Some finite double integral involving general class of polynomials, special functions , Aleph-function and multivariable I-function

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ABSTRACT

The aim of the present document is to evaluate three triple double finite integrals involving general class of polynomials, special functions, Aleph-function and multivariable I-function. Importance of our findings lies in the fact that they involve the multivariable I-function, which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them. Further we establish some special cases.

KEYWORDS : I-function of several variables, double integrals, special function, general class of polynomials, Aleph-function, multivariable H-function.

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1.Introduction and preliminaries.

The aim of the present document is to evaluate three triple double finite integrals involving general class of polynomials, special functions, Aleph-function and multivariable I-function defined by Prasad [5]. We will study the particular case concerning the multivariable H-function defined by Srivastava et al [7].

The Aleph- function , introduced by Südland [8] et al , however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral :

$$\aleph(z)=\aleph_{P_i,Q_i,c_i;r}^{M,N}\left(z \;\middle|\; \begin{matrix} (\mathrm{a}_j,A_j)_{1,\mathfrak{n}},[c_i(a_{ji},A_{ji})]_{\mathfrak{n}+1,p_i;r} \\ (\mathrm{b}_j,B_j)_{1,m},[c_i(b_{ji},B_{ji})]_{m+1,q_i;r} \end{matrix}\right)=\frac{1}{2\pi\omega}\int_L \Omega_{P_i,Q_i,c_i;r}^{M,N}(s)z^{-s}\mathrm{d}s \tag{1.1}$$

for all z different to 0 and

$$\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)=\frac{\prod_{j=1}^{M}\Gamma(b_j+B_js)\prod_{j=1}^{N}\Gamma(1-a_j-A_js)}{\sum_{i=1}^{r}c_i\prod_{j=N+1}^{P_i}\Gamma(a_{ji}+A_{ji}s)\prod_{j=M+1}^{Q_i}\Gamma(1-b_{ji}-B_{ji}s)} \tag{1.2}$$

With :$|argz|<\frac{1}{2}\pi\Omega$ Where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0\,;i=1,\cdots,r$

For convergence conditions and other details of Aleph-function , see Südland et al [8].the serie representation of Aleph-function is given by Chaurasia et al [1].

$$\aleph_{P_i,Q_i,c_i;r}^{M,N}(z)=\sum_{G=1}^{M}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{P_i,Q_i,c_i,r}^{M,N}(s)}{B_Gg!}z^{-s} \tag{1.3}$$

With $s=\eta_{G,g}=\frac{b_G+g}{B_G}, P_i<Q_i, |z|<1$ and $\Omega_{P_i,Q_i,c_i;r}^{M,N}(s)$ is given in (1.2) (1.4)

The generalized polynomials defined by Srivastava [6], is given in the following manner :

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}[y_1,\cdots,y_s] = \sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}$$

$$A[N_1,K_1;\cdots;N_s,K_s]y_1^{K_1}\cdots y_s^{K_s} \tag{1.5}$$

Where $M_1,\cdots,M_s$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_s,K_s]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N_1)_{M_1K_1}}{K_1!}\cdots\frac{(-N_s)_{M_sK_s}}{K_s!}A[N_1,K_1;\cdots;N_s,K_s] \tag{1.6}$$

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1,\cdots,z_r) = I_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}_{rj};\alpha'_{rj},\cdots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'};\cdots;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj};\beta'_{rj},\cdots,\beta^{(r)}_{rj})_{1,q_r}:(b'_j,\beta'_j)_{1,q'};\cdots;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{array}\right) \tag{1.7}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(t_1,\cdots,t_r)\prod_{i=1}^{s}\phi_i(s_i)z_i^{s_i}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.8}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [5]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_i| < \frac{1}{2}\Omega_i\pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}}\alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}}\beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}}\beta_k^{(i)} + \left(\sum_{k=1}^{n_2}\alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2}\alpha_{2k}^{(i)}\right) +$$

$$+ \left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right) - \left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)} + \sum_{k=1}^{q_3}\beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.9}$$

where $i = 1,\cdots,r$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$I(z_1,\cdots,z_r)=0(\,|z_1|^{\gamma'_1},\cdots,|z_r|^{\gamma'_r}\,), max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$

$I(z_1,\cdots,z_r)\;=0(\,|z_1|,\cdots,|z_r|^{\beta'_s}\,), min(\,|z_1|\,,\cdots,|z_r|\,)\rightarrow\infty$

where $k=1,\cdots,z:\alpha'_k=min[Re(b_j^{(k)}/\beta_j^{(k)})], j=1,\cdots,m_k$ and

$\beta'_k=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})], j=1,\cdots,n_k$

We will use these following notations in this paper :

$$U=p_2,q_2;p_3,q_3;\cdots;p_{r-1},q_{r-1};V=0,n_2;0,n_3;\cdots;0,n_{s-1} \tag{1.10}$$

$$W=(p',q');\cdots;(p^{(r)},q^{(r)});X=(m',n');\cdots;(m^{(r)},n^{(r)}) \tag{1.11}$$

$$A=(a_{2k},\alpha'_{2k},\alpha''_{2k});\cdots;(a_{(r-1)k}),\alpha'{}_{(r-1)k},\alpha''_{(r-1)k},\cdots,\alpha_{(r-1)k}^{(r-1)}) \tag{1.12}$$

$$B=(b_{2k},\beta'_{2k},\beta''_{2k});\cdots;(b_{(r-1)k}),\beta'{}_{(r-1)k},\beta''_{(r-1)k},\cdots,\beta_{(r-1)k}^{(r-1)}) \tag{1.13}$$

$$\mathfrak{A}=(a_{sk};\alpha'{}_{rk},\alpha''_{rk},\cdots,\alpha^{r}_{rk}):\mathfrak{B}=(b_{rk};\beta'{}_{rk},\beta''_{rk},\cdots,\beta^{r}_{rk}) \tag{1.14}$$

$$A'=(a'_k,\alpha'_k)_{1,p'};\cdots;(a_k^{(r)},\alpha_k^{(r)})_{1,p^{(r)}};B'=(b'_k,\beta'_k)_{1,q'};\cdots;(b_k^{(r)},\beta_k^{(r)})_{1,q^{(r)}} \tag{1.15}$$

The multivariable I-function write :

$$I(z_1,\cdots,z_r)=\;I_{U:p_r,q_r;W}^{V;0,n_r;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\;;\mathfrak{A};\mathrm{A'} \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & \mathrm{B};\;\mathfrak{B};\;\mathrm{B'} \end{array}\right) \tag{1.16}$$

2 . Required integrals

The following integral ([9],p.33;[2],p.172, eq(27);[3],p.46, eq(5) and [4],p.71] will be required to establish our main results :

a) $$\int_0^1 x^{c-1}(1-x)^{-1/2}{}_2F_1(a,b;a+b+1/2;x)\mathrm{d}x=\frac{\pi\Gamma(c)\Gamma(a+b+1/2)\Gamma(c-a-b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)\Gamma(c-a+1/2)\Gamma(c-b+1/2)} \tag{2.1}$$

where Re(c) > 0, Re(2c-a-b) > -1

b) $$\int_0^{\pi}(sin\theta)^{\alpha-1}P_v^{-\mu}(cos\theta)\mathrm{d}\theta=\frac{2^{-\mu}\pi\Gamma\{(\alpha\pm\mu)/2\}}{\Gamma\{(\alpha+v+1)/2\}\Gamma\{(\alpha-v)/2\}\Gamma\{(\mu+v+2)/2\}\Gamma\{(\mu-v+1)/2\}} \tag{2.2}$$

provided that $Re(\alpha \pm \mu) > 0$

c) $\int_0^\pi J_\mu(\theta)^{\mu+1}(cos\theta)^{2\rho+1}\mathrm{d}\theta = 2^\rho\Gamma(\rho+1)\alpha^{-\rho+1}J_{\rho+\mu+1}(\alpha)$ (2.3)

where $Re(\rho) > -1; Re(\mu) > -1$

d) $\int_0^{\pi/2} e^{i(\alpha+\beta)\theta}(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}{}_2F_1(\gamma,\delta;\beta;e^{i\theta}cos\theta)\mathrm{d}\theta = \frac{e^{i\pi\alpha/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta-\delta-\gamma)}{\Gamma(\alpha+\beta-\gamma)\Gamma(\alpha+\beta-\delta)}$ (2.4)

where $min\{Re(\alpha), Re(\beta), Re(\alpha+\beta-\gamma-\delta)\} > 0$

3. Main integrals

1) $\int_0^1\int_0^\pi x^{c-1}(1-x)^{-1/2}\ {}_2F_1(a,b;a+b+1/2;x)(sin\theta)^{\alpha-1}P_\upsilon^{-\mu}(cos\theta)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}(sin\theta)^d)$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1x^{e_1}(sin\theta)^{f_1} \\ \cdots \\ \mathrm{y}_sx^{e_s}(sin\theta)^{f_s}\end{pmatrix} I\begin{pmatrix} \mathrm{z}_1x^{c_1}(sin\theta)^{d_1} \\ \cdots \\ \mathrm{z}_rx^{c_r}(sin\theta)^{d_r}\end{pmatrix}\mathrm{d}x\mathrm{d}\theta$$

$$= \frac{\pi^2 2^{-\mu}\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)\Gamma\{(\mu+\upsilon+2)/2\}\Gamma\{(\mu-\upsilon+1)/2\}}$$

$$= \sum_{G=1}^M\sum_{g=0}^\infty\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$I_{U:p_r+4,q_r+4;W}^{V;0,n_r+4;X}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A}\ ;\ (1-\mathrm{c}-\mathrm{c}'\eta_{G,g}-\sum_{i=1}^s K_ie_i;c_1,\cdots,c_r), \\ \cdots & \cdots \\ \mathrm{z}_r & \mathrm{B}\ ;\ (\frac{1}{2}-\mathrm{c}+\mathrm{a}-\mathrm{c}'\eta_{G,g}-\sum_{i=1}^s Ke_i;c_1,\cdots,c_r), \end{array}\right.$$

$$(\tfrac{1}{2}-\mathrm{c}+\mathrm{a}+\mathrm{b}-\mathrm{c}'\eta_{G,g}-\sum_{i=1}^s K_ie_i;c_1,\cdots,c_r),(1-\frac{\alpha+\mu+d\eta_{G,g}+\sum_{i=1}^s K_if_i}{2};\mathrm{d}'_1,\cdots,d'_r),$$
$$\cdots \qquad \cdots$$
$$(\tfrac{1}{2}-c+b-c'\eta_{G,g}-\sum_{i=1}^s K_ie_i;c_1,\cdots,c_r),\ (\tfrac{1}{2}-\frac{\alpha+\upsilon+d\eta_{G,g}+\sum_{i=1}^s K_if_i}{2};\mathrm{d}'_1,\cdots,d'_r),$$

$$\left.\begin{array}{c}(1-\frac{\alpha-\mu+d\eta_{G,g}+\sum_{i=1}^s K_if_i}{2};\mathrm{d}'_1,\cdots,d'_r),\mathfrak{A}:A' \\ \cdots \\ (1-\frac{\alpha-\upsilon+d\eta_{G,g}+\sum_{i=1}^s K_if_i}{2};\mathrm{d}'_1,\cdots,d'_r),\mathfrak{B}:B'\end{array}\right) \quad (3.1)$$

where $d'_i = \frac{d_i}{2}, i = 1,\cdots,r$. Provided that

a) $Re[c+c'\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_ie_i+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>-1$

b) $Re[c-a-b+c'\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_ie_i+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>1/2$

c) $Re[c-a+c'\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_ie_i+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>-1/2$

d) $Re[c-b+c'\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_ie_i+\sum_{i=1}^{r}c_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>-1/2$

e) $Re[\alpha\pm\mu+d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_if_i+\sum_{i=1}^{r}d_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

f) $Re[\alpha-\upsilon+d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_if_i+\sum_{i=1}^{r}d_i\min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>0$

g) $Re[\alpha\pm\upsilon+d\min_{1\leqslant j\leqslant M}\frac{b_j}{B_j}+\sum_{i=1}^{s}K_if_i+\sum_{i=1}^{r}d_i\min_{1\leqslant j\leqslant m_i}\frac{b_j^{(i)}}{\beta_j^{(i)}}]>-1$

h) $|argz_i|<\frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.9)

i) $|argz|<\frac{1}{2}\pi\Omega$ where $\Omega=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}\alpha_j-c_i(\sum_{j=M+1}^{Q_i}\beta_{ji}+\sum_{j=N+1}^{P_i}\alpha_{ji})>0$

2) $\int_0^1\int_0^\pi x^{c-1}(1-x)^{-1/2}\,{}_2F_1(a,b;a+b+1/2;x)(sin\theta)^{\mu+1}(cos\theta)^{2\rho+1}J_\mu(\alpha sin\theta)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}(sin\theta)^d)$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c}\mathrm{y}_1x^{e_1}(cos\theta)^{2f_1}\\ \cdot\cdot\cdot\\ \mathrm{y}_sx^{e_s}(cos\theta)^{2f_s}\end{array}\right)I\left(\begin{array}{c}\mathrm{z}_1x^{c_1}(cos\theta)^{2d_1}\\ \cdot\cdot\cdot\\ \mathrm{z}_rx^{c_r}(cos\theta)^{2d_r}\end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\frac{\pi2^\rho\alpha^{-\rho}\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}\sum_{g=0}^{\infty}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}\ \frac{(-)^g\alpha^{2g+\mu}}{2^{2g+\mu}g!}$$

$$z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}\ I_{U:p_r+3,q_r+3;W}^{V;0,n_r+3;X}\left(\begin{array}{c}\mathrm{z}_1\\ \cdot\cdot\cdot\\ \mathrm{z}_r\end{array}\middle|\begin{array}{l}\mathrm{A}\ ;\ (1\text{ - c-c'}\eta_{G,g}-\sum_{i=1}^{s}K_ie_i;c_1,\cdots,c_r),\\ \cdot\cdot\cdot\\ \mathrm{B}\ ;\ (\frac{1}{2}\text{ -c +a -c'}\eta_{G,g}-\sum_{i=1}^{s}Ke_i;c_1,\cdots,c_r),\end{array}\right.$$

$$(\tfrac{1}{2} \text{ - c+a+b-c'}\eta_{G,g} - \textstyle\sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r),$$
$$\cdots$$
$$(\tfrac{1}{2} \text{ -c +b -c'}\eta_{G,g} - \textstyle\sum_{i=1}^{s} K_i e_i; c_1, \cdots, c_r),$$

$$\left.\begin{matrix} (- \rho - d\eta_{G,g} - \sum_{i=1}^{s} K_i f_i; d_1, \cdots, d_r), \mathfrak{A} : A' \\ \cdots \\ (\text{-g-}\rho - \mu - 1 - d\eta_{G,g} - \sum_{i=1}^{s} K_i f_i; d_1, \cdots, d_r), \mathfrak{B} : B' \end{matrix}\right) \tag{3.2}$$

Provided that

a)$Re[c + c' \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i e_i + \sum_{i=1}^{r} c_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1$

b)$Re[c - a - b + c' \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i e_i + \sum_{i=1}^{r} c_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

c)$Re[c - a + c' \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i e_i + \sum_{i=1}^{r} c_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

d)$Re[c - b + c' \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i e_i + \sum_{i=1}^{r} c_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

e)$Re[\rho + d \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i f_i + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

f)$Re[g - \mu + \rho + d \min_{1 \leqslant j \leqslant M} \frac{b_j}{B_j} + \sum_{i=1}^{s} K_i f_i + \sum_{i=1}^{r} d_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}] > -2$

g) $|arg z_i| < \frac{1}{2}\Omega_i \pi$, where Ω_i is defined by (1.9)

h) $|arg z| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M} \beta_j + \sum_{j=1}^{N} \alpha_j - c_i(\sum_{j=M+1}^{Q_i} \beta_{ji} + \sum_{j=N+1}^{P_i} \alpha_{ji}) > 0$

3) $\int_0^1 \int_0^{\pi/2} x^{c-1}(1-x)^{-1/2} \; {}_2F_1(a, b; a + b + 1/2; x)(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1} \; {}_2F_1(\gamma, \delta; \beta; e^{i\theta} cos\theta)$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}(sin\theta)^d) S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \text{y}_1 x^{e_1} e^{if_1\theta}(cos\theta)^{f_1} \\ \cdots \\ \text{y}_s x^{e_s} e^{if_s\theta}(cos\theta)^{f_s} \end{pmatrix} I \begin{pmatrix} \text{z}_1 x^{c_1} e^{i\theta d_1}(cos\theta)^{d_1} \\ \cdots \\ \text{z}_r x^{c_r} e^{i\theta d_r}(cos\theta)^{d_r} \end{pmatrix} \text{d}x\text{d}\theta$$

$$= \frac{\pi e^{i\pi\alpha/2}\Gamma(\beta)\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)} \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} a_1 \frac{(-)^g \Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!} \; y_1^{K_1}\cdots y_s^{K_s}$$

$$z^{\eta_{G,g}}\, e^{i\pi(d\eta_{G,g}+\sum_{j=1}^{s}K_jf_j)} I_{U:p_r+4,q_r+4;W}^{V;0,n_r+4;X}\left(\begin{array}{c|l} z_1 e^{i\pi\rho_1/2} & \text{A};(1 - \text{c-c'}\eta_{G,g} - \sum_{j=1}^{s}K_je_j; c_1,\cdots,c_r), \\ \cdots & \qquad\cdots \\ z_r e^{i\pi\rho_r/2} & \text{B}; (\tfrac{1}{2} \text{ -c +a -c'}\eta_{G,g} - \sum_{j=1}^{s}Ke_i; c_1,\cdots,c_r), \end{array}\right.$$

$$(1 \text{ -}\alpha - d\eta_{G,g} - \textstyle\sum_{j=1}^{s}K_jf_j; d_1,\cdots,d_r), \qquad (\tfrac{1}{2} \text{ - c+a+b-c'}\eta_{G,g} - \textstyle\sum_{j=1}^{s}K_je_j; c_1,\cdots,c_r),$$
$$\cdots \qquad\qquad \cdots$$
$$(1\text{-}\alpha - \beta + \gamma - d\eta_{G,g} - \textstyle\sum_{j=1}^{s}K_jf_j; d_1,\cdots,d_r), \; (\tfrac{1}{2} \text{ -c +b -c'}\eta_{G,g} - \textstyle\sum_{j=1}^{s}K_je_j; c_1,\cdots,c_r),$$

$$\left.\begin{array}{l} (1 \text{ -}\alpha - \beta + \gamma + \delta - d\eta_{G,g} - \sum_{j=1}^{s}K_jf_j; d_1,\cdots,d_r), \mathfrak{A} : A' \\ \qquad\cdots \\ (1\text{-}\alpha - \beta + \delta - d\eta_{G,g} - \sum_{j=1}^{s}K_jf_j; d_1,\cdots,d_r), \mathfrak{B} : B' \end{array}\right) \tag{3.3}$$

Provided that

a) $Re[c + c' \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_ie_i + \sum_{i=1}^{r}c_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1$

b) $Re[c - a - b + c' \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_ie_i + \sum_{i=1}^{r}c_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

c) $Re[c - a + c' \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_ie_i + \sum_{i=1}^{r}c_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

d) $Re[c - b + c' \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_ie_i + \sum_{i=1}^{r}c_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > -1/2$

e) $Re[\alpha + \beta - \delta - \gamma + d \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_if_i + \sum_{i=1}^{r}d_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

f) $Re[\alpha + \beta - \delta + d \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_if_i + \sum_{i=1}^{r}d_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

g) $Re[\alpha + \beta - \gamma + d \min_{1\leqslant j\leqslant M}\frac{b_j}{B_j} + \sum_{i=1}^{s}K_if_i + \sum_{i=1}^{r}d_i \min_{1\leqslant j\leqslant m^{(i)}}\frac{b_j^{(i)}}{\beta_j^{(i)}}] > 0$

h) $|arg z_i| < \frac{1}{2}\Omega_i\pi$, where Ω_i is defined by (1.9) i)$|argz| < \frac{1}{2}\pi\Omega$ where $\Omega = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}\alpha_j - c_i(\sum_{j=M+1}^{Q_i}\beta_{ji} + \sum_{j=N+1}^{P_i}\alpha_{ji}) > 0$

Proof of (3.1)

We first express the general class of polynomial occuring on the L.H.S of (3.1) in series form with the help of (1.5), the Aleph-function in series form with the help of (1.3) and replace the multivariable I-function by its Mellin-Barnes contour integral with the help of (1.7). Now we interchange the order of summation and integrations, we obtain

$$\text{L.H.S of (3.1)} = \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\left(\int_0^1 x^{c+c'\eta_{G,g}+\sum_{i=1}^{s}e_iK_i+\sum_{i=1}^{r}c_is_i-1}(1-x)^{-1/2}\right.$$

$$\left.{}_2F_1(a,b;a+b+1/2;x)\mathrm{d}x\right)\left(\int_0^\pi (sin\theta)^{\alpha+d\eta_{G,g}+\sum_{i=1}^{s}f_iK_i+\sum_{i=1}^{r}d_is_i-1}P_\upsilon^{-\mu}(cos\theta)\mathrm{d}\theta\right)\mathrm{d}s_1\cdots\mathrm{d}s_r$$

Now, using the result (2.1) and (2.2) to evaluate the x-integral and θ-integral and reinterpreting the multiple contour integral so obtained in the form of multivariable Aleph-function withe help of (1.7), we obtain the desired result.
The result (3.2) and (3.1) can be proved by similar prrofs with the help of integrals given by (2.2), (2.3) and (2.4).

4. Multivariable H-function

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [7]. We have the following results.

1) $$\int_0^1\int_0^\pi x^{c-1}(1-x)^{-1/2}\ {}_2F_1(a,b;a+b+1/2;x)(sin\theta)^{\alpha-1}P_\upsilon^{-\mu}(cos\theta)\aleph_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}(sin\theta)^d)$$

$$S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\begin{pmatrix} \mathrm{y}_1x^{e_1}(sin\theta)^{f_1} \\ \cdot\cdot\cdot \\ \mathrm{y}_sx^{e_s}(sin\theta)^{f_s}\end{pmatrix} H\begin{pmatrix} \mathrm{z}_1x^{c_1}(sin\theta)^{d_1} \\ \cdot\cdot\cdot \\ \mathrm{z}_rx^{c_r}(sin\theta)^{d_r}\end{pmatrix}\mathrm{d}x\mathrm{d}\theta$$

$$= \frac{\pi^2 2^{-\mu}\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)\Gamma\{(\mu+\upsilon+2)/2\}\Gamma\{(\mu-\upsilon+1)/2\}}$$

$$= \sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]} a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_G g!}z^{\eta_{G,g}}y_1^{K_1}\cdots y_s^{K_s}$$

$$H^{0,n_r+4;X}_{p_r+4,q_r+4;W}\left(\begin{array}{c} z_1 \\ \cdots \\ z_r \end{array}\middle| \begin{array}{l} (1 - c - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r), \\ \cdots \\ (\frac{1}{2} - c + a - c'\eta_{G,g} - \sum_{i=1}^s K e_i; c_1, \cdots, c_r), \end{array}\right.$$

$$(\tfrac{1}{2} - c + a + b - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r), (1 - \frac{\alpha+\mu+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; d'_1, \cdots, d'_r),$$
$$\cdots \qquad \cdots$$
$$(\tfrac{1}{2} - c + b - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r), (\tfrac{1}{2} - \frac{\alpha+\upsilon+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; d'_1, \cdots, d'_r),$$

$$\left.\begin{array}{l} (1 - \frac{\alpha-\mu+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; d'_1, \cdots, d'_r), \mathfrak{A} : A' \\ \cdots \\ (1 - \frac{\alpha-\upsilon+d\eta_{G,g}+\sum_{i=1}^s K_i f_i}{2}; d'_1, \cdots, d'_r), \mathfrak{B} : B' \end{array}\right) \tag{4.1}$$

where $d'_i = \dfrac{d_i}{2}, i = 1, \cdots, r.$

which holds true under the same conditions as needed in (3.1) with $U = V = A = B = 0$

2) $$\int_0^1 \int_0^\pi x^{c-1}(1-x)^{-1/2} {}_2F_1(a, b; a+b+1/2; x)(sin\theta)^{\mu+1}(cos\theta)^{2\rho+1} J_\mu(\alpha sin\theta) \aleph^{M,N}_{P_i,Q_i,c_i;r'}(zx^{c'}(sin\theta)^d)$$

$$S^{M_1,\cdots,M_s}_{N_1,\cdots,N_s}\left(\begin{array}{c} y_1 x^{e_1}(cos\theta)^{2f_1} \\ \cdots \\ y_s x^{e_s}(cos\theta)^{2f_s} \end{array}\right) H \left(\begin{array}{c} z_1 x^{c_1}(cos\theta)^{2d_1} \\ \cdots \\ z_r x^{c_r}(cos\theta)^{2d_r} \end{array}\right) dx d\theta$$

$$= \frac{\pi 2^\rho \alpha^{-\rho}\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)} \sum_{G=1}^{M} \sum_{g=0}^{\infty} \sum_{K_1=0}^{[N_1/M_1]} \cdots \sum_{K_s=0}^{[N_s/M_s]} \sum_{g=0}^{\infty} a_1 \frac{(-)^g \Omega^{M,N}_{P_i,Q_i,c_i,r'}(\eta_{G,g})}{B_G g!} \frac{(-)^g \alpha^{2g+\mu}}{2^{2g+\mu} g!}$$

$$z^{\eta_{G,g}} y_1^{K_1} \cdots y_s^{K_s} \; H^{0,n_r+3;X}_{p_r+3,q_r+3;W}\left(\begin{array}{c} z_1 \\ \cdots \\ z_r \end{array}\middle| \begin{array}{l} (1 - c - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r), \\ \cdots \\ (\frac{1}{2} - c + a - c'\eta_{G,g} - \sum_{i=1}^s K e_i; c_1, \cdots, c_r), \end{array}\right.$$

$$(\tfrac{1}{2} - c + a + b - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r),$$
$$\cdots$$
$$(\tfrac{1}{2} - c + b - c'\eta_{G,g} - \sum_{i=1}^s K_i e_i; c_1, \cdots, c_r),$$

$$\left.\begin{array}{l} (- \rho - d\eta_{G,g} - \sum_{i=1}^s K_i f_i; d_1, \cdots, d_r), \mathfrak{A} : A' \\ \cdots \\ (-g - \rho - \mu - 1 - d\eta_{G,g} - \sum_{i=1}^s K_i f_i; d_1, \cdots, d_r), \mathfrak{B} : B' \end{array}\right) \tag{4.2}$$

which holds true under the same conditions as needed in (3.2) with $U = V = A = B = 0$

3) $$\int_0^1\int_0^{\pi/2} x^{c-1}(1-x)^{-1/2}\ {}_2F_1(a,b;a+b+1/2;x)(sin\theta)^{\alpha-1}(cos\theta)^{\beta-1}\ {}_2F_1\left(\gamma,\delta;\beta;e^{i\theta}cos\theta\right)$$

$$\aleph_{P_i,Q_i,c_i;r'}^{M,N}(zx^{c'}(sin\theta)^d)S_{N_1,\cdots,N_s}^{M_1,\cdots,M_s}\left(\begin{array}{c} \mathrm{y}_1x^{e_1}e^{if_1\theta}(cos\theta)^{f_1} \\ \cdot\cdot\cdot \\ \mathrm{y}_sx^{e_s}e^{if_s\theta}(cos\theta)^{f_s}\end{array}\right)H\left(\begin{array}{c} \mathrm{z}_1x^{c_1}e^{i\theta d_1}(cos\theta)^{d_1} \\ \cdot\cdot\cdot \\ \mathrm{z}_rx^{c_r}e^{i\theta d_r}(cos\theta)^{d_r}\end{array}\right)\mathrm{d}x\mathrm{d}\theta$$

$$=\frac{\pi e^{i\pi\alpha/2}\Gamma(\beta)\Gamma(a+b+1/2)}{\Gamma(a+1/2)\Gamma(b+1/2)}\sum_{G=1}^{M}\sum_{g=0}^{\infty}\sum_{K_1=0}^{[N_1/M_1]}\cdots\sum_{K_s=0}^{[N_s/M_s]}a_1\frac{(-)^g\Omega_{P_i,Q_i,c_i,r'}^{M,N}(\eta_{G,g})}{B_Gg!}\ y_1^{K_1}\cdots y_s^{K_s}$$

$$z^{\eta_{G,g}}\,e^{i\pi(d\eta_{G,g}+\sum_{j=1}^{s}K_jf_j)}H_{p_r+4,q_r+4;W}^{0,n_r+4;X}\left(\begin{array}{c} \mathrm{z}_1e^{i\pi\rho_1/2} \\ \cdot\cdot\cdot \\ \mathrm{z}_re^{i\pi\rho_r/2}\end{array}\right|\begin{array}{c}(1\text{ - c-c'}\eta_{G,g}-\sum_{j=1}^{s}K_je_j;c_1,\cdots,c_r), \\ \cdot\cdot\cdot \\ (\frac{1}{2}\text{ -c +a -c'}\eta_{G,g}-\sum_{j=1}^{s}Ke_i;c_1,\cdots,c_r),\end{array}$$

$$\begin{array}{cc}(1\text{ -}\alpha-d\eta_{G,g}-\sum_{j=1}^{s}K_jf_j;d_1,\cdots,d_r), & (\frac{1}{2}\text{ - c+a+b-c'}\eta_{G,g}-\sum_{j=1}^{s}K_je_j;c_1,\cdots,c_r), \\ \cdot\cdot\cdot & \cdot\cdot\cdot \\ (1\text{-}\alpha-\beta+\gamma-d\eta_{G,g}-\sum_{j=1}^{s}K_jf_j;d_1,\cdots,d_r), & (\frac{1}{2}\text{ -c +b -c'}\eta_{G,g}-\sum_{j=1}^{s}K_je_j;c_1,\cdots,c_r),\end{array}$$

$$\left.\begin{array}{c}(1\text{ -}\alpha-\beta+\gamma+\delta-d\eta_{G,g}-\sum_{j=1}^{s}K_jf_j;d_1,\cdots,d_r),\mathfrak{A}:A' \\ \cdot\cdot\cdot \\ (1\text{-}\alpha-\beta+\delta-d\eta_{G,g}-\sum_{j=1}^{s}K_jf_j;d_1,\cdots,d_r),\mathfrak{B}:B'\end{array}\right) \quad (4.3)$$

which holds true under the same conditions as needed in (3.3) with $U=V=A=B=0$

6. Conclusion

The I-function of several variables presented in this paper, is quite basic in nature. Therefore , on specializing the parameters of this function, we may obtain various other special functions o several variables such as multivariable H-function ,Fox's H-function,, Meijer's G-function, Wright's generalized Bessel function, Wright's generalized hypergeometric function, MacRobert's E-function, generalized hypergeometric function, Bessel function of first kind, modied Bessel function, Whittaker function, exponential function , binomial function etc. as its special cases, and therefore, various unified integral presentations can be obtained as special cases of our results.

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