Square Divisor Cordial, Cube Divisor Cordial and Vertex Odd Divisor Cordial Labeling of Graphs

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Abstract

In this paper we investigate square divisor cordial labeling, cube divisor cordial labeling and vertex odd divisor cordial labeling of $K_{1,1,n}$, $K_2 + mK_1$, Umbrella, $C_n^{(t)}$, $G =< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$, arbitrary supersubdivision of $K_{1,n}$, the graph obtained by duplication of an edge in $K_{1,n}$ and $K_{2,n} \odot u_2(K_1)$.
Key words: Square divisor cordial labeling, Cube divisor cordial labeling,Vertex odd divisor cordial labeling.
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1 Introduction

Here we consider simple, finite, connected and undirected graph $G = (V, E)$ with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Gross and Yellen[2]. The most recent findings on various graph labeling techniques can be found in Gallian[1]. The brief summary of definitions and other information which are necessary for the present investigation are provided below.

1.1 Definitions

Definition 1.1.1. *[8] Let $G = (V, E)$ be a graph. A bijection $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ is said to be divisor cordial labeling of graph G if the induced function $f^* : E(G) \to \{0, 1\}$ defined by*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)|f(v) \text{ or } f(v)|f(u). \\ 0; & \text{otherwise}. \end{cases}$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.
A graph which admits divisor cordial labeling is called a divisor cordial graph.

Definition 1.1.2. *[4] Let $G = (V, E)$ be a graph, $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ be a bijection and the induced function $f^* : E(G) \to \{0, 1\}$ is defined by*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } [f(u)]^2|f(v) \text{ or } [f(v)]^2|f(u). \\ 0; & \text{otherwise}. \end{cases}$$

Then function f is called square divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.
A graph which admits square divisor cordial labeling is called square divisor cordial graph.

Definition 1.1.3. *[3] Let $G = (V, E)$ be a graph, $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ be a bijection and the induced function $f^* : E(G) \to \{0, 1\}$ is defined by*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } [f(u)]^3|f(v) \text{ or } [f(v)]^3|f(u). \\ 0; & \text{otherwise}. \end{cases}$$

Then function f is called cube divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.
A graph which admits cube divisor cordial labeling is called cube divisor cordial graph.

Definition 1.1.4. *[5] Let $G = (V, E)$ be a graph, $f : V(G) \to \{1, 3, \dots, 2n - 1\}$ be a bijection and the induced function $f^* : E(G) \to \{0, 1\}$ is defined by*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u)|f(v) \text{ or } f(v)|f(u). \\ 0; & \text{otherwise}. \end{cases}$$

Then function f is called vertex odd divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$.
A graph which admits vertex odd divisor cordial labeling is called vertex odd divisor cordial graph.

Definition 1.1.5. *[1] A graph is called a tripartite graph if we can divide the vertex set of the graph into three disjoint non empty subsets V_1, V_2 and V_3 so that vertices in the same set are not adjacent to each other. The complete tripartite graph with $|V_1| = n_1, |V_2| = n_2, |V_3| = n_3$ is denoted by K_{n_1,n_2,n_3}.*

Definition 1.1.6. *[1] Umbrella is the graph obtained from fan by joining a path P_m to a middle vertex of path P_n in fan F_n. It is denoted by $U(m, n)$.*

Definition 1.1.7. *[1] The joinsum of complete bipartite graphs $< K_{1,n}^{(1)}, K_{1,n}^{(2)} \dots, K_{1,n}^{(t)} >$ is the graph obtained by starting with t copies of $K_{1,n}$ namely $K_{1,n}^{(1)}, K_{1,n}^{(2)} \dots, K_{1,n}^{(t)}$ and joining apex vertex of each pair $K_{1,n}^{(i)}$ and $K_{1,n}^{(i+1)}$ to a new vertex v_i where $1 \leq i \leq k-1$.*

Definition 1.1.8. *[1] Let G be a graph with n vertices and e edges. A graph H is said to be a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq n$ in such a way that the ends of e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from G.*

Definition 1.1.9. *[1] Duplication of an edge $e = uv$ of a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v\} - \{v'\}$ and $N(v') = N(v) \cup \{u\} - \{u'\}$.*

Definition 1.1.10. *[9] Let (V_1, V_2) be the bipartition of $K_{m,n}$, where $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. The graph $K_{m,n} \odot u_i(K_1)$ is defined by attaching a pendant vertex to the vertex u_i for some i.*

Notation 1.1. *[1] The one point union of t (≥ 1) cycles, each of length n is denoted by $C_n^{(t)}$.*

2 Main Results

Theorem 2.1 $K_{1,1,n}$ is square divisor cordial graph for each n.

Proof: Let u and v be the vertices of degree n and $u_1, u_2, \dots, u_n$ be the vertices of degree 2, $|V(K_{1,1,n})| = n + 2$, $|E(K_{1,1,n})| = 2n + 1$.

We define vertex labeling

$f : V(K_{1,1,n}) \to \{1, 2, 3, \dots, n + 2\}$ as follows

$f(u) = 1, f(v) = p$, where p is largest prime number.

The remaining vertices $u_1, u_2, \dots, u_n$ of $K_{1,1,n}$ can be labeled by remaining labels $2, 3, \dots, p-1, p+1, \dots, n$ in any order.

Then we get $e_f(1) = e_f(0) + 1$.

Thus $K_{1,1,n}$ is a square divisor cordial graph.

Illustration 2.1 Square divisor cordial labeling of graph $K_{1,1,6}$ is shown in *Fig. 1* as an illustration for the *Theorem 2.1*.

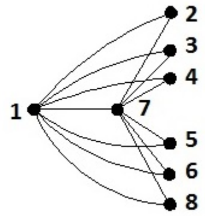

Figure 1: Square divisor cordial labeling of graph $K_{1,1,6}$

Corollary 2.1. *$K_{1,1,n}$ is a cube divisor cordial graph.*

Proof. The vertex labeling function can be defined same as in *Theorem 2.1*. One can observe that the vertex labeling function satisfies condition for cube divisor cordial labeling. Hence $K_{1,1,n}$ is a cube divisor cordial graph. □

Corollary 2.2. *$K_{1,1,n}$ is a vertex odd divisor cordial graph.*

Proof. The vertex labeling function can be defined similar as in *Theorem 2.1*. One can observe that the vertex labeling function satisfies condition for vertex odd divisor cordial labeling. Hence $K_{1,1,n}$ is a vertex odd divisor cordial graph. □

Theorem 2.2 $K_2 + mK_1$ is a square divisor cordial graph.

Proof: Let u and v be the vertices of degree $m + 1$ and $u_1, u_2, \dots, u_m$ be the vertices of degree 2 in $K_2 + mK_1$. Here $|V(K_2 + mK_1)| = m + 2$ and $|E(K_2 + mK_1)| = 2m + 1$.

We define vertex labeling

$f : V(K_2 + mK_1) \to \{1, 2, 3, \dots, m + 2\}$ as follows.

$f(u) = 1, f(v) = p$, where p is largest prime number and label the remaining labels to the remaining vertices $u_1, u_2, \dots, u_m$ in any order.

Here $|e_f(1)| = m + 1$ and $|e_f(0)| = m$.

Clearly it satisfies the condition $|e_f(1) - e_f(0)| \leq 1$.

Thus $K_2 + mK_1$ is a square divisor cordial graph.

Illustration 2.2 Square divisor cordial labeling of the graph $G = K_2 + 7K_1$ is shown in *Fig. 2* as an illustration for the *Theorem 2.2*.

Corollary 2.3. *$K_2 + mK_1$ is a cube divisor cordial graph.*

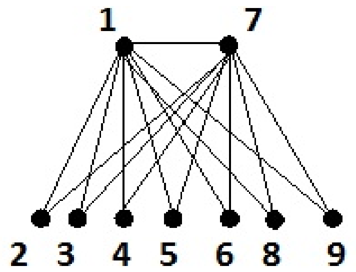

Figure 2: Square divisor cordial labeling of the graph $G = K_2 + 7K_1$

Corollary 2.4. *$K_2 + mK_1$ is a vertex odd divisor cordial graph.*

Theorem 2.3 Umbrella $U(n,3)$ is a square divisor cordial graph.

Proof: Let u be the vertex of degree n, v be the vertex of degree 2, w be the pendant vertex and $u_1, u_2, \dots, u_n$ be the vertices of path P_n. Here $|V(U(n,3))| = n+3$ and $|E(U(n,3))| = 2n+1$.
We define vertex labeling
$f : V(U(n,3)) \to \{1, 2, 3, \dots, n+3\}$ as per the following cases.
Case 1: n is odd.
$f(u) = 1, f(u_i) = i+1, 1 \leq i \leq n.$
$f(v) = n+2, f(w) = n+3.$
Here $e_f(1) = e_f(0) - 1.$
Case 2: n is even.
$f(u) = 1, f(u_i) = i+1, 1 \leq i \leq n.$
$f(v) = n+2, f(w) = n+3.$
Here $e_f(1) = e_f(0) + 1.$
In each case we have $|e_f(0) - e_f(1)| \leq 1$. Hence Umbrella $U(n,3)$ is a square divisor cordial graph.

Illustration 2.3 Square divisor cordial labeling of the graph $U(5,3)$ and $U(6,3)$ is shown in *Fig. 3 and Fig. 4* respectively as an illustration for *Theorem 2.3*.

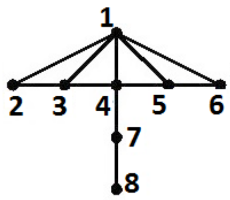

Figure 3: Square divisor cordial labeling of the graph $U(5,3)$

Corollary 2.5. *Umbrella $U(n,3)$ is a cube divisor cordial graph.*

Corollary 2.6. *Umbrella $U(n,3)$ is a vertex odd divisor cordial graph.*

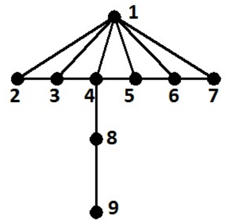

Figure 4: Square divisor cordial labeling of the graph $U(6,3)$

Theorem 2.4 $C_4^{(t)}$ is a square divisor cordial graph.

Proof: Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)}, 1 \leq i \leq t$ be the vertices of $C_4^{(t)}$ with $v_1^{(1)} = v_1^{(2)} = v_1^{(3)} = \dots = v_1^{(t)} = v$.
$|V(C_4^{(t)})| = 3t+1, |E(C_4^{(t)})| = 4t.$
We define vertex labeling
$f : V(C_4^{(t)}) \to \{1, 2, 3, \dots, 3t+1\}$ as follows.
$f(v) = 1, f(v_2^{(i)}) = 3i-1, f(v_3^{(i)}) = 3i,$
$f(v_4^{(i)}) = 3i+1, 1 \leq i \leq t.$
Here $e_f(1) = e_f(0) = 2t.$
Thus $C_4^{(t)}$ is a square divisor cordial graph.

Illustration 2.4 Square divisor cordial labeling of the graph $C_4^{(4)}$ is shown in *Fig. 5* as an illustration for *Theorem 2.4*.

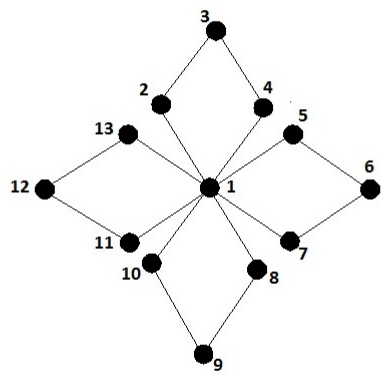

Figure 5: Square divisor cordial labeling of the graph $C_4^{(4)}$

Corollary 2.7. *$C_4^{(t)}$ is a cube divisor cordial graph.*

Corollary 2.8. *$C_4^{(t)}$ is a vertex odd divisor cordial graph.*

Theorem 2.5 The graph $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a square divisor cordial graph.

Proof: Let $G =< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$.
Let $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \dots v_n^{(1)}$ be the pendant vertices of $K_{1,n}^{(1)}$ and $v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \dots v_n^{(2)}$ be the pendant vertices

of $K_{1,n}^{(2)}$. Let $v_0^{(1)}$ and $v_0^{(2)}$ be the apex vertices of $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ respectively which are adjacent to a new common vertex say x.
Here $|V(G))| = 2n+3$, $|E(G)| = 2n+2$.
Now assign label 1 to $v_0^{(1)}$ and label $v_0^{(2)}$ by the largest prime number p such that $p \leq 2n+3$. Label the vertices $v_1^{(1)}, v_2^{(1)}, v_3^{(1)}, \ldots v_n^{(1)}, v_1^{(2)}, v_2^{(2)}, v_3^{(2)}, \ldots v_n^{(2)}, x$ by the remaining labels in any order.
Then we get $|e_f(1)| = |e_f(0)| = n+1$.
Thus $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ is a square divisor cordial graph.
Illustration 2.5 Square divisor cordial labeling of the graph $G =< K_{1,5}^{(1)}, K_{1,5}^{(2)} >$ is shown in *Fig. 6* as an illustration for *Theorem 2.5*.

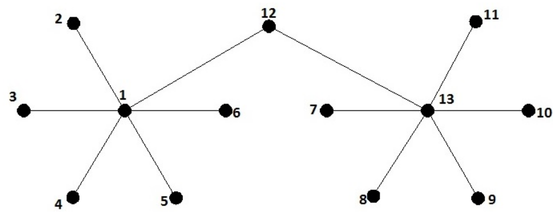

Figure 6: Square divisor cordial labeling of the graph $G =< K_{1,5}^{(1)}, K_{1,5}^{(2)} >$

Corollary 2.9. *The graph* $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ *is a cube divisor cordial graph.*

Corollary 2.10. *The graph* $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ *is a vertex odd divisor cordial graph.*

Theorem 2.6 The graph Arbitrary supersubdivision of $K_{1,n}$ is a square divisor cordial graph.
Proof: Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of $K_{1,n}$ and let e_i denote the edge v_0v_i of $K_{1,n}$ for $1 \leq i \leq n$.
Let G be the graph obtained by arbitrary supersubdivision of $K_{1,n}$ in which each edge e_i of $K_{1,n}$ is replaced by a complete bipartite graph K_{2,m_i} and u_{ij} be the vertices of m_i-vertices part, $1 \leq i \leq n, 1 \leq j \leq m_i$.
Observe that G has $2(m_1, m_2, \ldots, m_n)$ edges.
We define vertex labeling
$f : V(G) \to \{1, 2, 3, \ldots, n+2\}$ as $f(v_0) = 1$ and label the vertices $v_i, 1 \leq i \leq n$ by the last n consecutive prime numbers.
Assign remaining labels to the remaining vertices $u_{ij}, 1 \leq i \leq n, 1 \leq j \leq m_i$ in any order.
Here $e_f(1) = e_f(0)$. Thus arbitrary supersubdivision of $K_{1,n}$ is a square divisor cordial graph.
Illustration 2.6 Square divisor cordial labeling of arbitrary supersubdivision of $K_{1,4}$ is shown in *Fig. 7* as an illustration for *Theorem 2.6*.

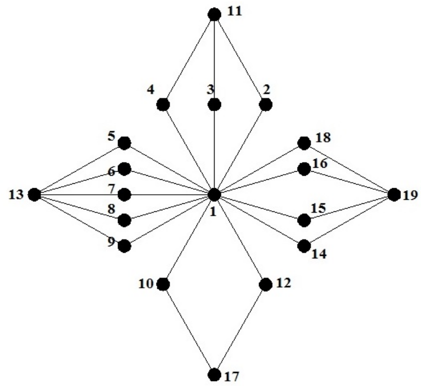

Figure 7: Square divisor cordial labeling of arbitrary supersubdivision of $K_{1,4}$

Remark 2.1. *Arbitrary supersubdivision of* $K_{1,n}$ *is nothing but the one vertex union of the complete bipartite graphs* K_{2,m_i}*, where* m_i *is arbitrary,* $1 \leq i \leq n$.

Corollary 2.11. *Arbitrary supersubdivision of* $K_{1,n}$ *is a cube divisor cordial graph.*

Corollary 2.12. *Arbitrary supersubdivision of* $K_{1,n}$ *is a vertex odd divisor cordial graph.*

Theorem 2.7 The graph obtained by duplication of an edge in $K_{1,n}$ is a square divisor cordial graph.
Proof: Let v_0 be the apex vertex and $v_1, v_2, \ldots, v_n$ be the successive pendant vertices of $K_{1,n}$.
Let G be the graph obtained by duplication of the edge $e = v_0v_n$ by a new edge $e' = v_0'v_n'$.
Hence in G, $deg(v_0) = n, deg(v_0') = n, deg(v_n) = 1, deg(v_n') = 1$ and $deg(v_i) = 2; 1 \leq i \leq n-1$,
$|V(G)| = n+3$, $|E(G)| = 2n$.
We define vertex labeling
$f : V(G) \to \{1, 2, 3, \ldots, n+3\}$ as follows.
$f(v_0) = 1, f(v_0') = p$, where p is the largest prime number. $f(v_n) = n+2, f(v_n') = n+3, f(v_i) = i+1; 1 \leq i \leq n-1$.
Here $e_f(1) = e_f(0) = n$.
Thus the graph obtained by duplication of an edge in $K_{1,n}$ is a square divisor cordial graph.
Illustration 2.7 Square divisor cordial labeling of the graph obtained by duplication of an edge in $K_{1,8}$ is shown in *Fig. 8* as an illustration for *Theorem 2.7*.

Corollary 2.13. *The graph obtained by duplication of an edge in* $K_{1,n}$ *is a cube divisor cordial graph.*

Corollary 2.14. *The graph obtained by duplication of an edge in* $K_{1,n}$ *is a vertex odd divisor cordial graph.*

Theorem 2.8 $K_{2,n} \odot u_2(K_1)$ is a square divisor cordial graph.
Proof: Let $G = K_{2,n} \odot u_2(K_1)$. Let $V = V_1 \cup V_2$ be the bipartition of $K_{2,n}$ such that $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, \ldots, v_n, w\}$.

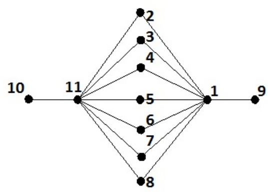

Figure 8: Square divisor cordial labeling of the graph obtained by duplication of an edge in $K_{1,8}$

Let w be the pendant vertex adjacent to u_2 in G.
$|V(G)| = n+3, |E(G| = 2n+1$.
We define vertex labeling
$f : V(G) \to \{1, 2, 3, \ldots, n+3\}$ as follows.
$f(u_1) = 1, f(u_2) = p$, where p is the largest prime number. Label the remaining labels to the remaining vertices $v_1, v_2, \ldots, v_n$ in any order.
Here $e_f(1) = n, e_f(0) = n+1$.
Thus $K_{2,n} \odot u_2(K_1)$ is a square divisor cordial graph.
Illustration 2.8 Square divisor cordial labeling of the graph $G = K_{2,5} \odot u_2(K_1)$ is shown in *Fig. 9* as an illustration for *Theorem 2.8.*

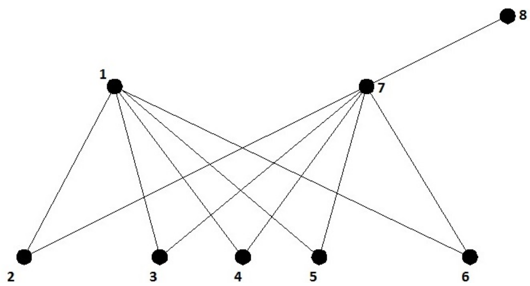

Figure 9: Square divisor cordial labeling of the graph $G = K_{2,5} \odot u_2(K_1)$

Corollary 2.15. *$K_{2,n} \odot u_2(K_1)$ is a cube divisor cordial graph.*

Corollary 2.16. *$K_{2,n} \odot u_2(K_1)$ is a vertex odd divisor cordial graph.*

3 Conclusion

In this paper, we prove several graphs in context of different graph operations admitting square divisor cordial labeling, cube divisor cordial labeling and vertex odd divisor cordial labeling. To investigate analogous results for different graphs is an open area of research. We have observed that many square divisor cordial graphs also admit cube divisor cordial labeling and odd vertex divisor cordial labeling. It is interesting to discuss the natural relation between these labelings, if any. The discussion and further scope of research in this area are left to the reader.

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