Certain finite double integrals involving biorthogonal polynomial, a general class of polynomials and multivariable I-function

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ABSTRACT
In this paper, we evaluate three finite double integrals involving varous products of biorthogonal polynomials, a general class of polynomial and multivariable I-function with general arguments. The integrals evaluated are quite general in nature and yield a number of new integrals as special cases.

KEYWORDS : I-function of several variables, finite double integral, H-function of several variables , general class of polynomials.

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1.Introduction and preliminaries.

In this paper, we evaluate three finite double integrals involving varous products of biorthogonal polynomials, a general class of polynomial and multivariable I-function defined by Prasad [5] with general arguments. We will study the case of multivariable H-function.
The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha'_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta'_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b'_j,\beta'_j)_{1,q'};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right) \tag{1.1}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(t_1,\cdots,t_r)\prod_{i=1}^{s}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.2}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [5]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) +$$

$$+ \left(\sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r} \beta_{rk}^{(i)}\right) \tag{1.3}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$I(z_1, \cdots, z_r) = 0(|z_1|^{\gamma'_1}, \cdots, |z_r|^{\gamma'_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$I(z_1, \cdots, z_r) = 0(|z_1|, \cdots, |z_r|^{\beta'_s}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$

where $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.4}$$

$$W = (p', q'); \cdots; (p^{(r)}, q^{(r)}); X = (m', n'); \cdots; (m^{(r)}, n^{(r)}) \tag{1.5}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k}); \cdots; (a_{(r-1)k}, \alpha'_{(r-1)k}, \alpha''_{(r-1)k}, \cdots, \alpha^{(r-1)}_{(r-1)k}) \tag{1.6}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k}); \cdots; (b_{(r-1)k}, \beta'_{(r-1)k}, \beta''_{(r-1)k}, \cdots, \beta^{(r-1)}_{(r-1)k}) \tag{1.7}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{rk}, \alpha''_{rk}, \cdots, \alpha^r_{rk}) : \mathfrak{B} = (b_{rk}; \beta'_{rk}, \beta''_{rk}, \cdots, \beta^r_{rk}) \tag{1.8}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \cdots; (a_k^{(r)}, \alpha_k^{(r)})_{1,p^{(r)}}; B' = (b'_k, \beta'_k)_{1,q'}; \cdots; (b_k^{(r)}, \beta_k^{(r)})_{1,q^{(r)}} \tag{1.9}$$

The multivariable I-function write :

$$I(z_1, \cdots, z_r) = I_{U:p_r,q_r;W}^{V;0,n_r;X} \left(\begin{array}{c|c} z_1 & \text{A } ;\mathfrak{A};\text{A'} \\ \cdot & \\ \cdot & \\ \cdot & \\ z_r & \text{B}; \mathfrak{B}; \text{B'} \end{array} \right) \tag{1.10}$$

Chai and Carlitz [1] studied the following pair of biorthogonal polynomials.

$$J_n^{(\alpha,\beta)}(x;k) = \frac{(\alpha+1)_{kn}}{n!} \sum_{j=0}^{n} (-)^j \binom{n}{j} \frac{(\alpha+\beta+n+1)_{kj}}{(\alpha+1)_{kj}} \left(\frac{1-x}{2}\right)^{kj} \tag{1.11}$$

and $$K_n^{(\alpha,\beta)}(x;k) = \frac{1}{n!} \sum_{j=0}^{n} (-)^j \binom{\beta+n}{j} \left(\frac{x-1}{2}\right)^j \left(\frac{x+1}{2}\right)^{n-j} \sum_{l=0}^{j} (-)^l \binom{j}{l} \left(\frac{\alpha+l+1}{k}\right)_n \tag{1.12}$$

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s] y_1^{K_1} \cdots y_s^{K_s} \tag{1.13}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex.

We shall require the following integrals for the evaluation of our main integrals :

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} J_n^{(\alpha,\beta)}(1-2x;k)\mathrm{d}x = \frac{(\alpha+1)_{kn}}{n!} \sum_{m=0}^{n} \frac{(-n)_m}{m!}$$

$$\times \frac{(\alpha+\beta+n+1)_{km}\Gamma(\mu)\Gamma(\lambda+km)}{(\alpha+1)_{km}\Gamma(\lambda+\mu+km)} \tag{1.14}$$

$$\int_0^1 x^{\lambda-1}(1-x)^{\mu-1} K_n^{(\alpha,\beta)}(1-2x;k)\mathrm{d}x = \frac{1}{n!} \sum_{m=0}^{n} (-)^m \binom{\beta+n}{m} \sum_{l=0}^{m} \frac{(-m)_l}{l!} \left(\frac{\alpha+l+1}{k}\right)_m$$

$$\frac{\Gamma(\lambda+m)\Gamma(\mu+n-m)}{\Gamma(\lambda+\mu+n)}$$ Where $Re(\lambda)>0, Re(\mu)>0, (\alpha)>-1, Re(\beta)>-1$ (1.15)

The above two integrals can be evaluated, if we make use of (1.9) and (1.10) respectively and the definition of Beta function.

2. Main integrals

Let $$A = \frac{(-N_1)_{M_1K_1}}{K_1!} \cdots \frac{(-N_t)_{M_tK_t}}{K_t!} A[N_1,K_1;\cdots;N_t,K_t] \tag{2.1}$$

m is a positive integer

First integral

$$\int_0^1 \int_0^1 x^{\lambda-1} y^{\rho-1} z^{\rho+2\sigma} (1-x)^{\mu-1} (1-y^2)^{\sigma-1} J_n^{(\alpha,\beta)}(1-2x;k)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c}\mathrm{a}_1 y^{u_1}(1-y^2)^{v_1}z^{u_1+2v_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{a}_t y^{u_t}(1-y^2)^{v_t}z^{u_t+2v_t}\end{array}\right) I\left(\begin{array}{c}\mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}(1-y^2)^{\delta_1}z^{\mu_1+2\delta_1}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}y^{\mu_r}(1-y^2)^{\delta_r}z^{\mu_r+2\delta_r}\end{array}\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{(\alpha+1)_{kn}}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}e^{i\pi(\rho+\mu(K_1+\cdots+K_t)/2)}$$

$$I_{U:p_r+4,q_r+2;W}^{V;0,n_r+4;X}\left(\begin{array}{c}\mathrm{z}_1 e^{i\pi\mu_1/2}\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r e^{i\pi\mu_r/2}\end{array}\middle|\begin{array}{c}\mathrm{A};(1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r),\\ \cdots\\ \cdots\\ \mathrm{B};(1\text{-}\lambda-\mu-kj;\sigma_1+\rho_1,\cdots,\sigma_r+\rho_r),\end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-\sum_{l=1}^{t}K_l u_l;\mu_1,\cdots,\mu_r),(2\sigma-2\sum_{l=1}^{t}K_l v_l;2\delta_1,\cdots,2\delta_r),\mathfrak{A}:A'\\ \cdots\\ \cdots\\ (1\text{-}\rho-2\sigma-\sum_{l=1}^{t}K_l(u_l+2v_l);\mu_1+2\delta_1,\cdots,\mu_r+2\delta_r),\mathfrak{B}:B'\end{array}\right) \quad (2.2)$$

where $z=\sqrt{1-y^2}+iy$ and

Provided that

a) $\min_{1\leqslant i\leqslant r}\{\mu_i,\delta_j,\rho_i,\sigma_i\}>0, Re(a)>-1, Re(\beta)>-1$

b) $Re[\rho+\sum_{i=1}^{r}\mu_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0\,;Re[\lambda+\sum_{i=1}^{r}\rho_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

and $Re[\mu+\sum_{i=1}^{r}\sigma_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0; Re[\sigma+\sum_{i=1}^{r}\delta_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

c) $|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.3)

Second integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+\delta}(1-x)^{\mu-1}(1-y^2)^{\delta/2-1}J_n^{(\alpha,\beta)}(1-2x;k)_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\begin{pmatrix} \mathrm{a}_1 y^{u_1} z^{u_1} \\ . \\ . \\ . \\ \mathrm{a}_t y^{u_t} z^{u_t} \end{pmatrix} I \begin{pmatrix} \mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}(yz)^{\mu_1} \\ . \\ . \\ . \\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}(yz)^{\mu_r} \end{pmatrix} \mathrm{d}x\mathrm{d}y$$

$$=\frac{(\alpha+1)_{kn}\Gamma(\delta)}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}$$

$$e^{i\pi(\rho+\mu(K_1+\cdots+K_t)/2)} I_{U:p_r+4,q_r+3;W}^{V;0,n_r+4;X}\left(\begin{matrix} \mathrm{z}_1 e^{i\pi\mu_1/2} \\ . \\ . \\ . \\ \mathrm{z}_r e^{i\pi\mu_r/2} \end{matrix}\ \middle|\ \begin{matrix} \mathrm{A}\ ;\ (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r) \\ \cdots \\ \cdots \\ \mathrm{B}\ ;\ (1\text{-}\lambda-\mu-kj;\sigma_1+\rho_1,\cdots,\sigma_r+\rho_r), \end{matrix}\right.$$

$$\left.\begin{matrix} ,(1\text{-}\rho-\sum_{l=1}^t K_l u_l;\mu_1,\cdots,\mu_r),(1-\rho-\delta+a+b-\sum_{l=1}^t K_l u_l;\mu_1,\cdots,\mu_r),\mathfrak{A}:A' \\ \cdots \\ \cdots \\ (1\text{-}\rho-\delta-\sum_{l=1}^t K_l u_l+a;\mu_1,\cdots,\mu_r),(1-\rho-\delta-\sum_{l=1}^t K_l u_l+b;\mu_1,\cdots,\mu_r),\mathfrak{B}:B' \end{matrix}\right) \quad (2.3)$$

where $z=\sqrt{1-y^2}+iy$ provided that

a) $\min\limits_{1\leqslant i\leqslant r}\{\mu_i,\delta_j,\rho_i,\sigma_i\}>0, Re(a)>-1, Re(\beta)>-1$

b) $Re[\rho+\sum_{i=1}^{r}\mu_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0\,;Re[\lambda+\sum_{i=1}^{r}\rho_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$

$Re[\mu+\sum_{i=1}^{r}\sigma_i\min\limits_{1\leqslant j\leqslant m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}]>0$ and $Re(\delta)>0, Re(\rho+\delta-a-b)>0$

c) $|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.3) and $Re(\delta)>0, Re(\rho+\delta-a-b)>0$

Third integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{2\rho-1}\zeta^{-\rho-\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}J_n^{(\alpha,\beta)}(1-2x;k)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\begin{pmatrix} \mathrm{a}_1 y^{2u_1}(1-y^2)^{v_1}\zeta^{-u_1-v_1} \\ . \\ . \\ . \\ \mathrm{a}_t y^{2u_t}(1-y^2)^{v_t}\zeta^{-u_t-v_t} \end{pmatrix} I \begin{pmatrix} \mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}y^{2\mu_1}(1-y^2)^{\delta_1}\zeta^{-\mu_1-\delta_1} \\ . \\ . \\ . \\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}y^{2\mu_r}(1-y^2)^{\delta_r}\zeta^{-\mu_r-\delta_r} \end{pmatrix} \mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{2}\frac{(\alpha+1)_{kn}}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}(1+b)^{-(\rho+\mu(K_1+\cdots+K_t))}$$

$$I_{U:p_r+4,q_r+2;W}^{V;0,n_r+4;X}\left(\begin{array}{c|c} z_1(1+b)^{-\mu_1} & \mathrm{A} ; (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r) \\ \cdot & \cdots \\ \cdot & \cdots \\ \cdot & \\ z_r(1+b)^{-\mu_r} & \mathrm{B} ; (1\text{-}\lambda-\mu-kj;\rho_1,\cdots,\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\rho-\sum_{l=1}^{t}K_l u_l;\mu_1,\cdots,\mu_r),(1-\sigma-\sum_{l=1}^{t}K_l v_l;\delta_1,\cdots,\delta_r),\mathfrak{A}:A' \\ \cdots \\ \cdots \\ (1\text{-}\rho-\sigma-\sum_{l=1}^{t}K_l(u_l+v_l);\mu_1+\delta_1,\cdots,\mu_r+\delta_r),\mathfrak{B}:B' \end{array}\right) \quad (2.4)$$

where $\zeta=(1+by^2), b>-1$ provided that

a) $\min\limits_{1\leqslant i\leqslant r}\{\mu_i,\delta_j,\rho_i,\sigma_i\}>0, Re(a)>-1, Re(\beta)>-1$

b) $Re[\rho+\sum_{i=1}^{r}\mu_i\min\limits_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0\,;Re[\lambda+\sum_{i=1}^{r}\rho_i\min\limits_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

and $Re[\mu+\sum_{i=1}^{r}\sigma_i\min\limits_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0; Re[\sigma+\sum_{i=1}^{r}\delta_i\min\limits_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

c) $|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.3)

remark : Similarly integrals involving $K_n^{(\alpha,\beta)}(1-2x;k)$ can also be evaluating.

Proof of the first integral

To establish the integral (2.2), we first use the definition of $S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}(.)$ and the I-function of r variables defined by Prasad [5] in Mellin-Barnes contour integral, changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Evaluating the resulting integral with the help of [3, p.450, eq.(4)]. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

The proof of the other integral formulas are similar to that of the first integral with the only difference that here we use other known integral [4,p.71, eq.(3.1.8)] and [2,p.10, eq.20]

3. Multivariable H-function

If $U=V=A=B=0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [7]. We have the following results.

First integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+2\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}J_n^{(\alpha,\beta)}(1-2x;k)$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c} \mathrm{a}_1 y^{u_1}(1-y^2)^{v_1}z^{u_1+2v_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{a}_t y^{u_t}(1-y^2)^{v_t}z^{u_t+2v_t} \end{array}\right) H\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}y^{\mu_1}(1-y^2)^{\delta_1}z^{\mu_1+2\delta_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}y^{\mu_r}(1-y^2)^{\delta_r}z^{\mu_r+2\delta_r} \end{array}\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{(\alpha+1)_{kn}}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}e^{i\pi(\rho+\mu(K_1+\cdots+K_t)/2)}$$

$$H_{p_r+4,q_r+2;W}^{0,n_r+4;X}\left(\begin{array}{c} \mathrm{z}_1 e^{i\pi\mu_1/2} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r e^{i\pi\mu_r/2} \end{array}\middle| \begin{array}{c} (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r), \\ \cdots \\ \cdots \\ (1\text{-}\lambda-\mu-kj;\sigma_1+\rho_1,\cdots,\sigma_r+\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} (1\text{-}\rho-\sum_{l=1}^{t}K_lu_l;\mu_1,\cdots,\mu_r),(2\sigma-2\sum_{l=1}^{t}K_lv_l;2\delta_1,\cdots,2\delta_r),\mathfrak{A}:A' \\ \cdots \\ \cdots \\ (1\text{-}\rho-2\sigma-\sum_{l=1}^{t}K_l(u_l+2v_l);\mu_1+2\delta_1,\cdots,\mu_r+2\delta_r),\mathfrak{B}:B' \end{array}\right) \quad (3.1)$$

where $z=\sqrt{1-y^2}+iy$ and

under the same conditions and notations that (2.2) with $U=V=A=B=0$

Second integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{\rho-1}z^{\rho+\delta}(1-x)^{\mu-1}(1-y^2)^{\delta/2-1}J_n^{(\alpha,\beta)}(1-2x;k)\,{}_2F_1[a,b;\delta;z\sqrt{1-y^2}]$$

$$S_{N_1,\cdots,N_t}^{M_1,\cdots,M_t}\left(\begin{array}{c} \mathrm{a}_1 y^{u_1}z^{u_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{a}_t y^{u_t}z^{u_t} \end{array}\right) H\left(\begin{array}{c} \mathrm{z}_1 x^{\rho_1}(1-x)^{\sigma_1}(yz)^{\mu_1} \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r x^{\rho_r}(1-x)^{\sigma_r}(yz)^{\mu_r} \end{array}\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{(\alpha+1)_{kn}\Gamma(\delta)}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}$$

$$e^{i\pi(\rho+\mu(K_1+\cdots+K_t)/2)}H^{0,n_r+4;X}_{p_r+4,q_r+3;W}\left(\begin{array}{c|c} z_1e^{i\pi\mu_1/2} & (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r) \\ \vdots & \cdots \\ & \cdots \\ z_re^{i\pi\mu_r/2} & (1\text{-}\lambda-\mu-kj;\sigma_1+\rho_1,\cdots,\sigma_r+\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c} ,(1\text{-}\rho-\sum_{l=1}^t K_lu_l;\mu_1,\cdots,\mu_r),(1-\rho-\delta+a+b-\sum_{l=1}^t K_lu_l;\mu_1,\cdots,\mu_r),\mathfrak{A}:A' \\ \cdots \\ \cdots \\ (1\text{-}\rho-\delta-\sum_{l=1}^t K_lu_l+a;\mu_1,\cdots,\mu_r),(1-\rho-\delta-\sum_{l=1}^t K_lu_l+b;\mu_1,\cdots,\mu_r),\mathfrak{B}:B' \end{array}\right) \quad (3.2)$$

where $z=\sqrt{1-y^2}+iy$ provided that

under the same conditions and notations that (2.2) with $U=V=A=B=0$

Third integral

$$\int_0^1\int_0^1 x^{\lambda-1}y^{2\rho-1}\zeta^{-\rho-\sigma}(1-x)^{\mu-1}(1-y^2)^{\sigma-1}J_n^{(\alpha,\beta)}(1-2x;k)$$

$$S^{M_1,\cdots,M_t}_{N_1,\cdots,N_t}\left(\begin{array}{c} \mathrm{a}_1y^{2u_1}(1-y^2)^{v_1}\zeta^{-u_1-v_1} \\ \vdots \\ \mathrm{a}_ty^{2u_t}(1-y^2)^{v_t}\zeta^{-u_t-v_t} \end{array}\right)H\left(\begin{array}{c} \mathrm{z}_1x^{\rho_1}(1-x)^{\sigma_1}y^{2\mu_1}(1-y^2)^{\delta_1}\zeta^{-\mu_1-\delta_1} \\ \vdots \\ \mathrm{z}_rx^{\rho_r}(1-x)^{\sigma_r}y^{2\mu_r}(1-y^2)^{\delta_r}\zeta^{-\mu_r-\delta_r} \end{array}\right)\mathrm{d}x\mathrm{d}y$$

$$=\frac{1}{2}\frac{(\alpha+1)_{kn}}{n!}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_t=0}^{[N_t/M_t]}\sum_{j=0}^{n}\frac{(-n)_j(\alpha+\beta+n+1)_{kj}}{j!(\alpha+1)_{kj}}Aa_1^{K_1}\cdots a_t^{K_t}(1+b)^{-(\rho+\mu(K_1+\cdots+K_t))}$$

$$H^{0,n_r+4;X}_{p_r+4,q_r+2;W}\left(\begin{array}{c|c} \mathrm{z}_1(1+b)^{-\mu_1} & (1\text{-}\mu;\sigma_1,\cdots,\sigma_r),(1-\lambda-kj;\rho_1,\cdots,\rho_r) \\ \vdots & \cdots \\ & \cdots \\ \mathrm{z}_r(1+b)^{-\mu_r} & (1\text{-}\lambda-\mu-kj;\rho_1,\cdots,\rho_r), \end{array}\right.$$

$$\left.\begin{array}{c}(1\text{-}\rho-\sum_{l=1}^{t}K_lu_l;\mu_1,\cdots,\mu_r),(1-\sigma-\sum_{l=1}^{t}K_lv_l;\delta_1,\cdots,\delta_r),\mathfrak{A}:A'\\ \cdots\\ \cdots\\ (1\text{-}\rho-\sigma-\sum_{l=1}^{t}K_l(u_l+v_l);\mu_1+\delta_1,\cdots,\mu_r+\delta_r),\mathfrak{B}:B'\end{array}\right) \tag{3.3}$$

where $\zeta = (1 + by^2), b > -1$

under the same conditions and notations that (2.2) with $U = V = A = B = 0$

5. Conclusion

The I-function of several variables defined by Prasad [5] presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariabble multivariable Fox's H-function, Fox's H-function , Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

References

[1] Chai W.A., Carlitz L. Biorthogonal condition for a class of polynomials, SIAM. Rev. 14 (1972), 494; ibid 15(1973), p. 670-672.

[2] Erdelyi A. et al Higher transcendental functions, Vol1 McGraw Hill, New York. (1953).

[3] Mac Robert T.M. Beta function formula and integrals involving E-functions, Math Ann, 142 (1961), p.450-452.

[4] Mathai A.M. And Saxena R.K. Generalized hypergeometric functions with applications in statistics and physical sciences. Lecture notes in Mathematics, Vol. 348, Springer Verlag, New York. (1973).

[5] Y.N. Prasad , Multivariable I-function , Vijnana Parishad Anusandhan Patrika 29 (1986) , page 231-237.

[6] Srivastava H.M. A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. Vol 77(1985), page183-191.

[7] H.M. Srivastava And R.Panda. Some expansion theorems and generating relations for the H-function of several complex variables. Comment. Math. Univ. St. Paul. 24(1975), p.119-137.

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