Some Common Fixed Point Theorem in Fuzzy Metric Space

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Abstract - *In this paper we are generalizing the results of Som [3,4] ,Mukherjee [2] and Shrivastava[1].*
Keywords - *semi-compatible and (B, T) is weak-compatible, Fuzzy Metric Space*.

MAIN RESULT

Theorem 1. Let A, B, S and T be self mappings of a complete fuzzy metric space (X, M, *) satisfying

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(b) one of A or S is continuous,

(c) the pair (A, S) is semi-compatible and (B, T) is weak-compatible,

(d) $aM(Ax, By, t) - bM(Sx, Ty, t) \geq \phi\,\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, By, t), M(Ty, Ax, t), M(Ty, By, t)\}$,

where $\phi : (R^+)^5 \rightarrow R^+$ is continuous and strictly increasing in each co-ordinate variable such that for all $x, y \in X$,

$a < b + 1$ and for any $\upsilon < 1$,

$\phi\,(\upsilon, \upsilon, a, \upsilon, a_2\upsilon, \upsilon) > \upsilon$, $a_1 + a_2 = 3$. Then A, B, S and T have a unique common fixed point in X.

Now, we are proving this theorem of shrivastava[1] for six mapping

Theorem 2. Let, A, B, S, T, P and Q be self mappings of a complete fuzzy metric space (X, M, *) satisfying

(a) $A(X) \subseteq PT(X), B(X) \subseteq QS(X)$,

(b) one of A or QS is continuous,

(c) the pair (A, QS) is semi compatible and (B, PT) is weak compatible,

(d) $aM(Ax, By, t) - bM(QSx, PTy, t) \geq \phi\,\{M(QSx, PTy, t), M(QSx, Ax, t),$

$$M(QSx, By, t), M(PTx, Ax, t), M(PTy, By, t)\},$$

where, ϕ is a continuous mapping $\phi : (R^+)^5 \rightarrow R$ satisfying $\upsilon < 1$,

$\phi\,(\upsilon, \upsilon, a_1\upsilon, a_2\upsilon, \upsilon) > \upsilon$, $a_1 + a_2 = 3$. Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Proof of theorem 1 , let x_o be any arbitrary point. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ then there exists $x_1, x_2 \in X$ such that

$$Ax_o = Tx_1 = y_1, \quad Bx_1 = Sx_2 = y_2 .$$

Inductively, construct two sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1},$$

$$y_{2n+2} = Bx_{2n+1} = Sx_{2n+2} \;; \quad n = 0, 1, 2, 3, \ldots$$

Let $M_n = M(y_n, y_{n+1}, t)$; $n = 0, 1, 2, 3, \ldots$

We claim that $\{M_n\}$ is a increasing sequence, suppose on the contrary that

$M_{2n} > M_{2n+1}$, for some n.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (d), we get

$$aM(Ax_{2n}, Bx_{2n+1}, t) - bM(Sx_{2n}, Tx_{2n+1}, t)$$

$$\geq \phi \{M(Sx_{2n}, Tx_{2n+1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n}, t), M(Tx_{2n+1}, Ax_{2n+1}, t)\}.$$

$$\Rightarrow \quad aM(y_{2n+1}, y_{2n+2}, t) - bM(y_{2n}, y_{2n+1}, t),$$

$$\geq \phi \{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, Y_{2n+1}, t), M(y_{2n}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+1}, t), M(y_{2n+1}, Y_{2n+2})\}$$

$$\Rightarrow \quad aM_{2n+1} - b\,M_{2n} \geq \phi \{M_{2n}, M_{2n}, M_{2n} + M_{2n+1}, 1, M_{2n+1}\}$$

$$> \phi \{M_{2n+1}, M_{2n+1}, 2M_{2n+1} + M_{2n+1}, M_{2n+1}\}$$

$$> M_{2n+1}$$

$$\Rightarrow \qquad M_{2n+1} > \frac{b}{a-1} M_{2n}$$

$$\Rightarrow \qquad M_{2n+1} > M_{2n} \qquad [\,\Theta\, a < b + 1\,]$$

which is a contradiction.

Thus $\{M_n\}$ is increasing sequence of positive real number in [0, 1] and therefore $\lim_{n\to\infty} M_n = 1$. Now, we show that $\{y_n\}$ is a cauchy sequence. Since $\lim_{n\to\infty} M_n = 1$, it is sufficient to show that $\{y_{2n}\}$ is a cauchy sequence.

Suppose that it is not so, then there is an $\varepsilon > 0$ such that for each integer 2k (k = 0, 1, 2, ...) there exists even integer 2nk and 2mk with 2k < 2nk < 2mk such that

$$M(y_{2nk}, y_{2mk}, t) \leq 1 - \varepsilon \quad ; \text{ for some } t > 0. \tag{1}$$

Let for each even integer 2k, 2mk be the least positive integer exceeding 2nk satisfying (1),

then $M(y_{2nk}, y_{2mk-2}, t) > 1 - \varepsilon$ and

$$M(y_{2nk}, y_{2mk}, t) \leq 1 - \varepsilon. \tag{2}$$

As such, for each even integer 2k, we have

$$1 - \varepsilon > M(y_{2nk}, y_{2mk}, t) \geq M(y_{2nk}, y_{2mk-2}, t/3) * M(y_{2mk-2}, y_{2mk-1}, t/3) * M(y_{2mk-1}, y_{2mk}, t/3).$$

so by (2) and as $k \to \infty$, we get

$$\lim_{k\to\infty} M(y_{2nk}, y_{2mk}, t) = 1 - \varepsilon. \tag{3}$$

Now, using (3) in the triangular inequalities

$$M(y_{2nk}, y_{2mk-1}, t) \geq M(y_{2nk}, y_{2mk}, t/2) * M(y_{2mk}, y_{2mk-1}, t/2)$$

and $M(y_{2nk+1}, y_{2mk-1}, t) \geq M(y_{2nk+1}, y_{2nk}, t/3) * M(y_{2nk}, y_{2mk}, t/3) * M(y_{2mk}, y_{2mk-1}, t/3).$

Taking $k \to \infty$, then

$$M(y_{2nk+1}, y_{2mk-1}, t) \geq 1 - \varepsilon * 1 = 1 - \varepsilon$$

and

$$M(y_{2nk+1}, y_{2mk-1}, t) \geq 1 * 1 - \varepsilon * 1 = 1 - \varepsilon.$$

Then

$$M(y_{2nk}, y_{2mk}) \geq M(y_{2nk}, y_{2nk+1}, t/2) * M(y_{2nk+1}, y_{2mk}, t/2)$$

$$= M(y_{2nk}, y_{2nk+1}, t/2) * M(Bx_{2nk}, Ax_{2mk-1}, t/2)$$

$$\geq M(y_{2nk}, y_{2nk+1}, t/2) * \frac{1}{a}\phi\ \{M(Sx_{2mk-1}, Tx_{2nk}, t/2), M(Sx_{2mk-1}, Ax_{2mk-1}, t/2), M(Sx_{2mk-1}, Bx_{2nk}, t/2),$$

$$M(Tx_{2nk}, Ax_{2mk-1}, t/2), M(Tx_{2nk}, Bx_{2nk}, t/2\} + \frac{b}{a} M(Sx_{2mk-1}, Tx_{2nk}, t/2)$$

$$\geq M(y_{2nk}, y_{2nk+1}, t/2) * \frac{1}{a} \phi \{M(y_{2mk-1}, y_{2nk}, t/2), M(y_{2mk-1}, y_{2mk}, t/2), M(y_{2mk-1}, y_{2nk+1}, t/2),$$

$$M(y_{2nk}, y_{2mk}, t/2), \{M(y_{2nk}, y_{2nk+1}\} + \frac{b}{a} M(y_{2mk-1}, y_{2nk}, t/2).$$

On taking $k \rightarrow \infty$

$$1 - \varepsilon \geq \frac{1}{a} \phi \{1-\varepsilon, 0, 1-\varepsilon, 1-\varepsilon, 0\} + \frac{b}{a}(1-\varepsilon)$$

$$> \frac{1}{a}(1-\varepsilon) + \frac{b}{a}(1-\varepsilon) = \frac{1+b}{a}(1-\varepsilon)$$

$$\Rightarrow 1-\varepsilon > 1-\varepsilon$$

which is a contradiction.

Hence $\{y_{2n}\}$ is a cauchy sequence in X. By completeness of X, $\{y_n\}$ converges to $z \in X$. Hence, the subsequences

$$\{Ax_{2n}\} \rightarrow z, \qquad \{Sx_{2n}\} \rightarrow z, \qquad (4)$$

$$\{Tx_{2n+1}\} \rightarrow z, \qquad \{Bx_{2n+1}\} \rightarrow z. \qquad (5)$$

Since the limit of a sequence in fuzzy metric space is unique we obtain that

$$Az = Sz$$

Step 1. Now, we will prove that $Az = z$. Suppose on the contrary $Az \neq z$.

By putting $x = z$, $y = x_{2n+1}$ in (d) we have

$$aM(Az, Bx_{2n+1}, t) - bM(Sz, Tx_{2n+1}, t)$$

$$\geq \phi \{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$$

$$\Rightarrow aM(Az, z, t) - bM(Az, z, t)$$

$$\geq \phi \{M(Az, z, t), M(Az, Az, t), M(Az, z, t), M(z, Az, t), M(z, z, t)\}$$

$$\geq \phi \{M(Az, z, t), 1, M(Az, z, t), M(Az, z, t), 1\}$$

$$\geq \phi \{M(Az, z, t), M(Az, z, t), 2M(Az, z, t), M(Az, z, t), M(Az, z, t)\}$$

$\Longrightarrow \quad (a - b)\, M(Az, z, t) > M(Az, z, t)$

which is a contradiction.

Hence $z = Az = Sz$.

Step 2. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that

$$z = Az = Tu.$$

Now, we have to prove that $z = Bu$, suppose on the contrary that $z \neq Bu$

Putting $x = x_{2n}$, $y = u$ in (d) we get.

$$aM(Ax_{2n}, Bu, t) - bM(Sx_{2n}, Tu, t)$$

$$\geq \phi\,\{M(Sx_{2n}, Tu, t), M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Bu, t), M(Tu, Ax_{2n}, t), M(Tu, Bu, t)\}.$$

On taking limit as $n \to \infty$ and using (4) we obtain that

$$aM(z, Bu, t) - bM(z, z, t) \geq \phi\,\{M(z, z, t), M(z, z, t), M(z, Bu, t), M(z, z, t), M(z, Bu, t)\}$$

$$\Rightarrow \quad aM(z, Bu, t) - b \geq \phi\,\{1, 1, M(z, Bu, t)\, 1, M(z, Bu, t)\}$$

$$aM(z, Bu, t) - bM(z, Bu, t) > \phi\,\{M(z, Bu, t), M(z, Bu, t), 2M(z, Bu, t), M(z, Bu, t), M(z, Bu, t)\}$$

$$(a - b)\, M(z, Bu, t) > M(z, Bu, t)$$

which is a contradiction.

Hence $z = Bu = Tu$ and the weak compatibility of (B, T) gives

$$TBu = BTu$$

$$\text{i.e. } Tz = Bz$$

Step 3. By putting $x = z$, $y = z$ in (d) and assuming $Az \neq Bz$, we have.

$$aM(Az, Bz, t) - bM(Sz, Tz, t)$$

$$\geq \phi\,\{M(Sz, Tz, t), M(Sz, Az, t), M(Sz, Bz, t), M(Tz, Az, t), M(Tz, Bz, t)\}$$

$$\Longrightarrow \quad aM(Az, Bz, t) - b\, M(Az, Bz, t)$$

$$\geq \phi\,\{M(Az, Bz, t), M(Az, Az, t), M(Az, Bz, t), M(Bz, Az, t), M(Tz, Tz, t)\}$$

$\Rightarrow$ (a-b) M(Az, Bz, t) $\geq$ ϕ {M(Az, Bz, t), 1,(M(Az, Bz, t), M(Az, Bz, t), 1}

$>$ ϕ {M(Az, Bz, t), M(Az, Bz, t), 2M(Az, Bz, t), M(Az, Bz, t), M(Az, Bz, t)}

$\Rightarrow$ (a - b) M(Az, Bz, t) $>$ M(Az, Bz, t)

Which is a contradiction. Hence Az = Bz.

Combining the result from **Steps 1, 2, 3** we obtain that

$$z = Az = Bz = Sz = Tz$$

Therefore z is a common fixed point of A, B, S and T.

Case 2. S is continuous

As S is continuous and (A, S) is semi-compatible, we have.

$$SAx_{2n} \rightarrow Sz, \; S^2x_{2n} \rightarrow Sz, \; ASx_{2n} \rightarrow Sz \qquad (6)$$

Thus, $\lim_{n\to\infty} SAx_{2n} = \lim_{n\to\infty} ASx_{2n} = Sz$

we prove Sz = z, suppose on the contrary that Sz $\neq$ z.

Step 4. Putting $x = Sx_{2n}$, $y = x_{2n+1}$ in (d)

$aM(ASx_{2n}, Bx_{2n+1}, t) – bM(SSx_{2n}, Tx_{2n+1}, t)$

$\geq \phi\{M(SSx_{2n}, Tx_{2n+1}, t), M(SSx_{2n}, ASx_{2n}, t), M(SSx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, ASx_{2n}, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$

$\Rightarrow$ aM(Sz, z, t) – bM(Sz, z, t)

$\geq$ ϕ {M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, Sz, t), M(z, z, t)}

$\geq$ ϕ {M(Sz, z, t), 1, M(Sz, z, t), M(Sz, z, t), 1}

$>$ ϕ {M(Sz, z, t), M(Sz, z, t), 2M(Sz, z, t), M(Sz, z, t), M(Sz, z, t)}

$\Rightarrow$ (a - b) M(Sz, z, t) $>$ M(Sz, z, t)

which is a contradiction. Hence Sz = z.

Step 5. By putting x = z, $y = x_{2n+1}$ in (d)

$aM(Az, Bx_{2n+1}, t) – bM(Sz, Tx_{2n+1}, t)$

$\geq \phi \{M(Sz, Tx_{2n+1}, t), M(Sz, Az, t), M(Sz, Bx_{2n+1}, t), M(Tx_{2n+1}, Az, t), M(Tx_{2n+1}, Bx_{2n+1}, t)\}$

$\Rightarrow aM(Az, z, t) – bM(z, z, t) \geq \phi \{M(z, z, t), M(z, Az, t), M(z, z, t), M(z, Az, t), M(z, z, t)\}$

$\Rightarrow aM(Az, z, t) – b \geq \phi \{1, M(Az, z, t), 1, M(Az, z, t), 1)$

$\Rightarrow aM(Az, z, t) –b (Az, z, t) > \phi \{M(Az, z, t), M(Az, z, t), 2M(Az, z, t), M(Az, z, t), M(Az, z, t)\}$

$\Rightarrow (a-b) M(Az, z, t) > M(Az, z, t)$

Which is a contradiction.

Hence Az = z = Sz.

Also Bz = Tz = z follows from **step 1, 2** we get that

z = Az = Bz = Sz = Tz.

Hence z is a common fixed point of A, B, S and T.

Uniqueness

Let z_1 and z_2 be two common fixed points of the A, B, S and T.

Then $z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1$ and $z_2 = Az_2 = Bz_2 = Sz_2 = Tz_2$.

Suppose $z_1 \neq z_2$. From (d), we have

$aM(Az_1, Bz_2, t) – bM(Sz_1, Tz_2, t) \geq \phi \{M(Sz_1, Tz_2, t), M(Sz_1, Az_1, t), M(Sz_1, Bz_2, t), M(Tz_2, Az_1, t),$

$M(Tz_2, Bz_2, t)\}$

$\Rightarrow aM(z_1, z_2, t) – bM(z_1, z_2, t) \geq \phi \{M(z_1, z_2, t), M(z_1, z_1, t), M(z_1, z_2, t), M(z_2, z_1, t), M(z_2, z_2, t)\}$

$\geq \phi \{M(z_1, z_2, t), 1, M(z_1, z_2, t), M(z_2, z_1, t), 1\}$

$> \phi \{M(z_1, z_2, t), M(z_1, z_2, t), 2M(z_1, z_2, t), M(z_2, z_2, t), M(z_1, z_2, t)\}$

$\Rightarrow (a-b) M(z_1, z_2, t) > M(z_1, z_2, t)$

which is a contradiction. Hence $z_1 = z_2$.

Thus z is a unique common fixed point of A, B, S and T.

By Theorem (1) the self mappings A, B, QS and PT have a unique

common fixed point i.e. Az = Bz = QSz = PTz = z. (7)

Similarly from (4) and (5)

$$\{Ax_{2n}\} \to z, \quad \{QSx_{2n}\} \to z, \tag{8}$$

$$\{PTx_{2n+1}\} \to z, \ \{Bx_{2n+1}\} \to z. \tag{9}$$

By putting x = Qz and y = x_{2n+1} in (d) and on assuming Qz $\neq$ z, we have

$$aM(AQz, Bx_{2n+1}, t) - bM(QSQz, PTx_{2n+1}, t)$$

$$\geq \phi\,\{M(QSQz, PTx_{2n+1}, t), M(QSQz, AQz, t), M(QSQz, Bx_{2n+1}, t), M(PTQz, AQz, t), M(PTx_{2n+1}, Bx_{2n+1}, t)\}.$$

As AQ = QA, QS = SQ, we have

$$aM(QAz, Bx_{2n+1}, t) - bM(QQSz, PTx_{2n+1}, t)$$

$$\geq \phi\,\{M(QQSz, PTx_{2n+1}, t), M(QQSz, QAz, t), M(QQSZ, Bx_{2n+1}, t), M(Q\,PTz, QAz, t), M(PTx_{2n+1}, Bx_{2n+1}, t)\}$$

Taking limit as n $\to \infty$ using (6), and (9) we have

$$aM(Qz, z, t) - bM(Qz, z, t)$$

$$\geq \phi\,\{M(Qz, z, t), M(Qz, Qz, t), M(Qz, z, t), M(Qz, Qz, t), M(z, z, t)\}$$

$$\Rightarrow (a-b)\, M(Qz, z, t) \geq \phi\,\{M(Qz, z, t), 2M(Qz, z, t), M(Qz, z, t), M(Qz, z, t), M(Qz, z, t)\}$$

$$\Rightarrow (a-b)\, M(Qz, z, t) > M(Qz, z, t)$$

which is a contradiction. Hence Qz = z.

Now QSz = SQz = Sz, gives Sz = z = Qz.

Again assuing Pz $\neq$ z and by putting x = x_{2n} and y = Pz in (4)

$$aM(Ax_{2n}, BPz, t) - bM(QSx_{2n}, PTPz)$$

$$\geq \phi\,\{(M(QSx_{2n}, PTPz, t), M(QSx_{2n}, Ax_{2n}, t), M(QSx_{2n}, BPz, t), M(PTx_{2n}, Ax_{2n}, t), M(PTPz, BPz, t)\}.$$

As PT = TP and BP = PB, we get that

$$aM(Ax_{2n}, PBz, t) - bM(QSx_{2n}, PPTz, t)$$

$\geq \phi$ {M(QSx_{2n}, PPTz, t), M(QSx_{2n}, Ax_{2n}, t), M(QSx_{2n}, PBz, t), M(PTx_{2n}, Ax_{2n}, t), M(PPTz, PBz, t)}

Taking n $\to \infty$, we get

aM(z, Pz, t) – bM(z, Pz, t)

$\geq \phi$ {M(z, Pz, t), M(z, z, t), M(z, Pz, t), M(z, z, t), M(Pz, z, t)}

$\Rightarrow$ (a-b) M(z, pz, t) $\geq \phi$ {M(z, pz, t), 2M(z, Pz, t), M(z, Pz, t), M(z, Pz, t), M(Pz, z, t)}

(a-b) M(z, Pz, t) > M(Pz, z, t)

which is a contradiction. Hence Pz = z.

Now, PTz = TPz = Tz gives pz = Tz = z.

Combining all these result, we obtain that

Az = Bz = Sz = Tz = Pz = Qz = z .

Hence z is a common fixed point of the mapping A, B, S, T, P and Q.

We can prove uniqueness of z on the same line as in theorem 5.2.

This complete the proof.

References

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