

Review Article - A study of some fixed point theorems for various types of maps

Masroor Mohammad¹, Rizwana Jamal² and Qazi Aftab Kabir³

^{1,3} Research Scholar Saifia Science College Bhopal, M.P.

² Assistant Professor Department of Mathematics Saifia Science College, Bhopal, M.P.

Abstract:

Fixed point theory off course entails the search for a combination of conditions on a set S and a mapping $T : S \rightarrow S$ which, in turn, assures that T leaves at least one point of S fixed, i.e. $x = T(x)$ for some $x \in S$. The theory has several rather well-defined (yet overlapping) branches. The purely topological theory as well as those topics which lie on the borderline of topology and functional analysis (e.g. those related to Leray-Schauder theory) have their roots in the celebrated theorem of L. E. J. Brouwer. This paper presents a review of the available literature on fixed point theorems for various types of maps.

Keywords: Fixed point theorems, multivalued mappings, nonexpansive mappings.

Introduction:

Fixed point theory is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. In the last 50 years, fixed point theory has been a flourishing area of research for many mathematicians. The origins of the theory, which date to the later part of the nineteenth century, rest in the use of successive approximations to establish the existence and uniqueness of solutions, particularly to differential equations. This method is associated with many celebrated mathematicians like Cauchy, Fredholm, Liouville, Lipschitz, Peano, and Picard. It is worth noting that the abstract formulation of Banach is credited as the starting point to metric fixed point theory. But the theory did not gain enough impetus till Felix Browder's major contribution to the development of the nonlinear functional analysis as an active and vital branch of mathematics. In recent years a number of excellent books, monographs, and surveys by distinguished authors about fixed point theory have appeared.

Some classical fixed point theorems for single valued nonexpansive mappings have been extended to multivalued mappings. The first results in this direction were established by Markin (1968) in a

Hilbert space setting and by Browder (1976) for spaces having a weakly continuous duality mapping. Lami Dozo (1973) generalized these results to a Banach space satisfying Opial's condition. By using Edelstein's method of asymptotic centers, Lim (1974) obtained a fixed point theorem for a multi-valued nonexpansive self-mapping in a uniformly convex Banach space. Kirk and Massa (1990) gave an extension of Lim's theorem proving the existence of a fixed point in a Banach space for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact. Many questions remain open (see [47] and [54]) about the existence of fixed points for multivalued nonexpansive mappings when the Banach space satisfies geometric properties which assure the existence of a fixed point in the single valued case, for instance, if X is a nearly uniformly convex space.

Review of Literature:

Husain and Latif (1991) proved some fixed point theorems for multi-valued contractive-type and nonexpansive type maps on complete metric spaces and on certain closed bounded convex subsets of Banach spaces. Hitzler and Seda (1999) discussed the semantics of disjunctive programs and databases and show how multivalued mappings and their fixed points arise naturally within this context. A number of fixed-point theorems for multivalued mappings are considered, some of which are already known and some of which are new. The notion of a normal derivative of a disjunctive program is introduced. Normal derivatives are normal logic programs which are determined by the disjunctive program. Thus, the well-known single-step operator associated with a normal derivative is single-valued, and its fixed points can be found by well-established means. It is shown how fixed points of the multivalued mapping determined by a disjunctive program relate to the fixed points of the single-step operators coming from its normal databases, and this point is discussed. Most of the results for multivalued mappings rest on corresponding, known results concerning fixed points of single-valued mappings. Since the latter results are

frequently referred to, they have been collected together for convenience in a survey which should be of independent interest as well as being preparatory for the later results. Miklaszewski (2001) proved that every Borsuk continuous set-valued map of the closed ball in the 3-dimensional Euclidean space, taking values which are one point sets or knots, has a fixed point. This result is a special case of the Gorniewicz Conjecture. Percup (2002) presented a new compactness method for operator inclusions in general, and for Hammerstein like inclusions, in particular. This method applies to acyclic multi-valued maps which satisfy a generalized compactness condition of Monch type. Kirk *et. al.*, (2003) extended Banach contraction principle to a case of cyclic contractive mappings. Banach's contraction principle is extended to multivalued mappings by Nadler (1969). Afterward, an interesting and rich fixed point theory for set-valued mapping was developed in many directions. The theory of multivalued mappings has applications in optimization problem, control theory, differential and integral theory, economics, informatics and in many branches of analysis. Moutawakil (2004) presented a generalization of the well-known nadler multi-valued contraction fixed point to the setting of symmetric spaces, obtains a new fixed point theorem for multivalued mappings in probabilistic spaces. Olantinwo (2009) established a fixed point theorem for multivalued operators in a complete b-metric space using the concept of Berinde and Berinde (2007) on multivalued weak contractions for the Picard iteration in a metric space. The main result generalizes, extends and improves some of the recent results of Berinde and Berinde (2007) as well as those of Daffer and Kaneko (1995) and also unifies several classical results pertaining to single and multivalued contractive mappings. Sintunavarat and Kumam (2012) quite recently, initiated the study of common fixed points for cyclic generalized multi-valued contraction mappings. The study of fixed points for mappings satisfying implicit relations is initiated in [42], [46], [9], [10], [3]. Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type, in fuzzy metric spaces and intuitionistic metric spaces. With this method, the proofs of some fixed point theorems are more simple. Also, the method allows the study of local and global properties of fixed point structures. Nashine *et. al.*, (2012) recently used a method from [1] and introduced an implicit relation-type-cyclic contractive for mappings in metric spaces for such mappings. The study of fixed points for multivalued mappings satisfying an implicit relation is initiated in [44], [43] and other papers. Popa (2015) proved a

general fixed point theorem for cyclic multi-valued mappings satisfying an implicit relation from [44] different from implicit relations used in [32] and [53], generalizing some results from [48], [35], [32], [34], [41], [19] and from other papers. Jungck (1976) proved a common fixed point theorem for commuting maps, generalizing the Banach's fixed point theorem. Sessa (1982) generalized the notion of commutativity and defined weak commutativity. Further, Jungck (1986) introduced more generalized commutativity, so called compatibility and generalized some results of Singh and Singh (1980) and Fisher (1983). Kaneko (1988) extended the concept of weakly commuting mappings for multi-valued set up and extended result of Jungck (1976). Kaneko and Sessa (1989) extended the concept of compatible mappings for multi-valued mappings and generalized the result of Kubiak (1985). Jungck and Rhoades (1998) extended weak compatibility in the settings of single-valued and multi-valued mappings. Pant (1994, 1998, 1999 and 1999) initiated the study of non compatible mappings and introduced R-weak commutativity of mappings. He also showed that for single-valued mappings pointwise R-weak commutativity is equivalent to weak compatibility at the coincidence points. Shahzad and Kamran (2001) and Singh and Mishra (2001) have independently extended the idea of R-weak commutativity to the settings of single and multi-valued mappings. Singh and Mishra (2001) introduced the notion of (IT)-commutativity for a hybrid pair of single valued and multi-valued mappings and showed that a pointwise R-weakly commuting hybrid pair need not be weakly compatible. However at the coincidence points pointwise R weak commutativity for hybrid pairs is equivalent to (IT) commutativity. Kamaran (2004) introduced the notion of T-weak commutativity for a single-valued and a multi-valued mapping and showed that it is weaker condition than (IT) commutativity and weak compatibility of hybrid pair. In their paper, Kaneko (1988) and Kaneko and Sessa (1989) have assumed a pair of single-valued and multi-valued mapping which are continuous at X and could prove the existence of a coincidence point. For the existence of a common fixed point an additional hypothesis is needed. They have also remarked whether or not the continuity of two mappings is really needed in the proof. Kubiacyk and Mustafa Ali (1996), Krzyska and Kubiacyk (1998) and many others have proved common fixed point theorems for multi-valued mappings. Asad and Ahmad (1999) extended the results of Fisher (1979) for multi-valued mappings using condition of weak commutativity or compatibility and proved that existence of common fixed point can be achieved by the continuity of the

single-valued mapping only, the continuity of the multi-valued mappings are not needed.

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