s-g INVERSE OF s-NORMAL MATRICES

R.Vijayakumar
Assistant professor of Mathematics,
Government Arts College(Autonomous), Kumbakonam – 612 002, India

Abstract

s-g inverse of a given square matrix is defined and its characterizations are obtained. s-hermitian idempotent matrix is defined. Properties of s-g inverse of an s-normal matrix are given.n

AMS classifications: 15A09, 15A57
Keywords:
s-normal, s-unitary, nilpotent, s-hermitian matrices.

1. Introduction:

Generalized inverse of a matrix was defined in [1] and its characterizations were obtained in [2]. Ann Lee initiated the study of secondary symmetric matrices in [3]. The concept of s-normal matrices was introduced and investigated in [4]. Some equivalent conditions on s-normal matrices were proved in [5]. S-unitary matrices was defined and analyzed in [6]. In this paper we describe s-g inverse of a square matrix, as the unique solution of a certain set of equations. This s-g inverse exists for particular kind of square matrices. We deal with s-g inverse of s-normal matrices. Let $C_{n\times n}$ denote the space of $n\times n$ complex matrices.

If $A=(a_{ij})\in C_{n\times n}$, then the secondary transpose of A, denoted by A^S, is defined as $A^S=(b_{ij})$, where $b_{ij}=a_{n-j+1,n-i+1}$. A^θ denotes the conjugate secondary transpose of A. i.e. $A^\theta=\overline{A}^S$.

Throughout this paper if $A\in C_{n\times n}$, then we assume that if $A\neq 0$, then $AA^\theta\neq 0$.

$$\text{(ie) } AA^\theta=0\Rightarrow A=0 \quad \text{........ ..(1)}$$

It is clear that the conjugate secondary transpose satisfies the following properties.

$$(A+B)^\theta=A^\theta+B^\theta$$
$$(\lambda A)^\theta=\bar{\lambda}A^\theta$$
$$(BA)^\theta=A^\theta B^\theta$$
$$(A^\theta)^\theta=A$$

Since $(BAA^\theta-CAA^\theta)(B-C)^\theta=(BA-CA)(BA-CA)^\theta$, by using (1), we can prove that

$$BAA^\theta=CAA^\theta\Rightarrow BA=CA \quad \text{.......... (2)}$$

Similarly, $$BA^\theta A=CA^\theta A\Rightarrow BA^\theta=CA^\theta \quad \text{.......... (3)}$$

Definition 1.1:

A matrix $A\in C_{n\times n}$ is said to be secondary normal (s-normal) if $AA^\theta=A^\theta A$.

Example 1.2:

$$A = \begin{pmatrix} 7+2i & 3 \\ 2 & 7+5i \end{pmatrix}$$ is an s-normal matrix.

Definition 1.3:

A matrix $A \in C_{n\times n}$ is said to be secondary unitary (s-unitary) if $AA^{\theta} = A^{\theta}A = I$.

Example 1.4:

$$A = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$ is an s-unitary matrix.

2. s-g inverse of a matrix

In this section, we find the condition for the existence of s-g inverse of a given square matrix.

Theorem 2.1:

The four equations

$$AXA = A \ldots\ldots\ldots(4)$$

$$XAX = X \ldots\ldots\ldots(5)$$

$$(AX)^{\theta} = AX \ldots\ldots\ldots\ (6)$$

$$(XA)^{\theta} = XA \ldots\ldots\ldots\ (7)$$

have unique solution for any $A \in C_{n\times n}$.

Proof:

First, we shall show that (5) and (6) are equivalent to the single equation

$$XX^{\theta}A^{\theta} = X \ldots\ldots\ldots\ (8)$$

By substituting (6) in (5), we get (8). Conversely, (8) implies $AXX^{\theta}A^{\theta} = AX$, the left hand side of which is s-hermitian. Hence (6) follows and substituting (6) in (8), we get (5)
Similarly, (4) and (7) are equivalent to the equation

$$XAA^{\theta} = A^{\theta} \ldots\ldots\ldots\ (9)$$

Thus to find a solution for the given set of equations, it is enough to find an X satisfying (8) and (9).
Now the expressions $A^{\theta}A, (A^{\theta}A)^2, (A^{\theta}A)^3 \ldots$ cannot all be linearly independent. (ie) there exists a relation

$$\lambda_1 A^{\theta}A + \lambda_2 (A^{\theta}A)^2 + \ldots + \lambda_k (A^{\theta}A)^k = 0 \ldots\ldots\ldots(10)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are not all zero. Let λ_r be the first non-zero λ (ie) $\lambda_1 = \lambda_2 = \ldots = \lambda_{r-1} = 0$.

Put $B = -\lambda_r^{-1}\left\{\lambda_{r+1} I + \lambda_{r+2} A^{\theta}A + \ldots + \lambda_k (A^{\theta}A)^{k-r-1}\right\}$.

Then (10) implies $B(A^{\theta}A)^{r+1} = (A^{\theta}A)^r$ and applying (2) and (3) repeatedly, we get

$BA^{\theta}AA^{\theta} = A^{\theta} \ldots\ldots\ldots\ (11)$.

If we take $X = BA^{\theta}$, then X satisfies (9) by (11).
Also (9) implies $A^{\theta}X^{\theta}A^{\theta} = A^{\theta}$ and hence $BA^{\theta}X^{\theta}A^{\theta} = BA^{\theta}$. Hence X also satisfies (8). Thus $X = BA^{\theta}$ is a solution for the set of equations.

Now let us prove that this X is unique. Suppose that X and Y satisfy (8) and (9). By substituting (7) in (5) and (6) in (4), we obtain $Y = A^{\theta}Y^{\theta}Y$ and $A^{\theta} = A^{\theta}AY$. Now

$$X = XX^{\theta}A^{\theta} = XX^{\theta}A^{\theta}AY = XAY = XAA^{\theta}Y^{\theta}Y = A^{\theta}Y^{\theta}Y = Y$$

Hence the solution X is unique.

Definition 2.2:

Let $A \in C_{n\times n}$. The unique solution of (4), (5), (6) and (7) is called the s-g inverse (secondary generalized inverse) of A and is written as $A^{\dagger_s}$.

Example 2.3:

If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $A^{\dagger_s} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$.

Note 2.4:

From (8) and (9), $A^{\dagger_s}$ satisfies

$$\left.\begin{array}{l} A^{\dagger_s}\left(A^{\dagger_s}\right)^{\theta}A^{\theta} = A^{\dagger_s} = A^{\theta}A^{\dagger_s\,\theta}A^{\dagger_s} \\ A^{\dagger_s}AA^{\theta} = A^{\theta} = A^{\theta}AA^{\dagger_s} \end{array}\right\} \text{..........(12)}$$

If λ is a scalar, then $\lambda^{\dagger_s}$ means λ^{-1} when $\lambda \neq 0$ and 0 if $\lambda = 0$.

3. s-g inverse of s-normal matrices

In this section s-hermitian idempotent matrix is defined. Properties of s-g inverse of a given matrix are given and some results on s-normal matrices are proved.

Theorem 3.1:

Let $A \in C_{n\times n}$. Then

(i) $(A^{\dagger_s})^{\dagger_s} = A$.

(ii) $A^{\theta\dagger_s} = A^{\dagger_s\theta}$.

(iii) If A is non singular, then $A^{\dagger_s} = A^{-1}$.

(iv) $(\lambda A)^{\dagger_s} = \lambda^{\dagger_s}A^{\dagger_s}$.

(v) $\left(A^{\theta}A\right)^{\dagger_s} = A^{\dagger_s}A^{\dagger_s\,\theta}$.

(vi) If U and V are s-unitary $(UAV)^{\dagger_s} = V^{\theta}A^{\dagger_s}U^{\theta}$.

(vii) If $A = \sum A_i$, where $A_i A_j^{\theta} = 0$ and $A_i^{\theta} A_j = 0$ whenever $i \neq j$, then $A^{\dagger s} = \sum A_i^{\dagger s}$.

(viii) If A is s-normal, then $A^{\dagger s} A = AA^{\dagger s}$ and $\left(A^n\right)^{\dagger s} = \left(A^{\dagger s}\right)^n$.

Proof:

(i) to (vi) can be proved by substituting the right hand side of each in the defining relations for the required s-g inverse in each case. To prove (v) we need (12) and to prove (vii) we require the fact that $A_i A_j^{\dagger s} = 0$ and $A_i^{\dagger s} A_j = 0$ if $i \neq j$ which follows from $A_j^{\dagger s} = A_j^{\theta} A_j^{\dagger s\,\theta} A_j^{\dagger s}$ and $A_i^{\dagger s} = A_i^{\dagger s} A_i^{\dagger s\,\theta} A_i^{\theta}$. (v) and (12) imply $A^{\dagger s} A = \left(A^{\theta} A\right)^{\dagger s} A^{\theta} A$ and $AA^{\dagger s} = \left(AA^{\theta}\right)^{\dagger s} AA^{\theta}$ and hence the first part of (viii) follows. The second part is a consequence of the first part.

Theorem 3.2:

A necessary and sufficient condition for the equation $AXB = D$ to have a solution is $AA^{\dagger s} DB^{\dagger s} B = D$, in which case the general solution is

$$X = A^{\dagger s} DB^{\dagger s} + Y - A^{\dagger s} AYBB^{\dagger s},$$

where Y is arbitrary.

Proof:

If X satisfies $AXB = D$, then $D = AXB = AA^{\dagger s} AXBB^{\dagger s} B = AA^{\dagger s} DB^{\dagger s} B$.

Conversely, if $D = AA^{\dagger s} DB^{\dagger s} B$, then $X = A^{\dagger s} DB^{\dagger s}$ is a particular solution of $AXB = D$, since $AXB = AA^{\dagger s} DB^{\dagger s} B = D$.

If $Y \in C_{n \times n}$, then any expression of the form $X = A^{\dagger s} DB^{\dagger s} + Y - A^{\dagger s} AYBB^{\dagger s}$ is a solution of $AXB = D$ and conversely, if X is a solution of $AXB = D$, then $X = A^{\dagger s} DB^{\dagger s} + X - A^{\dagger s} AXBB^{\dagger s}$ satisfies $AXB = D$. Hence the theorem.

Corollary 3.3:

The matrix equations $AX = B$ and $XD = E$ have a common solution if and only if each equation has a solution and $AE = BD$.

Proof:

It is easy to see that the conditions is necessary. Conversely, $A^{\dagger s} B$ and $ED^{\dagger s}$ are solutions of $AX = B$ and $XD = E$ and hence $AA^{\dagger s} B = B$ and $ED^{\dagger s} D = E$. Also $AE = BD$. By using these facts, it can be proved that $X = A^{\dagger s} B + ED^{\dagger s} - A^{\dagger s} AED^{\dagger s}$ is a common solution of the given equations.

Definition 3.4:

A matrix $E \in C_{n\times n}$ is said to be s-hermitian idempotent matrix if $EE^{\theta} = E$ (ie) $E = E^{\theta}$ and $E^2 = E$.

Theorem 3.5:

(i) $A^{\dagger_s}A$, $AA^{\dagger_s}$, $I - A^{\dagger_s}A$, $I - AA^{\dagger_s}$ are all s-hermitian idempotent.

(ii) K is idempotent $\Leftrightarrow$ there exist s-hermitian idempotents E and F such that $K = (FE)^{\dagger_s}$ in which case $K = EKF$.

Proof:

Proof of (i) is obvious. If K is idempotent, then $K^2 = K$. By (i) of theorem 3.1, $K = \left\{\left(K^{\dagger_s}K\right)\left(KK^{\dagger_s}\right)\right\}^{\dagger_s}$. Now if we take $E = KK^{\dagger_s}$ and $F = K^{\dagger_s}K$, they will satisfy our requirements. Conversely, if $K = (FE)^{\dagger_s}$ then $K = EFPEF$, where $P = (FE)^{\dagger_s\theta}(FE)^{\dagger_s}(FE)^{\dagger_s\theta}$ (Here we use the fact $Q^{\dagger_s} = Q^{\theta}\left(Q^{\dagger_s\theta}Q^{\dagger_s}Q^{\dagger_s\theta}\right)Q^{\theta}$ by (12)). Therefore $K = EKF$ and hence $K^2 = E(FE)^{\dagger_s}FE(FE)^{\dagger_s}F = E(FE)^{\dagger_s}F = K$. Hence K is idempotent.

Note 3.6:

(1) s-hermitian idempotent matrices are s-normal matrices.
(2) The s-g inverse of an s-hermitian idempotent matrix is also s-hermitian idempotent matrix.

Conclusion:

s-g inverse of a given square matrix is defined and the condition for the existence of s-g inverse of a matrix is obtained . s-g inverse of an s- normal matrix is classified. s-g inverses of other special type of matrices may also be obtained and their characterizations can be developed.

References:

[1]. Weddurburn, J.H.M., *Lectures on matrices*, colloq. Publ. Amer. Math. Soc. No.17, 1934.
[2]. Isarel-Adi Ben and Greville Thomas M E; Generalized *inverses: Theory and Applications*; A wiley interscience publications Newyork, 1974.
[3]. Ann Lee: Secondary *symmetric, secondary skew symmetric, secondary orthogonal matrices;* Period math. Hungary 7, 63-76, 1976.
[4]. S. Krishnamoorthy, R. Vijayakumar, *On s-normal matrices*, Journal of Analysis and Computation, Vol 2, 2009.
[5]. S. Krishnamoorthy, R. Vijayakumar, *Some characteristics on s-normal matrices*, International Journal of Computational and Applied mathematics, Vol.4, No.1, 49- 53, 2009.
[6] S. Krishnamoorthy, R. Vijayakumar, *Some equivalent conditions on s-normal matrices*, International Journal of Contemporary Mathematical Sciences, Vol.4, No.29, 1449- 1454, 2009.