Common Coupled Fixed Point Results for Generalized Rational Type Contractions in Complex Valued Metric Spaces

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Abstract

In this paper,we prove a common coupled fixed point results for generalized rational type contractions in complex valued metric spaces which generalize common coupled fixed point theorems due to Marwan Amin Kutbi et al.,[1].

Key words:Complex valued metric space; Coupled fixed point; Common coupled fixed point; Coupled coincidence point;

1 Introduction

Azam et al.[2] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition.Subsequently,Rouzkard and Imdad [3] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize,unify and complement the results of Azam et al.[2].Sintunavarat and Kumam [4] obtained common fixed point results by replacing constant of contractive condition to control functions.Recently,Klin-eam and Suanoom [5] extend the concept of complex valued metric spaces and generalized the results of Azam et al.[2] and Rouzkard and Imad [3].

The concept of coupled fixed point was first introduced by Bhaskar and Laxikantham [10] in 2006.Recently some researchers prove some coupled fixed point theorems in complex valued metric space in [11],[12]. In [1],Marwan Amin Kutbi et al.,gave a common coupled fixed point results for generalized contraction in complex valued metric space and proved following theorem.

Theorem 1.1. *Let (X, d) be a complete complex valued metric space,and let the mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x,y),T(u,v) \preceq \alpha\frac{d(x,u)+d(y,v)}{2} + \frac{\beta d(x,S(x,y)d(u,T(u,v))+\gamma d(u,S(x,y)d(x,T(u,v))}{1+d(x,u)+d(y,v)}$$

for all $x, y, u, v \in X$ and α, β and γ are non negative real with $\alpha+\beta+\gamma < 1$.Then S and T have a unique common coupled fixed point.

Theorem 1.2. *Let (X, d) be a complete complex valued metric space,and let the mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x,y),T(u,v) \preceq \begin{cases} \frac{\alpha(d(x,u)+d(y,v))}{2} + \\ \frac{\beta(d(x,S(x,y))d(S(x,y),T(u,v)))}{d(x,T(u,v))+d(u,S(x,y))+d(x,u)+d(y,v)} & if D \neq 0 \\ 0 & if D = 0. \end{cases}$$

for all $x, y, u, v \in X$,where $D = d(x, T(u, v)) + d(u, S(x, y)) + d(x, u) + d(y, v)$ and α, β are nonnegative reals with $\alpha + \beta < 1$.Then S and T have a unique common coupled fixed point.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \quad iff \quad Re(z_1) \leq Re(z_2), \quad Im(z_1) \leq Im(z_2).$$

Note that $0 \preceq z_1$ and $z_1 \neq z_2$,$z_1 \preceq z_2$ implies $|z_1| < |z_2|$.
The following definition is recently introduced by Azam et al.[2]

Definition 2.1. *Let X be a non empty set.Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies the following conditions:*

1. *$0 \preceq d(x, y)$,for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$;*
3. *$d(x, y) \preceq d(x, z) + d(z, y)$,for all $x, y, z \in X$.*

Then,d is called a complex valued metric on X,and (X, d) is called a complex valued metric space.

Example 2.1. *Let $X = \mathbb{C}$.Define the mapping $d\colon X \times X \to \mathbb{C}$ by*

$$d(x, y) = i|x - y|, \quad \forall \quad x, y \in X.$$

Then,(X, d) is a complex valued metric space.

Definition 2.2. *Let (X, d) be a complex valued metric space.*

1. *A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) := \{y \in X : d(x, y) \prec r\} \subseteq A$.*
2. *A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$,$B(x, r) \cap (A - \{x\}) \neq \phi$.*
3. *A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.*
4. *A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.*
5. *A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.*
6. *A sub-basis for a Hausdorff topology τ on X is a family $F := \{B(x, r) : x \in X \, and \, 0 \prec r\}$.*

Definition 2.3. *Let (X, d) be a complex valued metric space.A sequence $\{x_n\}$ in X is said to be*

1. *convergent to x,if for every $c \in \mathbb{C}$ with $o \prec c$ there is $k \in \mathbb{N}$ such that,for all $n > k$,$d(x_n, x) < c$.we denote this by $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$;*
2. *Cauchy,if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $k \in \mathbb{N}$ such that,for all $n > k$,$d(x_n, x_{n+m}) < c$,where $m \in \mathbb{N}$;*
3. *complete,if every Cauchy sequence in X converges in X.*

In[2],Azam et al.established the following two lemmas.

Lemma 2.2. [2] *Let (X, d) be a complex valued metric space,and let $\{x_n\}$ be a sequence in X.Then,$\{x_n\}$ converges to x if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

Lemma 2.3. [2] *Let (X, d) be a complex valued metric space,and let $\{x_n\}$ be a sequence in X.Then,$\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$,where $m \in \mathbb{N}$.*

Definition 2.4. [10] *An element $(x, y) \in X \times X$ is called a coupled fixed point of $T\colon X \times X \to X$ if*

$$x = T(x, y) \quad y = T(y, x)$$

Definition 2.5. [1] *An element $d(x, y) \in X \times X$ is called a coupled coincidence point of $S, T \colon X \times X \to X$ if*

$$S(x, y) = T(x, y), \quad S(y, x) = T(y, x).$$

Example 2.4. *An element $(x, y) \in X \times X \to X$ is called a coupled coincidence point of $S, T \colon X \times X \to X$ defined as $S(x, y) = x^2y^2$ and $T(x, y) = (\frac{4}{3}(x + y))$ for all $x, y \in X$.Then $(0, 0)$,$(1, 2)$,and $(2, 1)$ are coupled coincidence points of S and T.*

Example 2.5. *Let $X = \mathbb{R}$ and $S, T \colon X \times X \to X$ defined as $S(x, y) = x + y + sin(x + y)$ and $T(x, y) = x + y + xy + cos(x + y)$ for all $x, y \in X$.Then $(0, \frac{\pi}{4})$ and $(\frac{\pi}{4}, 0)$ are coupled coincidence points of S and T.*

Definition 2.6. [1] *An element $(x, y) \in X \times X$ is called a common coupled fixed point of $S, T \colon X \times X \to X$ if*

$$x = S(x, y) = T(x, y) \quad y = S(y, x) = T(y, x).$$

Example 2.6. *Let $X = \mathbb{R}$ and $S, T \colon X \times X$ defined as $S(x, y) = x(\frac{(x+(y-1)^2)}{2})$ and $T(x, y) = x(\sqrt{x^2 + y^2 + 4} - 2)$ for all $x, y \in X$.Then $(0, 0)$,$(1, 2)$ and $(2, 1)$ are common coupled fixed points of S and T.*

The purpose of this paper is to generalize the theorem of Marwan Amin Kutbi et al.,[1] to common coupled fixed point results for generalized rational type contractions in complex valued metric Spaces .

3 Main Results

Theorem 3.1. *Let (X, d) be a complete complex valued metric space,and let the mappings $S, T \colon X \times X \to X$ satisfy*

$$\begin{aligned} d(S(x,y),T(u,v) \preceq\, & a_1 \frac{d(x,u)+d(y,v)}{2} \\ & + a_2 \frac{d(x,S(x,y))d(u,T(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_3 \frac{d(u,S(x,y))d(x,T(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_4 \frac{d(S(x,y),T(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_5 \frac{d(S(x,y),T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_6 \frac{d(u,T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_7 \frac{d(u,S(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_8 \frac{d(u,S(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,S(x,y))} \\ & + a_9 \max\{d(u,S(x,y)), d(S(x,y),T(u,v))\} \end{aligned}$$

for all $x, y, u, v \in X$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_9 < 1$ and $a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1$.Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be arbitrary points in X.Define $x_{2k+1} = S(x_{2k}, y_{2k})$,$y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$,$y_{2k+2} = T(y_{2k+1}, x_{2k+1})$,for $k = 0, 1, \ldots$

Then,

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
&\preceq a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_3 \frac{d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_4 \frac{d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))d(x_{2k}, x_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_5 \frac{d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_6 \frac{d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_7 \frac{d(x_{2k+1}, S(x_{2k}, y_{2k}))d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_8 \frac{d(x_{2k+1}, S(x_{2k}, y_{2k}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k}))} \\
&+ a_9 \max\{d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))\} \\
&= a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_3 \frac{d(x_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_4 \frac{d(x_{2k+1}, x_{2k+2})d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_5 \frac{d(x_{2k+1}, x_{2k+2})d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_6 \frac{d(x_{2k+1}, x_{2k+2}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_7 \frac{d(x_{2k+1}, x_{2k+1})d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_8 \frac{d(x_{2k+1}, x_{2k+1})d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1})} \\
&+ a_9 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}
\end{aligned}$$

which implies that

$$(1-(a_2+a_4+a_5+a_6+a_9))|d(x_{2k+1},x_{2k+2})| \leq a\frac{|d(x_{2k},x_{2k+1})|}{2} + a_1\frac{|d(y_{2k},y_{2k+1})|}{2}$$

$$|d(x_{2k+1},x_{2k+2})| \leq a_1\frac{|d(x_{2k},x_{2k+1})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} + a_1\frac{|d(y_{2k},y_{2k+1})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} \tag{1}$$

Proceeding similarly one can prove that

$$|d(y_{2k+1},y_{2k+2})| \leq a_1\frac{|d(y_{2k},y_{2k+1})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} + a_1\frac{|d(x_{2k},x_{2k+1})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} \tag{2}$$

Adding (1) and (2),we get

$$\begin{aligned}|d(x_{2k+1},x_{2k+2})| + |d(y_{2k+1},y_{2k+2})| &\leq \frac{a_1}{(1-(a_2+a_4+a_5+a_6+a_9))} [|d(x_{2k},x_{2k+1})| + |d(y_{2k},y_{2k+1})|] \\ &= k[|d(x_{2k},x_{2k+1})| + |d(y_{2k},y_{2k+1})|].\end{aligned}$$

where $k = \frac{a_1}{(1-(a_2+a_4+a_5+a_6+a_9))} < 1$.
Also,

$$|d(x_{2k+2},x_{2k+3})| \leq a_1\frac{|d(x_{2k+1},x_{2k+2})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} + a_1\frac{|d(y_{2k+1},y_{2k+2})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} \tag{3}$$

$$|d(y_{2k+2},y_{2k+3})| \leq a_1\frac{|d(y_{2k+1},y_{2k+2})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} + a_1\frac{|d(x_{2k+1},x_{2k+2})|}{2(1-(a_2+a_4+a_5+a_6+a_9))} \tag{4}$$

Adding (3) and (5),we get

$$\begin{aligned}|d(x_{2k+2},x_{2k+3})| + |d(y_{2k+2},y_{2k+3})| &\leq \frac{a_1}{(1-(a_2+a_4+a_5+a_6+a_9))} [|d(x_{2k+1},x_{2k+2})| + |d(y_{2k+1},y_{2k+2})|] \\ &= k[|d(x_{2k+1},x_{2k+2})| + |d(y_{2k+1},y_{2k+2})|] \\ &\leq k^2[|d(x_{2k},x_{2k+1})| + |d(y_{2k},y_{2k+1})|].\end{aligned}$$

Continuing this way,we have

$$\begin{aligned}|d(x_n,x_{n+1})|+|d(y_n,y_{n+1})| &\leq k[|d(x_{n-1},x_n)|+|d(y_{n-1},y_n)|]\\ &\leq k^2[|d(x_{n-2},x_{n-1})|+|d(y_{n-2},y_{n-1})|]\\ &\leq \cdots \leq k^n[|d(x_0,x_1)|+|d(y_0,y_1)|]\end{aligned}$$

Now if $|d(x_n,x_{n+1})|+|d(y_n,y_{n+1})| = \delta_n$,then

$$\delta_n \leq k\delta_{n-1} \leq \cdots \leq k^n\delta_0.$$

Without loss of generality,we take $m > n$.Since $0 \leq k < 1$,so we get

$$\begin{aligned}|d(x_n,x_m)|+|d(y_n,y_m)| &\leq |d(x_n,x_{n+1})|+|d(y_n,y_{n+1})|\\ &+|d(x_{n+1},x_{n+2})|+|d(y_{n+1},y_{n+2})|+\dots\\ &+|d(x_{m-1},x_m)|+|d(y_{m-1},y_m)|\\ &\leq [k^n\delta_0+k^{n+1}\delta_0+\cdots+k^{m-1}\delta_0]\\ &\leq k^n[1+k+k^2+\dots]\delta_0\\ &= \frac{k^n}{1-k}\delta_0 \to 0 \quad as \quad n\to\infty.\end{aligned}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X.Since X is complete,there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.We now show that $x = S(x,y)$ and $y = S(y,x)$.We suppose on the contrary that $x \neq S(x,y)$ and $y \neq S(y,x)$ so that $0 \prec d(x,S(x,y)) = l_1$ and $0 \prec d(y,S(y,x)) = l_2$;we would then have

$$
\begin{aligned}
l_1 = d(x, S(x,y)) &\preceq d(x, x_{2k+2}) + d(x_{2k+2}, S(x,y)) \\
&\preceq d(x, x_{2k+2}) + d(S(x,y), T(x_{2k+1}, y_{2k+1})) \\
&\preceq d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x, S(x,y)) d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_3 \frac{d(x_{2k+1}, S(x,y)) d(x, T(x_{2k+1}, y_{2k+1}))}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_4 \frac{d(S(x,y), T(x_{2k+1}, y_{2k+1})) d(x, x_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_5 \frac{d(S(x,y), T(x_{2k+1}, y_{2k+1})) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_6 \frac{d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_7 \frac{d(x_{2k+1}, S(x,y)) d(x, x_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_8 \frac{d(x_{2k+1}, S(x,y)) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_9 \max\{d(x_{2k+1}, S(x,y)), d(S(x,y), T(x_{2k+1}, y_{2k+1}))\} \\
&= d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x, S(x,y)) d(x_{2k+1}, x_{2k+2})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_3 \frac{d(x_{2k+1}, x_{2k+1}) d(x_{2k}, x_{2k+2})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_4 \frac{d(S(x,y), x_{2k+2}) d(x, x_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_5 \frac{d(S(x,y), x_{2k+2}) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_6 \frac{d(x_{2k+1}, x_{2k+2}) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_7 \frac{d(x_{2k+1}, S(x,y)) d(x, x_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_8 \frac{d(x_{2k+1}, S(x,y)) d(y, y_{2k+1})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))} \\
&+ a_9 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|l_1| \leq |d(x, x_{2k+2})| &+ a_1 \frac{|d(x, x_{2k+1})| + |d(y, y_{2k+1})|}{2} \\
&+ a_2 \frac{|d(x, S(x,y))||d(x_{2k+1}, x_{2k+2})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_3 \frac{|d(x_{2k+1}, x_{2k+1})||d(x_{2k}, x_{2k+2})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_4 \frac{|d(S(x,y), x_{2k+2})||d(x, x_{2k+1})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_5 \frac{|d(S(x,y), x_{2k+2})||d(y, y_{2k+1})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_6 \frac{|d(x_{2k+1}, x_{2k+2})||d(y, y_{2k+1})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_7 \frac{|d(x_{2k+1}, S(x,y))||d(x, x_{2k+1})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_8 \frac{|d(x_{2k+1}, S(x,y))||d(y, y_{2k+1})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x,y))|} \\
&+ a_9 \max\{|d(x_{2k+1}, x_{2k+1})|, |d(x_{2k+1}, x_{2k+2})|\}
\end{aligned}
$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent to x and y,therefore by taking limit as $k \to \infty$ we get $|l_1| \leq 0$.Which is contradiction,so $|d(x, S(x,y))| = 0 \Rightarrow x = S(x,y)$. Similarly we can prove that $y = S(x,y)$.Also we can prove that $x = T(x,y)$ and $y = T(y,x)$.Thus (x,y) is a common coupled fixed point of S and T.
We now show that S and T have a unique common coupled fixed point.For this,assume that $(x_1, y_1) \in X \times X$ is a second common coupled fixed point of S and T.Then

$$
\begin{aligned}
d(x,x_1) &= d(S(x,y),T(x_1,y_1))\\
&\preceq a_1\frac{d(x,x_1)+d(y,y_1)}{2}\\
&+a_2\frac{d(x,S(x,y))d(x_1,T(x_1,y_1))}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_3\frac{d(x_1,S(x,y))d(x,T(x_1,y_1))}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_4\frac{d(S(x,y),T(x_1,y_1))d(x,x_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_5\frac{d(S(x,y),T(x_1,y_1))d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_6\frac{d(x_1,T(x_1,y_1))d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_7\frac{d(x_1,S(x,y))d(x,x_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_8\frac{d(x_1,S(x,y))d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,S(x,y))}\\
&+a_9\max\{d(x_1,S(x,y)),d(S(x,y),T(x_1,y_1))\}\\
&= a_1\frac{d(x,x_1)+d(y,y_1)}{2}\\
&+a_2\frac{d(x,x)d(x_1,x_1))}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_3\frac{d(x_1,x)d(x,x_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_4\frac{d(x,x_1)d(x,x_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_5\frac{d(x,x_1)d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_6\frac{d(x_1,x_1)d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_7\frac{d(x_1,x)d(x,x_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_8\frac{d(x_1,x)d(y,y_1)}{1+d(x,x_1)+d(y,y_1)+d(x_1,x)}\\
&+a_9\max\{d(x_1,x),d(x,x_1)\}
\end{aligned}
$$

Thus

$$|d(x,x_1)| \leq a_1 \frac{|d(x,x_1)| + |d(y,y_1)|}{2} + a_3 \frac{|d(x_1,x)||d(x,x_1)|}{|1+2d(x,x_1)+d(y,y_1)|} + a_4 \frac{|d(x,x_1)||d(x,x_1)|}{|1+2d(x,x_1)+d(y,y_1)|} + a_5 \frac{|d(x,x_1)||d(y,y_1)|}{|1+2d(x,x_1)+d(y,y_1)|} + a_7 \frac{|d(x_1,x)||d(x,x_1)|}{|1+2d(x,x_1)+d(y,y_1)|} + a_8 \frac{|d(x_1,x)||d(y,y_1)|}{|1+2d(x,x_1)+d(y,y_1)|} + a_9|d(x_1,x)|$$

Since $|1 + 2d(x, x_1) + d(y, y_1)| > |d(x, x_1)$,so we get

$$(1 - \frac{a_1}{2} - a_3 - a_4 - a_5 - a_7 - a_8 - a_9)|d(x,x_1)| \leq a_1 \frac{|d(y,y_1)|}{2}$$

$$|d(x,x_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)}|d(y,y_1)| \tag{5}$$

Similarly,

$$|d(y,y_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)}|d(x,x_1)| \tag{6}$$

Adding (5) and (6),we get

$$|d(x,x_1)| + |d(y,y_1)| \leq \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} [|d(y,y_1)| + |d(x,x_1)|]$$

$$[1 - \frac{a_1}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)}][|d(y,y_1)| + |d(x,x_1)|] \leq 0$$
$$\frac{2(1 - a_1 - a_3 - a_4 - a_5 - a_7 - a_8 - a_9)}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)}[|d(y,y_1)| + |d(x,x_1)|] \leq 0.$$

Since $a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1$.
Therefore

$$\frac{2(1 - a_1 - a_3 - a_4 - a_5 - a_7 - a_8 - a_9)}{(2 - a_1 - 2a_3 - 2a_4 - 2a_5 - 2a_7 - 2a_8 - 2a_9)} > 0$$

Hence
$[|d(y, y_1)| + |d(x, x_1)|] \leq 0$.
Which implies that $x = x_1$ and $y = y_1 \Rightarrow (x, y) = (x_1, y_1)$.
Thus,S and T have unique common coupled fixed point. □

Corollary 3.2. *Let (X, d) be a complete complex valued metric space,and let the mappings $T\colon X \times X \to X$ satisfy*

$$\begin{aligned} d(T(x,y),T(u,v) \preceq a_1 &\frac{d(x,u)+d(y,v)}{2} \\ &+ a_2 \frac{d(x,T(x,y))d(u,T(u,v))}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_3 \frac{d(u,T(x,y))d(x,T(u,v))}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_4 \frac{d(T(x,y),T(u,v))d(x,u)}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_5 \frac{d(T(x,y),T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_6 \frac{d(u,T(u,v))d(y,v)}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_7 \frac{d(u,T(x,y))d(x,u)}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_8 \frac{d(u,T(x,y))d(y,v)}{1+d(x,u)+d(y,v)+d(u,T(x,y))} \\ &+ a_9 \max\{d(u,T(x,y)),d(T(x,y),T(u,v))\} \end{aligned}$$

for all $x, y, u, v \in X$ and $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \geq 0$ with $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_9 < 1$ and $a_1 + a_3 + a_4 + a_5 + a_7 + a_8 + a_9 < 1$.Then S and T have a unique common coupled fixed point.

Proof. The proof follows from Theorem 3.1 by taking $S = T$. □

Theorem 3.3. *Let (X, d) be a complete complex valued metric space,and let the mappings $S, T\colon X \times X \to X$ satisfy*

$$\begin{aligned} d(S(x,y),T(u,v) \preceq a_1 &\frac{d(x,u)+d(y,v)}{2} \\ &+ a_2 \frac{d(x,S(x,y))d(S(x,y),T(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))+d(x,T(u,v))} \\ &+ a_3 \max\{d(u,S(x,y)),d(S(x,y),T(u,v))\} \end{aligned}$$

for all $x, y, u, v \in X$ and $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$.Then S and T have a unique common coupled fixed point.

Proof. Take two arbitrary points x_0, y_0 in X.Define $x_{2k+1} = S(x_{2k}, y_{2k})$,$y_{2k+1} = S(y_{2k}, x_{2k})$,$x_{2k+2} = T(x_{2k+1}, y_{2k+1})$ and $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$ for $k =$

$0, 1, 2 \ldots$ .Then,

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\
&\preceq a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, S(x_{2k}, y_{2k})) + d(x_{2k}, T(x_{2k+1}, y_{2k+1}))} \\
&+ a_3 \max\{d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))\} \\
&= a_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} \\
&+ a_2 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, x_{2k+1}) + d(x_{2k}, x_{2k+2})} \\
&+ a_3 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}
\end{aligned}$$

$$\begin{aligned}
|d(x_{2k+1}, x_{2k+2})| &\leq a_1 \frac{|d(x_{2k}, x_{2k+1})| + |d(y_{2k}, y_{2k+1})|}{2} \\
&+ a_2 \frac{|d(x_{2k}, x_{2k+1})||d(x_{2k+1}, x_{2k+2})|}{|1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k}, x_{2k+2})|} \\
&+ a_3 |d(x_{2k+1}, x_{2k+2})|
\end{aligned}$$

Since $|1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k}, x_{2k+2})| > |d(x_{2k}, x_{2k+1})|$,so we get

$$|d(x_{2k+1}, x_{2k+2})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k}, x_{2k+1})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| \tag{7}$$

Similarly we can prove

$$|d(y_{2k+1}, y_{2k+2})| \leq \frac{a_1}{2(1 - a_2 - a_3)} |d(y_{2k}, y_{2k+1})| + \frac{a_1}{2(1 - a_2 - a_3)} |d(x_{2k}, x_{2k+1})| \tag{8}$$

Adding (7) and (8),we get

$$\begin{aligned}
|d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})| &\leq \frac{a_1}{(1 - a_2 - a_3)} [|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})|] \\
&= k[|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})|]
\end{aligned}$$

where $k = \frac{a_1}{(1-a_2-a_3)} < 1$. Also

$$\begin{aligned} d(x_{2k+2}, x_{2k+3}) &= d(S(x_{2k+1}, y_{2k+1}), T(x_{2k+2}, y_{2k+2})) \\ &\preceq a_1 \frac{d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})}{2} \\ &+ a_2 \frac{d(x_{2k+1}, S(x_{2k+1}, y_{2k+1}))d(S(x_{2k+1}, y_{2k+1}), T(x_{2k+2}, y_{2k+2}))}{1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) + d(x_{2k+2}, x_{2k+2}) + d(x_{2k+1}, T(x_{2k+2}, y_{2k+2}))} \\ &+ a_3 \max\{d(x_{2k+2}, S(x_{2k+1}, y_{2k+1})), d(S(x_{2k+1}, y_{2k+1}), T(x_{2k+2}, y_{2k+2}))\} \\ &= a_1 \frac{d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})}{2} \\ &+ a_2 \frac{d(x_{2k+1}, x_{2k+2})d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) + d(x_{2k+2}, x_{2k+2}) + d(x_{2k+1}, x_{2k+3})} \\ &+ a_3 \max\{d(x_{2k+2}, x_{2k+2}), d(x_{2k+2}, x_{2k+3})\} \end{aligned}$$

$$\begin{aligned} |d(x_{2k+2}, x_{2k+3})| &\leq a_1 \frac{|d(x_{2k+1}, x_{2k+2})| + |d(y_{2k+1}, y_{2k+2})|}{2} \\ &+ a_2 \frac{|d(x_{2k+1}, x_{2k+2})||d(x_{2k+2}, x_{2k+3})|}{|1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) + d(x_{2k+1}, x_{2k+3})|} \\ &+ a_3 d(x_{2k+2}, x_{2k+3})| \end{aligned}$$

Since $|1+d(x_{2k+1}, x_{2k+2})+d(y_{2k+1}, y_{2k+2})+d(x_{2k+1}, x_{2k+3})| > |d(x_{2k+1}, x_{2k+2})|$,so we get

$$|d(x_{2k+2}, x_{2k+3})| \leq \frac{a_1}{2(1-a_2-a_3)}|d(x_{2k+1}, x_{2k+2})| + \frac{a_1}{2(1-a_2-a_3)}|d(y_{2k+1}, y_{2k+2})| \tag{9}$$

$$|d(y_{2k+2}, y_{2k+3})| \leq \frac{a_1}{2(1-a_2-a_3)}|d(y_{2k+1}, y_{2k+2})| + \frac{a_1}{2(1-a_2-a_3)}|d(x_{2k+1}, x_{2k+2})| \tag{10}$$

Adding (9) and (10),we get

$$\begin{aligned} |d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})| &\leq \frac{a_1}{(1-a_2-a_3)}[|d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})|] \\ &= k[|d(x_{2k+2}, x_{2k+3})| + |d(y_{2k+2}, y_{2k+3})|] \\ &= k^2[|d(y_{2k}, y_{2k+1})| + |d(x_{2k}, x_{2k+1})|] \end{aligned}$$

Continuing the same process,we get

$$\begin{aligned} |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| &\leq k[|d(x_{n-1}, x_n)| + |d(y_{n-1}, y_n)|] \\ &\leq k^2[|d(x_{n-2}, x_{n-1})| + |d(y_{n-2}, y_{n-1})|] \\ &\leq \cdots \leq k^n[|d(x_0, x_1)| + |d(y_0, y_1)|] \end{aligned}$$

If $|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n$.Then $\delta_n \leq k\delta_{n-1} \leq k^2\delta_{n-2} \leq \cdots \leq k^n\delta_0$. Without loss of generality,we take $m > n$.Since $0 \leq k < 1$,so we get

$$\begin{aligned} |d(x_n, x_m)| + |d(y_n, y_m)| &\leq |d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| \\ &+ |d(x_{n+1}, x_{n+2})| + |d(y_{n+1}, y_{n+2})| + \dots \\ &+ |d(x_{m-1}, x_m)| + |d(y_{m-1}, y_m)| \\ &\leq [k^n\delta_0 + k^{n+1}\delta_0 + \dots + k^{m-1}\delta_0] \\ &\leq k^n[1 + k + k^2 + \dots]\delta_0 \\ &= \frac{k^n}{1-k}\delta_0 \to 0 \quad as \quad n \to \infty. \end{aligned}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X.Since X is complete,there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.We now show that $x = S(x, y)$ and $y = S(y, x)$.We suppose on the contrary that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $0 \prec d(x, S(x, y)) = l_1$ and $0 \prec d(y, S(y, x)) = l_2$;we would then have

$$\begin{aligned} l_1 = d(x, S(x, y)) &\preceq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \\ &\preceq d(x, x_{2k+2}) + d(S(x, y), T(x_{2k+1}, y_{2k+1})) \\ &\preceq d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} \\ &+ a_2 \frac{d(x, S(x, y))d(S(x, y), T(x_{2k+1}, y_{2k+1}))}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) + d(x, T(x_{2k+1}, y_{2k+1}))} \\ &+ a_3 \max\{d(x_{2k+1}, S(x, y)), d(S(x, y), T(x_{2k+1}, y_{2k+1}))\} \\ &= d(x, x_{2k+2}) + a_1 \frac{d(x, x_{2k+1}) + d(y, y_{2k+1})}{2} \\ &+ a_2 \frac{d(x, S(x, y))d(S(x, y), x_{2k+2})}{1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) + d(x, x_{2k+2}))} \\ &+ a_3 \max\{d(x_{2k+1}, S(x, y)), d(S(x, y), x_{2k+2})\} \end{aligned}$$

which implies that

$$\begin{aligned} |l_1| = |d(x, S(x, y))| &\leq |d(x, x_{2k+2})| + a_1 \frac{|d(x, x_{2k+1})| + |d(y, y_{2k+1})|}{2} \\ &+ a_2 \frac{|d(x, S(x, y))||d(S(x, y), x_{2k+2})|}{|1 + d(x, x_{2k+1}) + d(y, y_{2k+1}) + d(x_{2k+1}, S(x, y)) + d(x, x_{2k+2})|} \\ &+ a_3 \max\{|d(x_{2k+1}, S(x, y))|, |d(S(x, y), x_{2k+2})|\} \end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are convergent to x and y,therefore by taking limit as $k \to \infty$ we get

$$|d(x, S(x, y))|(1 - a_2 - a_3) \leq 0$$

Since $a_1 + a_2 + a_3 < 1$.
Therefore,

$$1 - a_2 - a_3 > 0.$$

Hence

$$|l_1| = |d(x, S(x, y))| \leq 0.$$

Which is contradiction,so $|d(x, S(x, y))| = 0 \Rightarrow x = S(x, y)$.
Similarly we can prove that $y = S(y, x)$.Also we can prove that $x = T(x, y)$ and $y = T(y, x)$.Thus (x, y) is a common coupled fixed point of S and T.
We now show that S and T have a unique common coupled fixed point.For this,assume that $(x_1, y_1) \in X \times X$ is a second common coupled fixed point of S and T.Then

$$\begin{aligned} d(x, x_1) &= d(S(x, y), T(x_1, y_1)) \\ &\preceq a_1 \frac{d(x, x_1) + d(y, y_1)}{2} \\ &+ a_2 \frac{d(x, S(x, y))d(S(x, y), T(x_1, y_1))}{1 + d(x, x_1) + d(y, y_1) + d(x_1, S(x, y)) + d(x, T(x_1, y_1))} \\ &+ a_3 \max\{d(x_1, S(x, y)), d(S(x, y), T(x_1, y_1))\} \\ &= a_1 \frac{d(x, x_1) + d(y, y_1)}{2} \\ &+ a_2 \frac{d(x, x)d(x, , x_1)}{1 + d(x, x_1) + d(y, y_1) + d(x_1, x) + d(x, x_1))} \\ &+ a_3 \max\{d(x_1, x), d(x, x_1)\} \end{aligned}$$

Thus

$$|d(x, x_1)| \leq a_1 \frac{|d(x, x_1)| + |d(y, y_1)|}{2} + a_3 |d(x, x_1)|$$

Therefore,

$$|d(x, x_1)|(1 - \frac{a_1}{2} - a_3) \leq \frac{a_1}{2} |d(y, y_1)|$$

$$|d(x, x_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} |d(y, y_1)|. \tag{11}$$

Similarly,we can prove that

$$|d(y, y_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} |d(x, x_1)|. \tag{12}$$

Adding (11) and (12),we get

$$|d(x, x_1)| + |d(y, y_1)| \leq \frac{a_1}{2 - 2a_3 - a_1} [|d(x, x_1)| + |d(y, y_1)|]$$

$$(1 - \frac{a_1}{2 - 2a_3 - a_1})[|d(x, x_1)| + |d(y, y_1)|] \leq 0.$$

which is a contradiction because $a_1 + a_2 + a_3 < 1$.Thus,we get $x_1 = x$ and $y_1 = y$,which proves the uniqueness of common coupled fixed point of S and T. □

Corollary 3.4. *Let (X, d) be a complete complex valued metric space,and let the mappings $S\colon X \times X \to X$ satisfy*

$$\begin{aligned} d(S(x,y),S(u,v)) \preceq\, & a_1 \frac{d(x,u)+d(y,v)}{2} \\ & + a_2 \frac{d(x,S(x,y))d(S(x,y),S(u,v))}{1+d(x,u)+d(y,v)+d(u,S(x,y))+d(x,S(u,v))} \\ & + a_3 \max\{d(u,S(x,y)),d(S(x,y),S(u,v))\} \end{aligned}$$

for all $x, y, u, v \in X$ and $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$.Then S has a unique common coupled fixed point.

Proof. The proof follows from Theorem 3.3 by taking $T = S$. □

Example 3.5. *Suppose $X = [0, 1]$. Defined the function $d\colon X \times X \to \mathbb{C}$ by $d(x, y) = i|x - y|, \forall\ x, y \in X$.Clearly (X, d) is complex valued metric space.If We define two mappings $S, T\colon X \times X \to X$,as $S(x, y) = \frac{x+y}{4}$,$T(x, y) = \frac{x+y}{3}$ for each $x, y \in X$.Then it can be proved simply that the maps S and T satisfy the condition of Theorem 3.1 with $a_1 = \frac{1}{7}, a_2 = \frac{1}{6}, a_3 = \frac{1}{14}, a_4 = \frac{1}{15}, a_5 = \frac{1}{17}, a_6 = \frac{1}{16}, a_7 = \frac{1}{18}, a_8 = \frac{1}{19}, a_9 = \frac{1}{26}$.Hence $(0, 0)$ is a unique common coupled fixed point of S and T.*

References

[1] Marwan Amin Kutbi,Akbar Azam,Jamshaid Ahmad,and Cristina *Some Common Coupled fixed point results for generalized contraction in complex valued metric spaces*,J.of Applied Mathematics Vol.2013,Article ID 352927,10 pages.

[2] A. Azam B. Fisher and M. Khan *Common fixed point theorems in complex valued metric spaces* ,Numer.Funct.Anal.Optim.32(3),(2011) 243-253.

[3] F. Rouzkard,M. Imdad *Some common fixed point theorems on complex valued metric spaces*,Comput.Math.Appl.64(6) (2012)1866-1874.

[4] W. Sintunavarat and P. Kumam , *Generalized common fixed point theorems in complex valued metric spaces and applications*, J.Inequal.Appl.84 (2012).

[5] C.Klin-eam,and C.Suanoom,*Some common fixed point theorems for generalized-contractive-type mappings on complex-valued metric spaces*,Abstr.Appl.Anal.(2013) ID 604215.

[6] A.A,Mukheirmer,*Some common fixed point theorems in complex valued b-metric spaces*,The Scientific World Journal Vol.2014 (2014) Article ID 587825,6 pages.

[7] K.P.R.Rao, P.R.Swamy and J.R.Prasad, *A common fixed point theorem in complex valued b-metric spaces*,Bulletin of Mathematics and Statistics Research,1(1)(2013).

[8] R.K.Verma and H.K.Pathak, *Common fixed point theorems using property (E.A) in complex valued metric space*,Thai.J.Math.,Vol.11,No.2(2013),347-355.

[9] Naval Singh,Deepak Singh,Anil Badal and Vishal Joshi,*Fixed point theorems in complex valued metric spaces*,J.of the Egyptian Mathematical Soc.(2016)24,402-409.

[10] T.G.Bhaskar, and V.Laxikanthan*Fixed point theorem in partially ordered cone metric spaces and applications*,Nonlinear Analysis,TMA,65(2006),825-832.

[11] S.M.Kang,M.Kumar,P.Kumar,and S.Kumar*Coupled fixed point theorems in complex valued metric spaces*,In.Journal of Math.Analysis,Vol.7,(2013),no.46,2269-2277.

[12] Savitri,and N.Hooda *A common coupled fixed point theorem in complex valued metric space*,In.Journal of Computer Applications (0975-8887),Vol.109(2015), no.4,January 2015.

[13] M.Sarwar et al.*Common coupled fixed point theorems satisfying rational type contractive conditions in b-metric spaces*,Springerplus(2016),DOI 10.1186/s40064-016-1849-6.