A, b \leq a$ implies $b \in I$,
$(I3)$ $a \in I, b \in A$ implies $ab \in I$, $ba \in I$.

Definition 2.8. [3] The smallest ideal of A containing S is called the *ideal generated by* S and it is denoted by $< S >$. In particular if $S = \{a\}$, then we write $< a >$ for $< S >$ and we call $< a >$ as the principal ideal generated by a.

Theorem 2.9. [3] *For any* a *in* A, $< a >= \{t \in A \mid t \leq ma + xa + ay + x_1ay_1$ *for some nonnegative integer* m *and for some* $x,\ y,\ x_1,\ y_1 \in A\}$.

Theorem 2.10. [3] *If* I *and* J *are ideals of* A *then* $\{a \in A \mid a \leq x + y$, *for some* $x \in I, y \in J\}$ *is the join of* I *and* J.

Definition 2.11. [3] Let I and J be ideals of A. Then we define IJ to be the ideal generated by the set $\{ij \mid i \in I, j \in J\}$. i.e., $IJ =< \{ij \mid i \in I, j \in J\} >$.

Theorem 2.12. [3] $IJ = \{x \in A \mid x \leq ij$ *for some* $i \in I, j \in J\}$.

Remark 2.13. [3] If I and J are any two ideals of A then $IJ \subseteq I \bigcap J$.

Remark 2.14. [3] If I is an ideal of A then $I^n = \{x \in A \mid x \leq i^n$ for some $i \in I\}$.

3. Nilpotent & Prime Ideals

Through out this paper A stands for an sl-semiring with 0 least.

Definition 3.1. An ideal I of A is called *nilpotent* if there exists a positive integer n such that $I^n = \{0\}$.

Definition 3.2. An element a of A is called *nilpotent* if there is a positive integer n such that $a^n = 0$.

Remark 3.3. Let I be an ideal of A and n be a positive integer. Then $I^n = \{0\}$ if and only if $a^n = 0$ for all a in A.

Proof. Suppose $I^n = \{0\}$. Let $a \in I$. Then $a^n \in I^n$ and hence $a^n = 0$ for every $a \in I$.

Conversely suppose that $a^n = 0$ for all $a \in I$. Let $b \in I^n$. Then $\exists\, c \in I \ni b \leq c^n$. $\Rightarrow b \leq 0$. $\Rightarrow b = 0$. Hence $I^n = \{0\}$. □

Definition 3.4. A proper ideal P of A is called *prime* if and only if $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for any ideals I, J of A.

Remark 3.5. If P is a prime ideal of A then P contains every nilpotent ideal of A.

Proof. Let I be a nilpotent ideal of A. Then $\exists$ a positive integer $n \ni I^n = \{0\}$. $\Rightarrow$ $I^n \subseteq P$. Since P is a prime ideal of A, $I \subseteq P$. Hence the remark. □

Definition 3.6. An element a of A is called *strongly nilpotent* if and only if $\exists$ a positive integer $n \ni x_0ax_1ax_2.....x_{n-1}ax_n = 0$ for every $x_0, x_1, ..., x_n \in A$.

Theorem 3.7. *Every strongly nilpotent element of A is nilpotent. If A is a commutative sl-semiring with 0 least then an element a is strongly nilpotent if and only if a is nilpotent.*

Proof. Let a be a strongly nilpotent element of A. Then there exists a positive integer n such that $x_0ax_1ax_2.....x_{n-1}ax_n = 0$ for every $x_0, x_1, ..., x_n \in A$. $\Rightarrow$ $aaa......aa = 0$. $\Rightarrow a^{2n+1} = 0$ for some $2n + 1 \in N$. Hence a is nilpotent.

Suppose A is a commutative sl-semiring with 0 least and suppose that a is nilpotent. Then $\exists$ a positive integer $n \ni a^n = 0$. Let $x_0, x_1, ..., x_n \in A$. Consider $x_0ax_1ax_2.....x_{n-1}ax_n = x_0x_1....x_na^n = 0$. Hence a is strongly nilpotent. □

Theorem 3.8. *An element a of A is strongly nilpotent if and only if it is contained in a nilpotent ideal of A.*

Proof. Let I be a nilpotent ideal containing a. Then $I^n = \{0\}$ for some positive integer n and $a \in I$. Let $x_0, x_1, ..., x_n \in A$. Consider $x_0ax_1ax_2.....x_{n-1}ax_n \in I^n = \{0\}$. $\Rightarrow x_0ax_1ax_2.....x_{n-1}ax_n = 0$. Hence a is strongly nilpotent.

Conversely suppose that a is strongly nilpotent. Then there exists a positive integer n such that $x_0ax_1ax_2.....x_{n-1}ax_n = 0$ for every $x_0, x_1, ..., x_n \in A$. Take $I =< a >$. Since $a \in< a >$, $a \in I$. Now we prove I is a nilpotent ideal of A: Let $x \in I^{2n+1}$. Then $x \leq (t)^{2n+1}$ for some $t \in I$. $\Rightarrow x \leq (t)^{2n+1}$ where $t \leq ma + sa + ar + s_1ar_1$ for some nonnegative integer m, for some $s, r, s_1, r_1 \in A$. Since a is strongly nilpotent, we have $[Aa]^{n+1} = 0 = [aA]^{n+1} = [AaA]^n$ (Here $[Aa]^{n+1} = x_0ax_1ax_2.....x_{n-1}ax_n$). Now $x \leq t^{2n+1} \leq [ma + sa + ar + s_1ar_1]^{2n+1} = \sum(t_1t_2....t_{2n+1} \mid t_i \in \{ma, sa, ar, s_1ar_1\}) = 0$ and thus $x = 0$. Hence $I^{2n+1} = \{0\}$. Therefore every strongly nilpotent element is contained in a nilpotent ideal. □

4. Irreducible Ideals

In sl-semirings with 0 least we introduce the notion of an irreducible ideal as follows:

Definition 4.1. A proper ideal P of A is said to be an *irreducible ideal* if it is satisfies the condition: $I \bigcap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$ for any ideals I, J of A.

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Theorem 4.2. *For an ideal P in A, the following are equivalent:*
(1) P is an irreducible ideal of A,
(2) for any $a,\ b \in A$, $< a > \bigcap < b > \subseteq P$ implies $a \in P$ or $b \in P$.

Proof. (1) $\Rightarrow$ (2) : Suppose P is an irreducible ideal of A. Let $a,\ b \in A$ be such that $< a > \bigcap < b > \subseteq P$. Since $< a >$, $< b >$ are ideals of A and P is irreducible, $< a > \subseteq P$ or $< b > \subseteq P$. Hence $a \in P$ or $b \in P$.

(2) $\Rightarrow$ (1) : Suppose $< a > \bigcap < b > \subseteq P$ implies $a \in P$ or $b \in P$ for any a, b in A. Let I, J be ideals of A such that $I \bigcap J \subseteq P$. Suppose if $I \not\subseteq P$ and $J \not\subseteq P$. Then $\exists\ a \in I \setminus P$ and $b \in J \setminus P$. $\Rightarrow < a > \subseteq I$ and $< b > \subseteq J$. $\Rightarrow < a > \bigcap < b > \subseteq I \bigcap J \subseteq P$. $\Rightarrow < a > \bigcap < b > \subseteq P$. $\Rightarrow a \in P$ or $b \in P$, a contradiction. Hence $I \subseteq P$ or $J \subseteq P$. Hence the theorem. □

Remark 4.3. Any prime ideal of A is irreducible.

Proof. Let P be a prime ideal of A. Let I, J be ideals of A such that $I \bigcap J \subseteq P$. Since $IJ \subseteq I \bigcap J$, $IJ \subseteq P$. Since P is prime, $I \subseteq P$ or $J \subseteq P$. Hence the remark. □

Definition 4.4. Let a in A. Then an ideal P of A is said to be *value* of A if P is the maximal element in the family of all ideals of A not containing a.

Definition 4.5. An ideal P of A is called *semi maximal* if it is a value of some element of A.

Theorem 4.6. *Every semi maximal ideal of A is irreducible.*

Proof. Let P be a semi maximal ideal of A. Then $\exists$ an element $a \in A \ni P$ is a maximal element of the set $H = \{Q \mid Q$ is an ideal of $A, a \notin Q\}$. Let I, J be ideals of A such that $I \bigcap J \subseteq P$. Then $a \notin I \bigcap J$. $\Rightarrow a \notin I$ or $a \notin J$. $\Rightarrow I \in H$ or $J \in H$. $\Rightarrow I \subseteq P$ or $J \subseteq P$. Hence P is irreducible. □

We denote the set of all irreducible ideals of A by *spec$A$*. To each subset M of A, write $H(M) = \{P \in \text{spec}A \mid M \subseteq P\}$ and $S(M) = \text{spec}A \setminus H(M) = \{P \in \text{spec}A \mid M \not\subseteq P\}$.

In particular, if $M = \{a\}$, then we write $H(a)$ for $H(M)$ and $S(a)$ for $S(M)$.

Remark 4.7. $S(M) = S(< M >)$ for any subset M of A.

Theorem 4.8. *The set $\{S(I) \mid I$ is an ideal of $A\}$ is a topology on spec$A$. we call this topology as spectral topology.*

Proof. Clearly $S(0) = \emptyset$ and $S(A) = \text{spec}A$.
Now we prove $S(I \bigcap J) = S(I) \bigcap S(J)$ for any ideals I, J of A:
Consider $H(I \bigcap J) = \{P \in \text{spec}A \mid I \bigcap J \subseteq P\} = \{P \in \text{spec}A \mid I \subseteq P \text{ or } J \subseteq P\}$
$= \{P \in \text{spec}A \mid I \subseteq P\} \bigcup \{P \in \text{spec}A \mid J \subseteq P\} = H(I) \bigcup H(J)$.
Consider $S(I \bigcap J) = \text{spec}A \setminus H(I \bigcap J) = \text{spec}A \setminus [H(I) \bigcup H(J)]$
$= (\text{spec}A \setminus H(I)) \bigcap (\text{spec}A \setminus H(I)) = S(I) \bigcap S(J)$.
Now we prove $S(\bigvee I_\alpha) = \bigcup S(I_\alpha)$ for any family $\{I_\alpha\}$ of ideals of A:
Consider $H(\bigvee I_\alpha) = \{P \in \text{spec}A \mid \bigvee I_\alpha \subseteq P\} = \{P \in \text{spec}A \mid \bigcup I_\alpha \subseteq P\}$
(since $\bigvee I_\alpha \subseteq P \Leftrightarrow \bigcup I_\alpha \subseteq P$) $= \bigcap \{P \in \text{spec}A \mid I_\alpha \subseteq P\} = \bigcap H(I_\alpha)$.
Consider $S(\bigvee I_\alpha) = \text{spec}A \setminus H(\bigvee I_\alpha) = \text{spec}A \setminus \bigcap H(I_\alpha) = \bigcup (\text{spec}A \setminus H(I_\alpha))$
$= \bigcup S(I_\alpha)$.
Hence $\{S(I) \mid I$ is an ideal of $A\}$ is a topology on spec$A$. □

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Definition 4.9. An irreducible ideal P of A is called *minimal irreducible ideal* if P does not contain any irreducible ideal properly.

We denote the set of all minimal irreducible ideals of A by ΠA.

Lemma 4.10. *The intersection of any chain of irreducible ideals of A is an irreducible ideal of A.*

Proof. Let $\mathcal{C}$ be a chain of irreducible ideals of A. Take $P = \bigcap \mathcal{C}$. Let I, J be ideals of A such that $I \bigcap J \subseteq P$. Suppose if $I \nsubseteq P$ and $J \nsubseteq P$. Then $\exists\ P_1$, P_2 in $\mathcal{C} \ni I \nsubseteq P_1$ and $J \nsubseteq P_2$. Since $\mathcal{C}$ is a chain, either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. If $P_1 \subseteq P_2$ then $J \nsubseteq P_1$. $\Rightarrow I \bigcap J \nsubseteq P_1$. $\Rightarrow I \bigcap J \nsubseteq P$, a contradiction. If $P_2 \subseteq P_1$, similarly we get a contradiction. Hence $I \subseteq P$ or $J \subseteq P$. Hence P is irreducible. □

Theorem 4.11. *Every irreducible ideal contains a minimal irreducible ideal*

Proof. Let P be an irreducible ideal of A. Let $\mathcal{C}$ be the set of all irreducible ideals of A which are contained in P. Clearly $\mathcal{C}$ is nonempty and a poset under set inclusion. By the above Lemma, every chain in $\mathcal{C}$ has a lower bound in $\mathcal{C}$. Hence by Zorn's lemma, $\mathcal{C}$ has a minimal element say M. Therefore M is a minimal irreducible ideal such that $M \subseteq P$. □

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