Some Properties of Closed Range Operators

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Abstract: For the two bounded adjointable operators T and S with close ranges on Hilbert $\mathcal{A}$-modules, we demonstrate that, TS has closed range if and only if $\ker(T)+ran(S)$ be an orthogonal summand. Also, we conclude that TS has closed range if and only if $\ker(S^*)+ran(T^*)$ be an orthogonal summand. In addition we investigate the equivalence conditions for close range operators.

Keywords: Hilbert $\mathcal{A}$-module, projection, closed range, complemented submodules.

1. Introduction and preliminaries

Hilbert C^*-modules are routinely used as an important tool in various fields, such as the study of locally compact quantum groups and their representations, noncommutative geometry, KK-theory, and the study of completely positive maps between C^*-algebras. As given in [6] a (left) *pre-Hilbert C^*-module* over a (not necessarily unital) C^*-algebra $\mathcal{A}$ is defined as a left $\mathcal{A}$-module E, equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{A}$, which is $\mathcal{A}$-linear in terms of the first variable and has the properties:

$$\langle x, y\rangle = \langle y, x\rangle^*, \quad \langle x, x\rangle \geq 0 \text{ with equality if and only if } x = 0.$$

Here after, we assume that the linear structures of $\mathcal{A}$ and E are compatible. A pre-Hilbert $\mathcal{A}$-module E is called a *Hilbert $\mathcal{A}$-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. If E and F be two Hilbert $\mathcal{A}$-modules then the set of the all ordered pairs of the direct *orthogonal sum of E and F*, which is denoted by $E \oplus F$, is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle(x_1, y_1),(x_2, y_2)\rangle = \langle x_1, x_2\rangle_E + \langle y_1, y_2\rangle_F$. A Hilbert $\mathcal{A}$-submodule of a Hilbert $\mathcal{A}$-module F is a direct orthogonal summand if E a long with its orthogonal complement $E^\perp$ in F gives rise to an $\mathcal{A}$-linear isometric isomorphism of $E \oplus E^\perp$ and F. Some interesting results about orthogonally complemented submodules can be found in [2], [3]. For the basic theory of Hilbert C^*-modules we refer interested readers to the book by E. C. Lance [6] and to the respective chapters in the monographic publications [4], [8], [10].

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For convenience, we consider $\mathcal{A}$ as an arbitrary C^*-algebra (i.e. not necessarily unital). Since we deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by lower case letters. We use the notations Dom(.), Ker(.) and Ran(.) for domain, kernel and range of operators, respectively. Also, $L(X)$ stands for the set of all continuous linear operator from X to itself. If E and F be Hilbert $\mathcal{A}$-modules and W is an orthogonal summand in $E \oplus F$, P_W stands for the orthogonal projection of $E \oplus F$ onto W, and P_E and P_F denote the canonical projections onto the first and second factors of $E \oplus F$.

The set of all $\mathcal{A}$-linear maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that the equality

$$\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E \tag{1.1}$$

holds for any $x \in E,\ y \in F$ is denoted by $B(E, F)$. The operator T^* is the *adjoint operator* of T. The existence of an adjoint operator T^* for any $\mathcal{A}$-linear operator $T : E \to F$ implies that each adjointable operator is necessarily bounded and $\mathcal{A}$-linear in the sense that $T(ax) = aT(x)$ for any $a \in \mathcal{A}$, $x \in E$, and similarly for T^*. In fact, the equality (1.1) is holds for any elements of E and F, so E belongs to the domain of T.

Lemma 1.1. *Suppose X is a Hilbert A -module and $T \in L(X)$ is selfadjoint, then T has closed range if and only if $0 \notin acc\sigma(T)$. Where $acc\sigma(T)$, denote the accumulent point of the spectrum of T.*

2. The product of projections and operators

Theorem 2.1. *Suppose X is a Hilbert $\mathcal{A}$ -module and t , k be self adjoint operators in $L(X)$ with has closed range and k is invertable and $t = kf$ then f has closed range.*

Proof. For this end by lemma 1.1 we show that 0 does not belong to the accumulation spectrum of f. we know that $0 \notin acc\sigma(t)$ then we claim $0 \notin acc\sigma(k)$. if for all $\{\lambda_n\}sp(t)$ be a sequence such that λ_n doesn't tend 0 then, have $k^{-1}t = f$ which implies that $0 \notin acc\sigma(f)$ then f has closed range. □

Lemma 2.2. *Suppose X is a Hilbert A -module and P , Q are orthogonal projections in $L(X)$. Then $P - Q$ has closed range then $P + Q$ has closed range.*

Proof. we know $P - Q$ has closed range and also $P^2 - Q^2$ has closed range hence

$$P^2 - Q^2 = (P - Q)(P + Q)$$

hence $P + Q$ has closed range. for inverse $P + Q$ has closed range, hence $P^2 + Q^2$ has closed range and by $P^2 + Q^2 = (P + iQ)(P - iQ)$ then by assumption $P + iQ$ has closed range then $P + Q$ is also. □

Theorem 2.3. *If E be a Hilbert $\mathcal{A}$-modules and t and k , f be in $L(E)$ such that t and f has closed range and $t = kf$, then $\ker(k)$ is a complemented submodule of E.*

Proof. by assumption t and f has closed range and by Theorem 3.2 of [6] we have $\ker(f)$ and $\ker(t)$ are complemented submodule of E hence $E = \ker(t) \oplus \ker(t)^\perp$ and $E = ran(f) \oplus ran(f)^\perp$.

Now we have $k(ran(f) \oplus ran(f)^{\perp})$ by linearity k we have $k(ran(f)) + k(ran(f)^{\perp})$, it is enough to show $k(ran(f)) \cap k(ran(f)^{\perp}) = \{0\}$ if $z \in k(ran(f)) \cap k(ran(f)^{\perp})$ then there exists $x = f(x') \in ran(f)$ and $y \in ran(f)^{\perp}$ such that $k(y) = k(f(x')) = z$ we know that $\ker(f)$ is a complemented submodule of E and $x' \in E$, if $x' \in \ker(f)$ then $k(0) = z = 0$ other wise if $x' \in \ker(f)^{\perp}$ then $x = f(x') \in f(\ker(f)^{\perp}) \subseteq ran(f)$ also $k(x) - k(y) = k(x - y) = 0$ then implies $x - y \in \ker k$ and also $y \in ran(f)^{\perp} \subseteq E$, again if $y \in \ker(f)$ then $k(0) = z = 0$ other wise if $y \in \ker(f)^{\perp}$ then $f(y) \in f(\ker(f)^{\perp}) \subseteq ran(f)$ and $f(y) \in f(ran(f)^{\perp})$ hence $f(y) \in f(ran(f)^{\perp}) \cap f(\ker(f)^{\perp})$ this argument implies

$$y \in ran(f)^{\perp} \cap \ker(f)^{\perp}$$

on the other hand

$$z = k(y) \in k(ran(f)) \cap k(ran(f)^{\perp})$$

then

$$y \in ran(f) \cap ran(f)^{\perp}$$

then by completeness $ran(f)$ then $y = 0$ and $k(ran(f)) \cap k(ran(f)^{\perp}) = \{0\}$ only remain that $k(ran(f)) \oplus k(ran(f)^{\perp}) = ran(ranf) \oplus (ran(ranf))^{\perp}$ □

Remark 2.4. If E, F are two Hilbert $\mathcal{A}$-modules and t and k , f be in $L(E, F)$ with has closed range and $t = kf$, if t and k orthogonal complement of E in F then f orthogonal complement of E in F.

Proposition 2.5. *If X be Hilbert $\mathcal{A}$-modules and suppose that $T \in L(X)$ be orthogonal projection has closed range if and only if $Ker(T)$ is orthogonally complemented in X.*

Proof. with Theorem 3.2 of [6] For the converse, we have $X = \ker T \oplus (\ker T)^{\perp}$ for all $0 \neq x \in Ker(T)^{\perp}$ have $T(x) \neq 0$ hence $\{\| Tx \| : x \in Ker(T)^{\perp}\} > 0$ then remain complete by Lemma 2.4 of [9]. □

Suppose M and N are submodule of a Hilbert $\mathcal{A}$-module E, then $(M + N)^{\perp} = M^{\perp} \cap N^{\perp}$ In particular, if P and Q be projections on Hilbert $\mathcal{A}$-module X. If $PQ = QP$, then PQ is a projection and $\ker PQ = \ker P + \ker Q$ and

$$ranPQ = ranP \cap ranQ = (\ker P)^{\perp} \cap (\ker Q)^{\perp} = (\ker P + \ker Q)^{\perp}$$

hence $E = ranPQ \oplus \ker PQ = (\ker P + \ker Q) \oplus (\ker P + \ker Q)^{\perp}$ and also we have $\ker P = Ran(1 - P)$ hence

$$E = (Ran(1 - P) + \ker Q) \oplus (\ker P + \ker Q)^{\perp}$$

hence $Ran(1 - P) + \ker Q$ is an orthogonal summand by [9] Theorem 2.6 implies that $(1 - P)Q$ has closed range.

Lemma 2.6. *Suppose X is a Hilbert $\mathcal{A}$ -module and P , Q are orthogonal projections in $L(X)$ and $P - Q$ has closed range then*

$$X = (\ker(P + Q) \cup \ker(P - Q)) \oplus (\ker(P - Q)^{\perp} \cup (\ker(P + Q)^{\perp})$$

Proof. Since $P-Q$ has closed range then $\ker(P-Q)$ is a complemented submodule of X, then $\ker(P-Q)\oplus(\ker(P-Q))^{\perp}=X$ hence by lemma 2.2, we have $\ker(P+Q)\oplus(\ker(P+Q))^{\perp}=X$ if $x\in\ker(P+Q)\Rightarrow(P+Q)x=0\Rightarrow Px+Qx=0$ with $\ker(P+Q)=\{x\in X:x\in(\ker P\cap\ker Q)\cup(Px=-Qx)\}$ and $\ker(P-Q)=\{x\in X:x\in(\ker P\cap\ker Q)\cup(Px=Qx)\}$ then

$$\ker(P^2-Q^2)=\{x\in X:x\in(\ker P\cap\ker Q)\cup(Px=\pm Qx)\}$$

hence

$$\ker(P+Q)\cap\ker(P-Q)=\{x\in X:x\in(\ker P\cap\ker Q)\}$$

$$\ker(P^2-Q^2)=\{x\in X:x\in(\ker P\cap\ker Q)\cup(Px=-Qx)\cup(Px=Qx)\}$$

we know that P and Q projection then

$$X=\ker(P^2-Q^2)\oplus(\ker(P^2-Q^2))^{\perp}=(\ker(P+Q)\cup\ker(P-Q))\oplus(\ker(P-Q)^{\perp}\cup(\ker(P+Q)^{\perp})$$

□

Proposition 2.7. *Let P and Q be projections on Hilbert $\mathcal{A}$-module X and $RanP\perp RanQ$ then $P+Q$ is an orthogonal projection*

Proof. we know that $\ker P\oplus RanP=X$ and $\ker Q\oplus RanQ=X$ then for each z in X we can uniquely write $z=x+y$ with x in $\ker P$ and y in $RanP$, hence $Q(\ker P\oplus RanP)$

$$QP(z)=QP(x+y)=Q(Px+Py)=Q(Px)+Q(Py)=Q(0)+Q(Py)=0$$

then $(P+Q)^2=P+Q$ □

3. Closeness of the range of the products

Proposition 3.1. *Suppose E, F, G are Hilbert $\mathcal{A}$-modules and $S\in L(E,F)$ and $T\in L(F,G)$ are bounded adjointable operators and TS has closed ranges and T is an isometric $\mathcal{A}$-linear map with complemented range then S has closed range.*

Proof. Let TS has closed range, so $ranS^*T^*$ has closed range and $ranS^*T^*=ranS^*T^*TS$, from T is an isometric $\mathcal{A}$-linear map with complemented range, we have $T^*T=1_E$, hence $ranS^*T^*=ranS^*S$ has closed range and implies that S has closed range. □

In this example show that dense range can not implies closed range

Example 3.2. Let E, F are Hilbert $\mathcal{A}$-modules an where $\{\varepsilon_n\}$ is orthonormal bases for H. Let $\{\lambda_n\}$ be a sequence of non-zero scalars such that $\lambda_n\to 0$ as $n\to 0$. Let $T\in L(F,E)$ For x in E and y in F, define $\theta_{x,y}:F\to E$ by

$$\theta_{x,y}(z)=x<y,z>\qquad(z\in F)$$

It is clear that $\theta_{x,y}\in L(F,E)$ be defined by

$$Tz=\sum_{j=1}^{\infty}\lambda_j x<y,z>$$

Note that T is a bounded linear operator, each λ_j is an eigenvalue of T and 0 is an accumulation point of the eigne spectrum of T. Consequently, $ran(T)$ is not closed in E, we observation that $ranT$ is dense.

Lemma 3.3. *Let E and F are Hilbert $\mathcal{A}$-modules and $T \in L(E,F)$ has closed range Then $T + \alpha I$ is uniformly bounded for $\alpha > 0$.*

Proof. if $\alpha \in acc\sigma(T^*T)$ then $T^*T - \alpha I$ is invertable and $(T^*T - \alpha I)^{-1}$ exists, therefore by let $k_\alpha = (T^*T - \alpha I)T$ we have $k_\alpha^* k_\alpha = (T^*T - \alpha I)^{-2} T^*T$ Since $k_\alpha^* k_\alpha$ is self adjoint, it follows that

$$\| k_\alpha^* k_\alpha \| = \| k_\alpha \|^2 = r_\sigma(k_\alpha^* k_\alpha) = \sup\{\frac{\lambda}{(\lambda + \alpha)^2} : \lambda \in \sigma(T^*T)\}$$

Taking $0 < \alpha < \| T \|^2$ and

$$f_\alpha(t) = \frac{t}{(t+\alpha)^2} \quad , \qquad t \in (0, \| T \|^2]$$

we see that f_α attains its maximum at $t = \alpha$, i.e.,

$$\sup_{0 < \alpha \leq \|T\|^2} f_\alpha(t) = f_\alpha(\alpha) = \frac{1}{4\alpha}$$

Now suppose that 0 is an accumulation point of $\sigma(T^*T)$. Then there exists a sequence λ_n of distinct elements in $\sigma(T^*T)$ such that $\lambda_n \to 0$ as $n \to \infty$. Hence,

$$\| k_{\lambda_n} \|^2 = f_{\lambda_n}(\lambda_n) = \frac{1}{4\lambda_n} \to \infty \quad as \quad n \to \infty$$

Thus, the family $\{k_\alpha : \alpha > 0\}$ is not uniformly bounded. Hence, $ran(T)$ is not closed. □

Proposition 3.4. *Let E, F are Hilbert $\mathcal{A}$-modules and T_α be a net in $L(E,F)$ are bounded adjointable operators T_α converges to T and T_α has closed range Then T has closed range.*

Proof. If T_α has closed ranges then $\ker T_\alpha \oplus (\ker T_\alpha)^\perp = E$ and $\ker T_\alpha = \{x \in E : T_\alpha(x) = 0\}$ and $T_\alpha \to T$, hence $T(x) = 0$ if and only if $T_\alpha(x) = 0$ when $\alpha \to \infty$ hence $\ker T_\alpha = \ker T$ when $\alpha \to \infty$ $\ker(T)$ is orthogonally complemented in E and

$$\inf\{\| T_\alpha x \| : x \in Ker(T_\alpha)^\perp and \| x \| = 1\} = \inf\{\| Tx \| : x \in Ker(T)^\perp and \| x \| = 1\} > 0.$$

then T has closed range. □

Definition 3.5. An element T in $L(E,F)$ is called a partial isometry, if $T |_{(\ker T)^\perp}$ is an isometry; that is, for every $x \in (\ker T)^\perp$, $\| Tx \| = \| x \|$.

Theorem 3.6. *If E, F are Hilbert $\mathcal{A}$-modules and $T \in L(E,F)$ partial isometry then T has a closed range.*

Proof. To see that $ranT$ is closed, suppose $y \in \overline{ranT}$. Then there exists a sequence of element $\{x_n\}$ in E such that $y = \lim_{n\to\infty} T(x_n)$. hence T is a partial isometry we have T^*T is a projection in $L(E)$, Then

$$TT^*y = \lim_{n\to\infty} TT^*T(x_n) = \lim_{n\to\infty} T(x_n) = y$$

Hence $y \in ranT$ so $ranT$ is closed. □

Proposition 3.7. *Let E be Hilbert $\mathcal{A}$-module and S, T be in $L(E)$ are self-adjoint operators such that $TS = ST$ has closed range then $|TS|$ have closed range.*

Proof. then $TS \in L(E)$ is normal, by $(TS)(TS)^* = (TS)^*(TS)$ we have $TS^2T = ST^2S$ hence, where we have $|c| = (c^*c)^{\frac{1}{2}}$, this implies that $|TS| = |ST|$ Thus TS is normal, $\ker TS = \ker S^*T^*$. It follows that

$$\ker(TS) = \ker(ST) = \ker(|TS|) = \ker(|ST|)$$

Now by proposition 3.7 of [6] we have

$$ST(E) = \overline{ST(E)} = \overline{(TS)^*(E)} = \overline{(TS)^*(TS)(E)} = \overline{TS^2T(E)} = \overline{(ST)^*(E)} = TS(E)$$

then $ran(TS) = ran(ST) = ran(|TS|) = ran(|ST|)$ □

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