Successive Occurrence of a Non-Zero Digit in All Base *b* Natural Numbers Less Than b^n

Neeraj Anant Pande

Associate Professor, Department of Mathematics & Statistics,
Yeshwant Mahavidyalaya (College), Nanded – 431602, Maharashtra, INDIA

Abstract — *Considering base b number system for any positive integer b > 1, all successive positive integers less than b^n, for any positive integer n, are under consideration. Successive occurrences of digit 1 in initial numbers less than b^n in base b are analyzed here. The formulae for the number of occurrences of successive 1's, first and last instance in base power ranges are developed. This is done for all occurrences of multiple number of successive 1's and finally all results are generalized for such successive occurrences of all non-zero digits. Tabulated values are provided for computer friendly base b = 16.*

Keywords — *General base, Natural numbers, non-zero digit, successive occurrences.*

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I. Introduction

We consider positive integers

$$1, 2, 3, \cdots .$$

This is that part of mathematics which is most commonly used.

In our usual course, decimal system is used [1]. It is number system with base 10, having 10 digits, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. As such any number system with base $b > 1$ is considerable and it will have exactly b number of digits. For $b \leq 10$, we denote these digits by $0, 1, 2, \cdots , b - 1$. For $b > 10$, we denote first 10 digits by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and remaining digits by successive alphabets $A, B, C, \cdots , X$ where X is not necessarily alphabet X but merely a symbol to designate the alphabet at number equal to 9 ahead of its own position in the list of alphabets. Every number system exhibits many interesting properties [9].

For simplicity, the term number is used to mean the natural number. We consider all numbers in base b in range $1 - b^n$, except b^n, for $n \in N$, the numbers under consideration being m, with $1 \leq m < b^n$, the last number b^n being not taken as it contains $n + 1$ of digits.

II. Successive Occurrence of Digit 1

All types of occurrences of digit 0 in natural numbers are well-known [5], [6], [7].

The first natural number digit 1 is having unique property that it is in number systems with all bases $b > 1$.

As example of a general base b, we have chosen computer system base $b = 16$. Then we have determined this count of successive occurrence of single 1 and also double 1's in numbers less than 16^{15}, which is little more than a one quintillion (10^{18}), by using Java program and these counts are as given below. Although the base of numbers under analysis is 16, we have used usual base 10 while writing their count!

TABLE I : Number of Hexadecimal Numbers with Single and Double Successive 1's in their Digits

Sr. No.	Numbers Range Less Than	Number of Numbers in Base 16 with single (Successive) 1	Number of Numbers in Base 16 with two Successive 1's
1.	16^1	1	0
2.	16^2	30	1
3.	16^3	675	30
4.	16^4	13,500	675
5.	16^5	253,125	13,500
6.	16^6	4,556,250	253,125
7.	16^7	79,734,375	4,556,250
8.	16^8	1,366,875,000	79,734,375
9.	16^9	23,066,015,625	1,366,875,000
10.	16^{10}	384,433,593,750	23,066,015,625
11.	16^{11}	6,343,154,296,875	384,433,593,750
12.	16^{12}	103,797,070,312,500	6,343,154,296,875
13.	16^{13}	1,686,702,392,578,125	103,797,070,312,500

Sr. No.	Numbers Range Less Than	Number of Numbers in Base 16 with single (Successive) 1	Number of Numbers in Base 16 with two Successive 1's
14.	16^{14}	27,246,730,957,031,250	1,686,702,392,578,125
15.	16^{15}	437,893,890,380,859,360	27,246,730,957,031,250

In the first range $1 \le m < 16^1 = 16$, we know that single 1 is seen once as a number itself. Its count is ${}^{1\text{-}(1\text{-}1)}C_1 15^{1\text{-}1} = 1$. Single occurrence is treated as successive by absence of non-successive character!

During second range span $1 \le m < 16^2 = 256$, single 1 occurs 30 times. Its 15 instances are in numbers

1, 21, 31, 41, 51, 61, 71, 81, 91, A1, B1, C1, D1, E1 and F1

at unit's places and 15 are in numbers

10, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E and 1F

at ten's places. So, within the second occurrence block, the count is ${}^{2\text{-}(1\text{-}1)}C_1 15^{2\text{-}1} = 2 \times 15 = 30$. Again being solo, it is considered to be successive.

In this range, double 1 occurs just once in number 11. Clearly this is of successive type and the count can be seen as ${}^{2\text{-}(2\text{-}1)}C_1 15^{2\text{-}2} = 1 \times 1 = 1$.

In the third range, $1 \le m < 16^3 = 4{,}096$, single 1 comes 675 times in various positions. It is in numbers

1, 21, 31,⋯, F1, 201, 221,⋯, 2F1, 301, 321,⋯, 3F1,⋯, F01, F21,⋯, FF1,

at unit's places, in numbers

12, 13, 14,⋯, 1F, 210, 212,⋯, 21F, 310, 312,⋯, 31F,⋯, F10, F12,⋯, F1F,

at ten's places and in

100, 102, 103,⋯, 10F, 120, 122, 123,⋯, 130, 132, 133,⋯, 13F,⋯, 1F0, 1F2, 1F3,⋯, 1FF

at hundred's places.

This occurrence in third block is ${}^{3\text{-}(1\text{-}1)}C_1 15^{3\text{-}1} = 3 \times 15^2 = 3 \times 225 = 675$ times, all of them being considered successive.

In this range, successive double 1's are in

211, 311,⋯, F11,

at unit's and ten's places and in

110, 112, 113,⋯, 11F

at ten's and hundred's places.

This count is ${}^{3\text{-}(2\text{-}1)}C_1 15^{3\text{-}2} = 2 \times 15 = 30$.

Also successive triple 1's occur once in 111.

Every number in above table has such explanation.

In fact, there is a straightforward formula for this count.

Notation : We introduce the generalized notation ${}^{S}_{1}\underset{b}{O}{}^{n}_{r}$ for number of base b numbers less than b^n with r number of successive 1's.

Theorem 1 : If r, n and b are positive integers with $r \le n$ and $b > 1$, then the number of numbers in base b containing exactly r number of successive digit 1's in the range $1 \le m < b^n$ is

$$ {}^{S}_{1}\underset{b}{O}{}^{n}_{r} = {}^{n-(r-1)}C_1 (b-1)^{n-r}. $$

Proof. Let n, r and b be positive integers with $r \le n$ and $b > 1$. There are in total b digits available to occupy n places in all numbers in base b within range $1 \le m < b^n$. We want r successive places to be occupied by digit 1. The various choices for these r successive places for digit 1 will be ${}^{n\text{-}(r\text{-}1)}C_1$ in number. Now for each such choice, remaining $n - r$ places are to be occupied by any of the remaining $(b - 1)$ digits except 1 and there are $(b - 1)^{n-r}$ choices for each of that. This totals to ${}^{n\text{-}(r\text{-}1)}C_1(b - 1)^{n-r}$ and hence ${}^{S}_{1}\underset{b}{O}{}^{n}_{r} = {}^{n-(r-1)}C_1 (b-1)^{n-r}$. This completes the proof of the theorem.

With same choice of $b = 16$, the table given above is now extended to higher occurrences of successive 1's by using this formula. Again counts are given in decimal base.

TABLE III : NUMBER OF HEXADECIMAL NUMBERS WITH MULTIPLE SUCCESSIVE 1'S IN THEIR DIGITS

Sr. No.	Number Range <	Number of Numbers in Base 16 with 3 Successive 1's	Number of Numbers in Base 16 with 4 Successive 1's	Number of Numbers in Base 16 with 5 Successive 1's
1.	16^3	1	0	0
2.	16^4	30	1	0
3.	16^5	675	30	1

Sr. No.	Number Range <	Number of Numbers in Base 16 with 3 Successive 1's	Number of Numbers in Base 16 with 4 Successive 1's	Number of Numbers in Base 16 with 5 Successive 1's
4.	16^6	13,500	675	30
5.	16^7	253,125	13,500	675
6.	16^8	4,556,250	253,125	13,500
7.	16^9	79,734,375	4,556,250	253,125
8.	16^{10}	1,366,875,000	79,734,375	4,556,250
9.	16^{11}	23,066,015,625	1,366,875,000	79,734,375
10.	16^{12}	384,433,593,750	23,066,015,625	1,366,875,000
11.	16^{13}	6,343,154,296,875	384,433,593,750	23,066,015,625
12.	16^{14}	103,797,070,312,500	6,343,154,296,875	384,433,593,750
13.	16^{15}	1,686,702,392,578,125	103,797,070,312,500	6,343,154,296,875

TABLE IIII : CONTINUED …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 6 Successive 1's	Number of Numbers in Base 16 with 7 Successive 1's	Number of Numbers in Base 16 with 8 Successive 1's
1.	16^6	1	0	0
2.	16^7	30	1	0
3.	16^8	675	30	1
4.	16^9	13,500	675	30
5.	16^{10}	253,125	13,500	675
6.	16^{11}	4,556,250	253,125	13,500
7.	16^{12}	79,734,375	4,556,250	253,125
8.	16^{13}	1,366,875,000	79,734,375	4,556,250
9.	16^{14}	23,066,015,625	1,366,875,000	79,734,375
10.	16^{15}	384,433,593,750	23,066,015,625	1,366,875,000

TABLE IIV : CONTINUED …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 9 Successive 1's	Number of Numbers in Base 16 with 10 Successive 1's	Number of Numbers in Base 16 with 11 Successive 1's
1.	16^9	1	0	0
2.	16^{10}	30	1	0
3.	16^{11}	675	30	1
4.	16^{12}	13,500	675	30
5.	16^{13}	253,125	13,500	675
6.	16^{14}	4,556,250	253,125	13,500
7.	16^{15}	79,734,375	4,556,250	253,125

TABLE IV : CONTINUED …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 12 Successive 1's	Number of Numbers in Base 16 with 13 Successive 1's	Number of Numbers in Base 16 with 14 Successive 1's	Number of Numbers in Base 16 with 15 Successive 1's
1.	16^{12}	1	0	0	0
2.	16^{13}	30	1	0	0
3.	16^{14}	675	30	1	0
4.	16^{15}	13,500	675	30	1

III. FIRST SUCCESSIVE OCCURRENCE OF DIGIT 1

Irrespective of base for representation, the first number containing 1 is naturally 1. For 2 successive 1's, the first instance is 11, for 3 it is 111 and so on. These notations are in their respective bases and true in all bases but their values in decimal are different depending on base *b*. They are formulated as follows.

Formula 1 : If n, r and b are natural numbers with $b > 1$, then the first occurrence of r number of successive 1's in base b numbers in range $1 \le m < b^n$ is

$$f = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1} \left(1 \times b^j\right), & \text{if } r \le n \end{cases}.$$

IV. LAST SUCCESSIVE OCCURRENCE OF DIGIT 1

Taking base $b = 16$, following are the last successive occurrences of 1.

TABLE IIVI : LAST HEXADECIMAL NUMBERS WITH MULTIPLE SUCCESSIVE 1'S IN THEIR DIGITS IN VARIOUS BASE POWER RANGES

Sr. No.	Number Range < → Last Number in Base 16 with Successive ↓	16^1	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9
1.	1 1	1	F1	FF1	F,FF1	FF,FF1	FFF,FF1	F,FFF,FF1	FF,FFF,FF1	FFF,FFF,FF1
2.	2 1's	-	11	F11	F,F11	FF,F11	FFF,F11	F,FFF,F11	FF,FFF,F11	FFF,FFF,F11
3.	3 1's	-	-	111	F,111	FF,111	FFF,111	F,FFF,111	FF,FFF,111	FFF,FFF,111
4.	4 1's	-	-	-	1,111	F1,111	FF1,111	F,FF1,111	FF,FF1,111	FFF,FF1,111
5.	5 1's	-	-	-	-	11,111	F11,111	F,F11,111	FF,F11,111	FFF,F11,111
6.	6 1's	-	-	-	-	-	111,111	F,111,111	FF,111,111	FFF,111,111
7.	7 1's	-	-	-	-	-	-	1,111,111	F1,111,111	FF1,111,111
8.	8 1's	-	-	-	-	-	-	-	11,111,111	F11,111,111
9.	9 1's	-	-	-	-	-	-	-	-	111,111,111

Here is how they come.

Formula 2 : If n, r and b are natural numbers with $b > 1$, then the last occurrence of r number of successive 1's in base b numbers in range $1 \le m < b^n$ is

$$l = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1} \left(1 \times b^j\right) + \begin{cases} 0 & \text{, if } r = n \\ \sum_{j=r}^{n-1} \left((b-1) \times b^j\right), & \text{if } r < n \end{cases} \end{cases}.$$

The formulae 1 and 2 here are precisely same as those in [8]. This similarity resembles similarity of those in [2] with [3].

V. EXTENSION TO OTHER NON-ZERO DIGITS

We finish by observing an important fact that the discussion done for occurrences of successive digit 1's is applicable parallely for all other non-zero digits. If the non-zero digit of interest is d, where $1 \le d < b$, the following are extensions of results derived so far.

Notation : Further generalized notation ${}^{S}_{d}\underset{b}{O}{}^{n}_{r}$ stands for number of base b numbers less than b^n with r number of successive digits d's.

Theorem 2 : If r, n, d, and b are positive integers with $r \le n$ and $1 \le d < b$, then the number of base b numbers containing exactly r number of successive digit d's in the range $1 \le m < b^n$ is

$${}^{S}_{d}\underset{b}{O}{}^{n}_{r} = {}^{n-(r-1)}C_1 (b-1)^{n-r}$$

Formula 3 : If n, r, d and b are natural numbers with $1 \le d < b$, then the first occurrence of r number of successive digit d's in numbers in base b in range $1 \le m < b^n$ is

$$f = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1} \left(d \times b^j \right), & \text{if } r \le n \end{cases}.$$

Formula 4 : If n, r, d and b are natural numbers with $1 \le d < b$, then the last occurrence of r number of successive digit d's in numbers in base b in range $1 \le m < b^n$ is

$$l = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1} \left(d \times b^j \right) + \begin{cases} 0 & \text{, if } r = n \\ \sum_{j=r}^{n-1} \left((b-1) \times b^j \right), & \text{if } r < n \end{cases} \end{cases}.$$

Remark : All the work here has generalized the earlier results in [3].

The number sequences coming for number of increasing number of successive non-zero digits in increasing base powers are successive progressing.

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