On the Stability of Affine Functional Equations in Various Spaces

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Abstract* — *In this paper, we study the Hyers-Ulam-Rassias stability of the following affine functional equation
$f(2x+y)+f(x+2y)+f(x)+f(y)=4f(x+y)$
on 2- Banach space, Random normed space and Intuitionistic random normed space.

Keywords — 2-Banach spaces, RN-space, IRN – space, affine functional equations.

I. Introduction

In 1940, Stanislaw M. Ulam [10], triggered the study of stability problems for various functional equations. He presented a number of important unsolved problems. One of the interesting problem in the theory of non-linear analysis concerning the stability of homomorphism was as follows:

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$, for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$, for all $x\in G_1$? If the answer is affirmative, we would say that equation of homomorphism $H(xy)=H(x)H(y)$ is stable.

In 1941, D. H. Hyers [3] was the first mathematician to present the result concerning the stability of functional equations on Banach spaces. The generalized version of D. H. Hyers [3] result was given by famous Greece mathematician Th. M. Rassias [11] in 1978. The stability paper [12] given by Th. M. Rassias has significantly influenced in the development of stability of functional equations. Further, in 1994, P. Gavruta [6] provided a further generalization in which he replaced the bound $\theta(\|x\|^p+\|y\|^p)$ by a general function $\phi(x,y)$ for the existence of unique linear mapping. The functional equation

$$f(2x+y)+f(x+2y)+f(x)+f(y)=4f(x+y) \tag{1.1}$$

is called as affine functional equation. Since the function the affine mapping f (i.e. it is a sum between a constant and a additive function) is the solution of the above functional equation (1.1). Thus, it is called the affine functional equation. Recently, L. Cadariu et. al. [6] proved the stability of the above said functional equation on abelian groups using direct and fixed point approach.

This paper is organized as follows: In Section 1, we adopt some usual terminology, notations and conventions which will be used later in the next sections. In Section 2 we establish the Hyers-Ulam-Rassias stability of the affine functional equation (1.1) in 2-Banach space. In Section 3, we establish the Hyers-Ulam-Rassias stability of the affine functional equation (1.1) in random normed space. Further in the last section we established the Hyers-Ulam-Rassias stability in IRN-spaces.

In 1960s, S. Gahler [7, 8, 9] introduced the concept of linear 2-normed spaces.

Definition 1.1. Let A be a linear space over $\Re$ with $\dim A>1$ and let $\|.,.\|:A\times A\to\Re$ be a function satisfying the following properties:

(a) $\|x,y\|=0$ if and only if x and y are linearly dependent,

(b) $\|x,y\|=\|y,x\|$,

(c) $\|\lambda x,y\|=|\lambda|\|x,y\|$,

(d) $\|x,y+z\|\le\|x,y\|+\|x,z\|$,

for all $x,y,z\in A$ and $\lambda\in\Re$. Then the function $\|.,.\|$ is called a 2-norm on A and the pair $(A,\|.,.\|)$ is called a linear 2-normed space. Sometimes the condition (d) called the triangle inequality.

In 2011, W. G. Park [13] introduces a basic property of linear 2-normed spaces as follows.

Lemma 1.2. Let $(A,\|.,.\|)$ be a linear 2-normed space. If $\|x,y\|=0$ for all $y\in A$, then $x = 0$.

In the 1960's, S. Gahler and A. White [1, 2, 9] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space A is called a Cauchy sequence if there are two points $y,z\in A$ such that y and z are linearly independent, $\lim_{l,m\to\infty}\|x_l - x_m, y\| = 0$ and $\lim_{l,m\to\infty}\|x_l - x_m, z\| = 0$.

Definition 1.4. A sequence $\{x_n\}$ in a linear 2-normed space A is called a convergent sequence if there is an $x\in A$ such that $\lim_{n\to\infty}\|x_n - x, y\| = 0$ for all $y\in A$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n\to\infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Lemma 1.5. For a convergent sequence $\{x_n\}$ in a linear 2-normed space A, $\lim_{n\to\infty}\|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for all $y\in A$.

Definition 1.6. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Throughout this paper, let A be a normed linear space and B a 2-Banach space.

II. Stability of the Affine Functional Equation (1.1) in 2- Banach Space

Theorem 2.1. Let A be a linear space, B be a 2-Banach space and $f: A\to B$ be a mapping with $0\le\mu<\infty$ and $0<p<2$ satisfying the inequality

$$\|f(2x+y)+f(x+2y)+f(x)+f(y)-4f(x+y),\ z\| \le \mu(\|x\|^p+\|y\|^p) \qquad (2.1)$$

for all $x,y\in A$ and $z\in B$. Then, there exists a unique mapping $Q: A\to B$ such that

$$\|f(x)-Q(x), y\| \le \frac{2^{(l+1)(m-l)}-2^{lp(m-l)}}{2(2-2^p)}\mu\|x\|^p \qquad (2.2)$$

for all $x\in A$ and $y\in B$.

Proof: Let $y = 0$ in (2.1), we get

$$\|f(2x)-2f(x),\ z\| \le \mu\|x\|^p$$

$$\left\|\frac{f(2x)}{2}-f(x),\ z\right\| \le \frac{1}{2}\mu\|x\|^p \qquad (2.3)$$

for all $x\in A$ and $z\in B$. Substituting $2x$ at the place of x and dividing by 2 in the above inequality (2.3), we get

$$\left\|\frac{f(2^2x)}{2^2}-\frac{f(2x)}{2}, z\right\| \le \frac{1}{2.2}\mu\|2x\|^p \qquad (2.4)$$

for all $x\in A$ and $z\in B$. Again replacing x with $2^i x$ and dividing by 2^i in inequality (2.3), we have

$$\left\|\frac{f(2^{i+1}x)}{2^{i+1}}-\frac{f(2^ix)}{2^i},\ z\right\| \le \frac{1}{2.2^i}\mu\|2^ix\|^p \le \frac{1}{2.2^i}2^{ip}\mu\|x\|^p \qquad (2.5)$$

for all $x\in A$, $z\in B$ and $i\ge 0$. Now, in order to prove that the sequence $\left\{\frac{f(2^nx)}{2^n}\right\}$ is a convergent sequence, let us consider l, m be two positive number with $l<m$, such that

$$\left\|\frac{f(2^mx)}{2^m}-\frac{f(2^lx)}{2^l},\ z\right\| \le \frac{\mu}{2}\sum_{i=l}^{m-1}\frac{2^{ip}}{2^i}\mu\|x\|^p \qquad (2.6)$$

for all $x\in A$, $z\in B$ and $i\ge 0$. Taking limit on both sides of (2.6), we get

$$\lim_{m,l\to\infty}\left\|\frac{f(2^mx)}{2^m}-\frac{f(2^lx)}{2^l}, z\right\| = 0$$

for all $x\in A$ and all $z\in B$. Which implies that the sequence $\left\{\frac{f(2^nx)}{2^n}\right\}$ is a Cauchy sequence in B. Since the space B is 2- Banach space, the sequence $\left\{\frac{f(2^nx)}{2^n}\right\}$ is convergent also. Therefore, we may define a quadratic mapping $Q: A\to B$ defined by $Q(x)=\lim_{n\to\infty}\frac{f(2^nx)}{2^n}$ for all $x\in A$. Now, to prove that the mapping $Q: A\to B$ also satisfies the functional equation (1.1), by using Lemma 1.5 and the inequality (2.1), we have

$$\|Q(2x+y)+Q(x+2y)+Q(x)+Q(y)-4Q(x+y),\ z\|$$

$$= \lim_{i\to\infty}\frac{1}{2^i}\|f(2.2^ix+2^iy)+f(2^ix+2.2^iy)+f(2^ix) + f(2^iy)-4f(2^ix+2^iy),\ z\|$$

$$\le \lim_{i\to\infty}\frac{1}{2^i}\mu(\|2^ix\|^p+\|2^iy\|^p)$$

$$\le \lim_{i\to\infty}\frac{2^{ip}}{2^i}\mu\|x\|^p + \lim_{i\to\infty}\frac{2^{ip}}{2^i}\|y\|^p = 0$$

$$\|Q(2x+y)+Q(x+2y)+Q(x)+Q(y)-4Q(x+y), z\| = 0$$

for all $x,y\in A$ and all $z\in B$. In order to prove the inequality (2.2) that is the main result of theorem 2.1, by using (2.6), we have

$$\|f(x)-Q(x), y\| = \lim_{m\to\infty}\left\|f(x)-\frac{f(2^mx)}{2^m}, y\right\| \le \frac{2^{(l+1)(m-l)}-2^{lp(m-l)}}{2(2-2^p)}\mu\|x\|^p$$

for all $x\in A$ and all $y\in B$. Now, to prove the uniqueness of the mapping $Q: A\to B$, let us consider another mapping $Q^1: A\to B$ which satisfies the inequality (1.1), we have

$$\left\|Q(x)-Q^1(x),y\right\|=\frac{1}{2^n}\left\|Q(2^n x)-Q^1(2^n x),y\right\|$$

$$\leq\frac{1}{2^n}(\left\|Q(2^n x)-f(2^n x),y\right\|+\left\|Q^1(2^n x)-f(2^n x),y\right\|)$$

which tends to zero as $n\to\infty$ for all $x\in A$ and all $y\in B$. Hence by Lemma 1.2 we conclude that $Q(x)=Q^1(x)$ for all $x\in A$. Which proves the uniqueness of $Q:A\to B$. Hence the theorem.

Theorem 2.2. Let A be a linear space, B be a 2-Banach space and $f:A\to B$ be a mapping with $0\leq\mu<\infty$ and $p>2$ satisfying the inequality

$$\left\|f(2x+y)+f(x+2y)+f(x)+f(y)-4f(x+y),z\right\|$$
$$\leq\mu(\left\|x\right\|^p+\left\|y\right\|^p)\qquad(2.7)$$

for all $x,y\in A$ and $z\in B$. Then, there exists a unique mapping $Q:A\to B$ such that

$$\left\|f(x)-Q(x),y\right\|\leq\frac{2^{lp(m-l)}-2^{(l)(m-l)}}{2(2^p-2)}\mu\left\|x\right\|^p\qquad(2.8)$$

for all $x\in A$ and $y\in B$.

Proof: Putting $y=0$ in (2.7), we get

$$\left\|f(2x)-2f(x),z\right\|\leq\mu\left\|x\right\|^p$$

$$\left\|f(x)-2f\left(\frac{x}{2}\right),z\right\|\leq\mu\left\|\frac{x}{2}\right\|^p\qquad(2.9)$$

for all $x\in A$ and $z\in B$. Substituting $x/2$ at the place of x and multiplying by 2 in the above inequality (2.9), we get

$$\left\|2f\left(\frac{x}{2}\right)-2^2f\left(\frac{x}{2^2}\right),z\right\|\leq2\mu\left\|\frac{x}{2^2}\right\|^p$$

for all $x\in A$ and $z\in B$. Again replacing x with $x/2^i$ and multiplying by 2^i in inequality (2.9), we have

$$\left\|2^i f\left(\frac{x}{2^i}\right)-2^{i+1}f\left(\frac{x}{2^{i+1}}\right),z\right\|\leq2^i\mu\left\|\frac{x}{2^{i+1}}\right\|^p$$

for all $x\in A$, $z\in B$ and $i\geq0$. Now in order to prove that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a convergent sequence, let us consider l, m be two positive number with $l<m$, such that

$$\left\|2^m f(\frac{x}{2^m})-2^l f(\frac{x}{2^l}),z\right\|\leq\mu\sum_{i=l}^{m-1}\frac{2^i}{2^{ip}}\left\|x\right\|^p\qquad(2.10)$$

for all $x\in A$, $z\in B$ and $i\geq0$. Taking limit on both sides of (2.10), we get

$$\lim_{m,l\to\infty}\left\|2^m f(\frac{x}{2^m})-2^l f(\frac{x}{2^l}),z\right\|=0$$

for all $x\in A$ and all $z\in B$. Which implies that the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence in B. Since the space B is 2- Banach space, the sequence $\left\{2^n f(\frac{x}{2^n})\right\}$ is convergent also. Therefore, we may define a mapping $Q:A\to B$ defined by $Q(x)=\lim_{n\to\infty}2^n f(\frac{x}{2^n})$ for all $x\in A$. Further, the remaining proof of this theorem is similar as the proof of Theorem 2.1.

III. Random Normed Stability of the Affine Functional Equation Using Fixed Point Approach

In this section, we shall prove the Hyers-Ulam-Rassias stability of affine functional equation (1.1) in random normed space using fixed point approach.

Definition 3.1 [16] A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly a t – norm) if T satisfies the following conditions :

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1)=a$ for all $a\in[0,1]$;

(d) $T(a,b)\leq T(c,d)$ whenever $a\leq c$ and $b\leq d$ for all $a,b,c,d\in[0,1]$.

Typical examples of continuous t-norm are
$T(a,b)=ab$, $T(a,b)=\max(a+b-1,0)$ and
$T(a,b)=\min(a,b)$

Definition 3.2 [15] A Random Normed space (briefly RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

(RN2) $\mu_{\alpha x}(t)=\mu_x(t/|\alpha|)$ for all x in X, $\alpha\neq0$ and all $t\geq0$;

(NN3) $\mu_{x+y}(t+s)\geq T(\mu_x(t),\mu_y(s))$ for all x, $y\in X$ and all $t,s\geq0$

Definition 3.3 Let (X,μ,T) be an RN- space

1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $t>0$ and $\varepsilon>0$, there exists a positive integer N such that $\mu_{x_n-x}(t)>1-\varepsilon$ whenever $n\geq N$.

2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for every t>0 and ε>0, there exists a positive integer N such that $\mu_{x_n - x_m}(t) > 1-\varepsilon$ whenever $n \geq m \geq N$.
3) An RN- space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 3.4 [16] If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then

$Lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Theorem 3.5 (Fixed point alternative). Let (X, d) be a complete generalized metric space and a contractive mapping J: $X \to X$, with the Lipschitz constant L. Then, for each given element $x \in X$, either

(A$_1$) $d(J^n x, J^{n+1}x) = +\infty$ for all $n \geq 0$,

Or

(A$_2$) There exists a natural n_0 such that:

(A$_{20}$) $d(J^n x, J^{n+1}x) = +\infty$ for all $n \geq n_0$,

(A$_{21}$) The sequence $(J^n x)$ is convergent to a fixed point y^* of J;

(A$_{22}$) y^* is the unique fixed point of J in the set

$$Y = \{y \in X, d(J^{n_0}x, y) < +\infty\};$$

(A$_{23}$) $d(y, y^*) \leq \dfrac{1}{1-L} d(y, Jy)$, for all $y \in Y$.

The following lemmas will be used in the proof of theorem (3.1)

Lemma 3.6 [17] Let (X, d) be a complete generalized metric space and let $A : X \to X$ be a strict contraction with the Lipschitz constant L such that $d(x_0, A(x_0)) < +\infty$ for some $x_0 \in X$. Then A has a unique fixed point in the set $Y = \{y \in X, d(x_0, y) < \infty\}$ and the sequence $(A^n(x))_{n\in N}$ converges to the fixed point x^* for every $x \in Y$. Moreover, $d(x_0, A(x_0)) \leq \delta$ implies $d(x^*, x_0) \leq \delta/1-L$.

Lemma 3.7 [18, 19] d_G is a complete generalized metric on E.

Theorem 3.1 Let X be a real linear space, 'f' be a mapping from X into a complete random normed space (Y, μ, T_M) with $f(0) = 0$ and let $\Phi : X \times X \to D^+$ be a mapping satisfying

$$\Phi_{2x,2y}(\alpha t) \geq \Phi_{x,y}(t) \qquad (3.1)$$

for all $x, y \in X$ and for all t>0. If

$$\mu_{(f(2x+y)+f(x+2y)+f(x)+f(y)-4f(x+y))}(t) \geq \Phi_{x,y}(t) \qquad (3.2)$$

for all x, $y \in X$, then there exists a unique mapping $g : X \to Y$ such that

$$\mu_{g(x)-f(x)}(t) \geq \Phi_{x,0}((2-\alpha)t) \qquad (3.3)$$

for all $x \in X$ and for all t>0, and $g(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$.

Proof: Let us consider $y = 0$ in (3.2), we get

$$\mu_{(f(2x)-2f(x))}(t) \geq \Phi_{x,0}(t) \quad \text{for all } x \in X \qquad (3.4)$$

therefore

$$\mu_{(\frac{1}{2}f(2x)-f(x))}(t) = \mu_{\frac{1}{2}(f(2x)-2f(x))}(t)$$

$$= \mu_{(f(2x)-2f(x))}(2t) \geq \Phi_{x,0}(2t) \text{ for all } x \in X \text{ and } t>0 \qquad (3.5)$$

Let us consider the set $S = \{g : X \to Y\}$ and the mapping d_G in S defined by

$$d_G(f, g) = \inf\{u \in R^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t), \forall x \in X, t > 0\}$$

Now using Lemma 3.7, (d_G, S) is a complete generalized metric space. Let us consider the linear mapping $J : S \to S$ such that

$$Jg(x) = \frac{g(2x)}{2}$$

We claim that the mapping J is a strictly contractive self mapping of S with the Lipschitz constant $\alpha/2$. Let g and h be two mappings in S such that $d_G(g,h) < \varepsilon$, then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \Phi_{x,0}(2t) \quad \text{for all } x \in X \text{ and } t>0$$

Hence $\mu_{Jg(x)-Jh(x)}(\frac{\alpha}{2}\varepsilon t) = \mu_{\frac{1}{2}(g(2x)-h(2x))}(\frac{\alpha}{2}\varepsilon t)$

$$= \mu_{(g(2x)-h(2x))}(\alpha\varepsilon t)$$

$$\geq \Phi_{2x,0}(2\alpha t)$$

for all $x \in X$, t>0

Since $\Phi_{2x,0}(2\alpha t) \geq \Phi_{x,0}(2t)$,

$\mu_{Jg(x)-Jh(x)}(\frac{\alpha}{2}\varepsilon t) \geq \Phi_{x,0}(2t)$, that is $d_G(g,h) < \varepsilon$

$$\Rightarrow d_G(Jg, Jh) < \frac{\alpha}{2}\varepsilon$$

which implies that $d_G(Jg, Jh) < \frac{\alpha}{2} d_G(g,h)$ for all g, h in S. Now, it follows from (3.5) that $d_G(f, Jf) \leq 1$. Using the Lemma 3.6, we show the existence of a fixed point of J, that is the existence of a mapping $g : X \to Y$ satisfying the following:

(i) g is a fixed point of J, that is $g(2x) = 2g(x)$ for all x in X.

(ii) Since for any x in X and t>0, $d_G(J^n f, g) < \varepsilon$ implies $\mu_{u(x)-v(x)}(t) \geq \Phi_{x,0}(\frac{2t}{\varepsilon})$, from $d_G(J^n f, g) \to 0$, it follows that the $\lim_{n\to\infty} \dfrac{f(2^n x)}{2^n} = g(x)$ for all x in X.

(iii) Also, $d_G(f,g) \le \frac{1}{1-L} d_G(f,Jf)$ implies that the inequality

$$d_G(f,g) \le \frac{1}{1-\frac{\alpha}{2}} d_G(f,Jf)$$

and so,

$$\mu_{f(x)-g(x)}\left(\frac{2t}{2-\alpha}\right) \ge \Phi_{x,0}(2t)$$

for all $t>0$ and for all $x \in X$. It follows that

$$\mu_{f(x)-g(x)}(t) \ge \Phi_{x,0}((2-\alpha)t)$$

for all $x \in X$ and all $t>0$.

The mapping g is also unique, it follows from the fact that g is the unique fixed point of J with the property that, if there is a $T \in]0, \infty[$ such that

$$\mu_{f(x)-g(x)}(Tt) \ge \Phi_{x,0}(2t)$$

for all $x \in X$ and for all $t>0$.

This completes the proof of theorem.

IV. Intuitionistic Random Normed Stability of Affine Functional Equation (1.1).

In this section, we prove the Hyers-Ulam-Rassias stability of the affine functional equation (1.1) in intuitionistic random normed space.

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of Intuitionistic random normed space as in [15, 16, 18, 20, 21, 22, 23, 24].

Definition 4.1 A measure distribution function is a function $\mu : R \to [0, 1]$ which is left continuous, non-decreasing on R, $\inf_{t\in R}\mu(t) = 0$ and $\sup_{t\in R}\mu(t) = 1$.

We will denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0 & if \quad t \le 0 \\ 1 & if \quad t > 0 \end{cases}$$

If X is a nonempty set, then $\mu : X \to D$ is called a probabilistic measure on X and $\mu(x)$ is denoted by μ_x.

Definition 4.2. A non-measure distribution function is a function $\nu: R \to [0, 1]$ which is right continuous, non-decreasing on R, $\inf_{t\in R} \nu(t) = 0$ and $\sup_{t\in R} \nu(t) = 1$.

We will denote by B the family of all non-measure distribution functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1 & if \quad t \le 0 \\ 0 & if \quad t > 0 \end{cases}$$

If X is a nonempty set, then $\nu : X \to B$ is called a probabilistic non-measure on X and $\nu(x)$ is denoted by ν_x.

Lemma 4.3 [25, 26] Consider the set L^* and operation $\le_{L^*}$ defined by :

$L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2 \; and, x_1 + x_2 \le 1\}$,

$(x_1,x_2) \le_{L^*} (y_1,y_2) \Leftrightarrow x_1 \le y_1, x_2 \ge y_2$,

$\forall (x_1,x_2),(y_1,y_2) \in L^*$

Then $(L^*, \le_{L^*})$ is a complete lattice.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

In section 1, we presented classical t-norm. Using the lattice $(L^*, \le_{L^*})$, these definitions can be straightforwardly extended.

Definition 4.4 [25] A triangular norm (t-norm) on L^* is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions:

(a) $(\forall x \in L^*)(T(x,1_{L^*}) = x)$ (boundary condition);

(b) $(\forall (x,y) \in (L^*)^2)(T(x,y) = T(y,x))$ (commutativity);

(c) $(\forall (x,y,z) \in (L^*)^3)(T(x,T(y,z)) = T(T(x,y),z))$ (associativity);

(d) $(\forall (x,x^1,y,y^1) \in (L^*)^4)(x \le_{L^*} x^1 and,$ $y \le_{L^*} y^1 \Rightarrow T(x,y) \le_{L^*} T(x^1,y^1))$ (monotonicity).

If $(L^*, \le_{L^*}, T)$ is an abelian topological monoid with unit 1_{L^*} then T is said to be a continuous t-norm.

Definition 4.5 [25] A continuous t-norm T on L^* is said to be continuous t-represent able if there exist a continuous t-norm * and a continuous t-conorm ◊ on [0, 1] such that, for all

$x = (x_1, x_2), y = (y_1, y_2) \in L^*, T(x, y) = (x_1 * y_1, x_2 \lozenge y_2)$.

For example,

$T(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$

and $M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$

are continuous t-representable

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$.

Now, we define a sequence T^n recursively by

$T^1 = T$ and

$T^n(x^{(1)}, \ldots, x^{(n+1)}) = T(T^{n-1}(x^{(1)}, \ldots, x^{(n)}), x^{(n+1)})$, $\forall n \ge 2, x^{(i)} \in L^*$.

Definition 4.6 A negator on L^* is any decreasing mapping $N: L^* \to L^*$ satisfying $N(1_{L^*}) = 0_{L^*}$ and $N(0_{L^*}) = 1_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on [0, 1] is a decreasing function N: [0, 1] → [0, 1] satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on [0, 1] defined by $N_s(x) = 1 - x$, $\forall x \in [0, 1]$.

Definition 4.7 Let μ and ν be measure and non-measure distribution functions from $X \times (0, +\infty)$ to [0, 1] such that $\mu_x(t) + \nu_x(t) \le 1$ for all $x \in X$ and $t>0$. The triple $(X, P_{\mu,\nu}, T)$ is said to be an intuitionistic random normed space (briefly IRN-space) if X is a vector space, T is continuous t-represent able and $P_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions for all $x, y \in X$ and $t, s>0$,

(a) $P_{\mu,\nu}(x, 0) = 0_{L^*}$;

(b) $P_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;

(c) $P_{\mu,\nu}(\alpha x, t) = P_{\mu,\nu}(x, t/|\alpha|)$ for all $a \neq 0$;

(d) $P_{\mu,\nu}(x + y, t + s) \geq_{L^*} T(P_{\mu,\nu}(x, t), P_{\mu,\nu}(y, s))$.

In this case, $P_{\mu,\nu}$ is called an intuitionistic random norm. Here, $P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t))$.

Example 4.8 Let $(X, \|\cdot\|)$ be a normed space. Let $T(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$
for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and let μ, ν be measure and non-measure distribution
functions defined by

$$P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right),$$

for all $t \in \mathrm{R}^+$

Then $(X, P_{\mu,\nu}, T)$ is an IRN-space.

Definition 4.9 (1) A sequence $\{x_n\}$ in an IRN-space $(X, P_{\mu,\nu}, T)$ is called a Cauchy sequence if, for any $\varepsilon>0$ and $t>0$, there exists an $n_0 \in N$ such that

$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$ for all $n, m \geq n_0$,

where N_s is the standard negator.

(2) The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{P_{\mu,\nu}} x$) if

$P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t>0$.

(3) An IRN-space $(X, P_{\mu,\nu}, T)$ is said to be complete if every Cauchy sequence in X is
convergent to a point $x \in X$.

Now, we prove the main result of this section as follows:

Theorem 4.10 Let X be a linear space and $(Y, P_{\mu,\nu}, T)$ be a complete IRN-space. Let $f: X \to Y$ be a mapping with $f(0) = 0$ for which there exists $\xi, \zeta : X \times X \to D^+$, where $\xi(x, y)$ is denoted by $\xi_{x,y}(t)$, $\zeta(x, y)$ is denoted by $\zeta_{x,y}(t)$ and $(\xi_{x,y}(t), \zeta_{x,y}(t))$ denoted by $Q_{\xi,\zeta}(x, y, t)$ with the property

$$P_{\mu,\nu}(f(2x+y) + f(x+2y) + f(x) + f(y) - 4f(x+y), t) \geq_{L^*} Q_{\xi,\zeta}(x,y,t) \tag{4.1}$$

If

$$T_{k=1}^{\infty}(Q_{\xi,\zeta}(2^{n+k-1}x, 0, 2^n t)) = 1_{L^*} \tag{4.2}$$

$$\lim_{n\to\infty} Q_{\xi,\zeta}(2^n x, 2^n y, 2^n t) = 1_{L^*} \tag{4.3}$$

for every $x, y \in X$ and $t>0$, then there exists a unique mapping Q: $X \to Y$ such that

$$P_{\mu,\nu}(f(x) - Q(x), t) \geq_{L^*} T_{k=1}^{\infty}(Q_{\xi,\zeta}(2^{k-1}x, 0, t)). \tag{4.4}$$

Proof: - Let $0<\alpha<2$ and $y = 0$ in (4.1), we get,

$$P_{\mu,\nu}(f(2x) - 2f(x), t) \geq_{L^*} Q_{\xi,\zeta}(x,0,t)$$

$$P_{\mu,\nu}\left(\frac{f(2x)}{2} - f(x), \frac{t}{2}\right) \geq_{L^*} Q_{\xi,\zeta}(x,0,2t) \tag{4.5}$$

Therefore, it follows that replacing x with $2^k x$, we have

$$P_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{(k+1)}} - \frac{f(2^k x)}{2^k}, \frac{t}{2^k}\right) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, 2t) \tag{4.6}$$

which shows that

$$P_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{(k+1)}} - \frac{f(2^k x)}{2^k}, t\right) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, 2^{(k+1)}t) \tag{4.7}$$

that is

$$P_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{(k+1)}} - \frac{f(2^k x)}{2^k}, \frac{t}{2^{(k+1)}}\right) \geq_{L^*} Q_{\xi,\zeta}(2^k x, 0, t) \tag{4.8}$$

for all $k \in N$ and $t>0$. As $1 > \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots.. \frac{1}{2^n}$ with the single inequality, it follows that

$$P_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), t\right) \geq_{L^*} P_{\mu,\nu}\left(\frac{f(2^n x)}{2^n} - f(x), \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} t\right)$$

$$\geq_{L^*} T_{k=0}^{n-1}\left(P_{\mu,\nu}\left(\frac{f(2^{k+1}x)}{2^{(k+1)}} - \frac{f(2^k x)}{2^k}, \frac{t}{2^{k+1}}\right)\right)$$

$$\geq_{L^*} T_{k=0}^{n-1}(Q_{\mu,\nu}(2^{k-1}x, 0, t)) \tag{4.9}$$

To prove the convergence of the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$, replacing x with $2^m x$ in the above inequality (4.9), we get

$$P_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{2^{(n+m)}} - \frac{f(2^m x)}{2^m}, t\right)$$

$$\geq_{L^*} T_{k=0}^{n-1}(Q_{\mu,\nu}(2^{k+m-1}x, 0, 2^m t)) \tag{4.10}$$

since the right hand side of the inequality tends to 1_{L^*} as m tends to ∞, hence the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence since Y is a complete IRN-space. Therefore, we may define

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \text{ for all } x \text{ in } X.$$

Now, to show that Q is a quadratic mapping, Replacing x and y with $2^n x$ and $2^n y$ respectively in (4.1), it follows that

$$P_{\mu,\nu}\left(\frac{f(2^n(2x+y))}{2^n} + \frac{f(2^n(x+2y))}{2^n} + \frac{f(2^n x)}{2^n} + \frac{f(2^n y)}{2^n} - 4\frac{f(2^n(x+y))}{2^n}, t\right)$$

$$\geq_{L^*} Q_{\xi,\zeta}(2^n x, 2^n y, 2^n t)$$

taking the limit as $n \to \infty$, we find that Q satisfies (4.1) for all $x, y \in X$. Taking limit as $n \to \infty$, inequality (4.9) implies (4.4).

To prove the uniqueness of the quadratic mapping Q, let us assume that there exists a quadratic function Q^1 which satisfies (4.4). Obviously, we have $Q(2^n x) = 2^n Q(x)$ and $Q^1(2^n x) = 2^n Q^1(x)$ for all $x \in X$ and $n \in N$. Hence, it follows from (4.2) and (4.4) that

$$P_{\mu,\nu}(Q(x) - Q^1(x), t) \geq_{L^*} P_{\mu,\nu}(Q(2^n x) - Q^1(2^n x), 2^n t)$$

$$P_{\mu,\nu}(Q(x) - Q^1(x), t) \geq_{L^*} T_{\mu,\nu}(P_{\mu,\nu}(Q(2^n x) - f(2^n x), 2^{n-1} t), P_{\mu,\nu}(f(2^n x) - Q^1(2^n x), 2^{n-1} t))$$

$$P_{\mu,\nu}(Q(x) - Q^1(x), t) \geq_{L^*} T(T_{k=1}^{\infty}(Q_{\xi,\zeta}(2^{n+k-1} x, 0, 2^n t), T_{k=1}^{\infty}(Q_{\xi,\zeta}(2^{n+k-1} x, 0, 2^n t)) = T(1_{L^*}, 1_{L^*}) = 1_{L^*}$$

for all x in X. Hence the desired result.

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