A Discrete Host Commensal Species with Limited Resources and Mortality Rate for the Commensal

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***Abstract* - This paper deals with an investigation on a Discrete Host-Commensal species with limited resources and mortality rate for the commensal. The model comprises of a commensal (S_1), host (S_2) that would benefit S_1, without getting effected either positively or adversely. The model is characterized by a couple of first order non-linear ordinary differential equations. All possible equilibrium points are identified based on the model equations at two stages and criteria for their stability are discussed.**

KeyWords - Commensal, Equilibrium point, Host, Oscillates, Stable.

1. INTRODUCTION

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [4] and by Volterra [5]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Meyer [6], Kushing [7], Paul colinvaux [8], Kapur [9] etc. The ecological interactions can be broadly classified as Prey – predation, Competition, Commensalim, Ammensalism, Neutralism and so on. N.C.Srinivas [10] studied competitive eco-systems of two species and three species with limited and unlimited resources. Later, Lakshminarayan [11], Lakshminarayan and Pattabhi Ramacharyulu [12] studied Prey-predator ecological models with a partial cover for the prey and alternate food for the predator. Stability analysis of competitive species was carried out by Archana Reddy, Pattabhi Ramacharyulu and Krishna Gandhi [3] and by Bhaskara Rama Sarma and Pattabhi Ramacharyulu [13], while Ravindra Reddy [15] investigated mutualism between two species. Recently Phani Kumar [14] studied some mathematical models of ecological commensalism and Acharyulu [1, 2] investigated Ammensalism between two species.

The present investigation is a study of a discrete host-commensal species with limited resources and mortality rate for the commensal. Figure-1 shows the schematic sketch of the system under investigation.

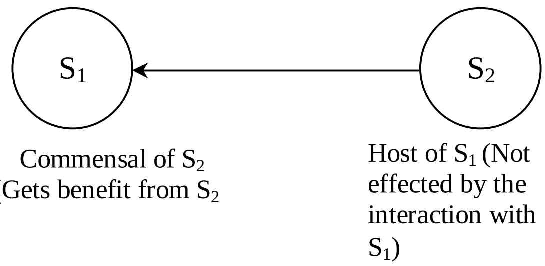

Fig 1: Schematic sketch of the system

Commensalism is a symbiotic interaction between two populations where one population (S_1) gets benefit from (S_2) while the other (S_2) is neither harmed nor benefited due to the interaction with (S_1). The benefited species (S_1) is called the commensal and the other (S_2) is called the host. Examples of commensalism are (i) A squirrel in an oak tree gets a place to live and food for its survival, while the tree remains neither benefited nor harmed. (ii)The clownfish shelters among the tentacles of the sea anemone, while the sea anemone is not effected.

2. BASIC EQUATIONS

The model equations for two species host-commensal species with limited resources and mortality rate for the commensal is given by the following system of first order non-linear differential equations employing the following notation.

NOTATION ADOPTED:

$N_1(t)$: The population strength of commensal species (S_1)
$N_2(t)$: The population strength of host species (S_2)
t : Time instant
a_i : Natural growth rates of S_i, i = 1, 2
a_{ii} : Self inhibition coefficient of S_i, i = 1, 2
a_{12}: Commensal coefficient of S_1

$K_i = \frac{a_i}{a_{ii}}$: Carrying capacities of S_i, i = 1,2

If the death rate is greater than the birth rate for any species, we continue to use the same notation as natural growth rate with negative sign for the rate of difference.

Further the variables N_1, N_2 are non-negative and the model parameters a_1, a_2, a_{11}, a_{12}, a_{22} are assumed to be non-negative constants.

The derivative form of the basic model equations for the growth rates of S_1, S_2 are

$$\frac{dN_1}{dt} = -a_1 N_1 - a_{11} N_1^2 + a_{12} N_1 N_2 \quad (1)$$

$$\frac{dN_2}{dt} = a_2 N_2 - a_{22} N_2^2 \quad (2)$$

The discrete form of the equations (1) and (2) is

$$N_1(t+1) = \alpha_1 N_1(t) - a_{11} N_1^2(t) + a_{12} N_1(t) N_2(t)$$

$$N_2(t+1) = \alpha_2 N_2(t) - a_{22} N_2^2(t)$$

where $\alpha_1 = 1 - a_1, \alpha_2 = 1 + a_2$

3. EQUILIBRIUM STATES

Stage I:

The system under investigation has three equilibrium states given by

$N_i(t+1) = N_i(t)$, i = 1,2

(i) Fully washed out state

E_0: $\overline{N}_1 = 0, \overline{N}_2 = 0$

(ii) The state in which only the host survives and the commensal is washed out

E_1: $\overline{N}_1 = 0, \overline{N}_2 = K_2$

(iii) Co-existent state (Commensal and host both survive)

E_2: $\overline{N}_1 = \frac{a_{12}K_2}{a_{11}} - K_1, \overline{N}_2 = K_2$

This state exists only when $\frac{a_{12}K_2}{a_{11}} > K_1$

4. STABILITY OF EQUILIBRIUM STATES

4.1 Stability of $E_0(0,0)$:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = 0$

i.e. $N_1(t+r) = 0$, $N_2(t+r) = 0$, where r is an integer

4.2 Stability of $E_1(0, K_2)$:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots\ldots = 0$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots\ldots = K_2$

i.e. $N_1(t+r) = 0$, $N_2(t+r) = K_2$, where r is an integer

Hence, $E_1(0, K_2)$ is **stable**.

4.3 Stability of E_2 $(\overline{N}_1, \overline{N}_2)$:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots. = \frac{a_{12}K_2}{a_{11}} - K_1$ (2)

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = K_2$

i.e. $N_1(t+r) = \frac{a_{12}K_2}{a_{11}} - K_1$, $N_2(t+r) = \ldots. = K_2$,

where r is an integer (3)

Hence, E_2 $(\overline{N}_1, \overline{N}_2)$ is **stable.**

(4)

At this stage all the three equilibrium states E_0, E_1, E_2 are **stable**.

(5)

Stage II:

The system under investigation has thirteen equilibrium states given by

$N_i(t+2) = N_i(t)$, i = 1, 2

(i) Fully washed out state

E_0: $\overline{N}_1 = 0, \overline{N}_2 = 0$. (6)

(ii) States in which only the host survives and the commensal is washed out.

E_1: $\overline{N}_1 = 0, \overline{N}_2 = K_2$

E_2: $\overline{N}_1 = 0, \overline{N}_2 = \frac{(a_2 + 2) + \sqrt{a_2^2 - 4}}{2a_{22}}$,

when $a_2 > 2$

E_3: $\overline{N}_1 = 0,\ \overline{N}_2 = \frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$,

when $a_2 > 2$

E_4: $\overline{N}_1 = 0,\ \overline{N}_2 = \frac{2}{a_{22}}$, when $a_2 = 2$

(iii) Co-existent states (or) Normal steady states.

E_5: $\overline{N}_1 = \frac{a_{12}K_2}{a_{11}} - K_1,\ \overline{N}_2 = K_2$

E_6: $\overline{N}_1 = \frac{(b_1+2) + \sqrt{b_1^2-4}}{2a_{11}},\ \overline{N}_2 = K_2$, when $b_1>2$, where $b_1 = a_{12}K_2 - a_1$

E_7: $\overline{N}_1 = \frac{(b_1+2) - \sqrt{b_1^2-4}}{2a_{11}},\ \overline{N}_2 = K_2$, when $b_1 > 2$

E_8: $\overline{N}_1 = \frac{2}{a_{11}},\ \overline{N}_2 = K_2$, when $b_1 = 2$

E_9: $\overline{N}_1 = \frac{2a_{12}}{a_{11}a_{22}} - K_1,\ \overline{N}_2 = \frac{2}{a_{22}}$, when $a_2 = 2$

E_{10}: $\overline{N}_1 = \frac{(d_1+2) + \sqrt{d_1^2-4}}{2a_{11}},\ \overline{N}_2 = \frac{2}{a_{22}}$, when $a_2 = 2$ and $d_1 > 2$,

where $d_1 = \frac{2a_{12}}{a_{22}} - a_1$

E_{11}: $\overline{N}_1 = \frac{(d_1+2) - \sqrt{d_1^2-4}}{2a_{11}},\ \overline{N}_2 = \frac{2}{a_{22}}$,

when $a_2 = 2$ and $d_1 > 2$

E_{12}: $\overline{N}_1 = \frac{2}{a_{11}},\ \overline{N}_2 = \frac{2}{a_{22}}$, when $a_2 = 2$ and $d_1 = 2$

5. STABILITY OF EQUILIBRIUM STATES

The stability of the equilibrium points E_0, E_1 and E_5 have been discussed already in the stage I. Now we will discuss the stability of equilibrium points except these three points in this stage.

5.1 Stability of E_2:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots\ldots = 0$, i.e. $N_1(t+r) = 0$ where r is an integer.

$N_2(t) = N_2(t+2) = N_2(t+4) = \ldots\ldots = \frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$

$N_2(t+1) = N_2(t+3) = N_2(t+5) = \ldots = \frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$

i.e. $N_2(t+2r) = \frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$ and

$N_2(t+\overline{2r+1}) = \frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$

where r is an integer.

Hence, E_2 **oscillates** finitely between

$\frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$ and $\frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$, where $a_2>2$

5.2 Stability of E_3:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$, i.e. $N_1(t+r) = 0$, where r is an integer.

$N_2(t) = N_2(t+2) = N_2(t+4) = \ldots = \frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$, where $a_2>2$

$N_2(t+1) = N_2(t+3) = N_2(t+5) = \ldots = \frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$

i.e. $N_2(t+2r) = \frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$ and

$N_2(t+\overline{2r+1}) = \frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$

where r is an integer.

Hence, E_3 **oscillates** between

$\frac{(a_2+2) - \sqrt{a_2^2-4}}{2a_{22}}$ and $\frac{(a_2+2) + \sqrt{a_2^2-4}}{2a_{22}}$, where $a_2>2$

5.3 Stability of E_4:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = 0$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{2}{a_{22}}$

$N_1(t+r) = 0$, $N_2(t+r) = \frac{2}{a_{22}}$, where r is an integer. Hence, E_4 is **Stable**.

5.4 Stability of E_6:

$N_1(t) = N_1(t+2) = N_1(t+4) = \ldots =$

$\frac{(b_1+2)+\sqrt{b_1^2-4}}{2a_{11}}$, $b_1>2$

$N_1(t+1) = N_1(t+3) = N_1(t+5) = \ldots =$

$\frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}}$

i.e. $N_1(t+2r) = \frac{(b_1+2)+\sqrt{b_1^2-4}}{2a_{11}}$ and

$$N_1(t+\overline{2r+1}) = \frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}}$$

where r is an integer

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = K_2$

i.e. $N_2(t+r) = K_2$, where r is an integer

Hence, E_6 **oscillates** finitely between

$$\frac{(b_1+2)+\sqrt{a_1^2-4}}{2a_{11}} \text{ and } \frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}}$$

5.5 Stability of E_7:

$N_1(t) = N_1(t+2) = N_1(t+4) = \ldots =$

$\frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}}$

$N_1(t+1) = N_1(t+3) = N_1(t+5) = \ldots =$

$\frac{(b_1+2)+\sqrt{b_1^2-4}}{2a_{11}}$

i.e. $N_1(t+2r) = \frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}}$ and

$$N_1(t+\overline{2r+1}) = \frac{(b_1+2)+\sqrt{b_1^2-4}}{2a_{11}}$$

where r is an integer

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = K_2$

$N_2(t+r) = K_2$, where r is an integer

Hence, E_{17} **oscillates** between

$$\frac{(b_1+2)-\sqrt{b_1^2-4}}{2a_{11}} \text{ and } \frac{(b_1+2)+\sqrt{b_1^2-4}}{2a_{11}}$$

5.6 Stability of E_8:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = \frac{2}{a_{11}}$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = K_2$

i.e. $N_1(t+r) = \frac{2}{a_{11}}$ and $N_2(t+r) = K_2$, where r is an integer

Hence, E_8 is **stable**

5.7 Stability of E_9:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = \frac{2a_{12}}{a_{11}a_{22}} - K_1$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{2}{a_{22}}$

i.e. $N_1(t+r) = \frac{2a_{12}}{a_{11}a_{22}} - K_1$ and $N_2(t+r) = \frac{2}{a_{22}}$,

where r is an integer

Hence, E_9 is **stable**

5.8 Stability of E_{10}:

$N_1(t) = N_1(t+2) = N_1(t+4) = \ldots =$

$\frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}}$, $d_1>2$

$N_1(t+1) = N_1(t+3) = N_1(t+5) = \ldots =$

$\frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}}$, $d_1>2$

i.e. $N_1(t+2r) = \frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}}$ and

$$N_1(t+\overline{2r+1}) = \frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}}, \; d_1>2$$

where r is an integer

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{2}{a_{22}}$

$N_2(t+r) = \frac{2}{a_{22}}$, where r is an integer

Hence, E_{10} **oscillates** finitely between

$$\frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}} \text{ and }$$

$$\frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}}$$

5.9 Stability of E_{11}:

$N_1(t) = N_1(t+2) = N_1(t+4) = \ldots =$

$$\frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}}$$

$N_1(t+1) = N_1(t+3) = N_1(t+5) = \ldots =$

$$\frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}}$$

i.e. $N_1(t+2r) = \frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}}$ and

$$N_1(t+\overline{2r+1}) = \frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}}$$

where r is an integer

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{2}{a_{22}}$

i.e. $N_2(t+r) = \frac{2}{a_{22}}$, where r is an integer

Hence, E_{11} **oscillates** finitely between

$$\frac{(d_1+2)-\sqrt{d_1^2-4}}{2a_{11}} \text{ and } \frac{(d_1+2)+\sqrt{d_1^2-4}}{2a_{11}}$$

5.10 Stability of E_{12}:

$N_1(t) = N_1(t+1) = N_1(t+2) = \ldots = \frac{2}{a_{11}}$

$N_2(t) = N_2(t+1) = N_2(t+2) = \ldots = \frac{2}{a_{22}}$

i.e. $N_1(t+r) = \frac{2}{a_{11}}$ and $N_2(t+r) = \frac{2}{a_{22}}$, where r is an integer

Hence, E_{12} is **stable**.

At this stage, of all thirteen equilibrium points, only the seven equilibrium points are **stable.**

6. CONCLUSIONS

The present paper deals with the study on discrete host-commensal species with limited resources and mortality rate for the commensal. The model comprises of a commensal (S_1), a host (S_2) that benefit S_1, without getting effected either positively or adversely. All possible equilibrium points of the model are identified based on the model equations at two stages.

Stage-I : $N_i(t+1) = N_i(t)$; i = 1, 2

Stage-II : $N_i(t+2) = N_i(t)$; i = 1, 2

In Stage-I there are only three equilibrium points, while the Stage-II there would be thirteen equilibrium points. All the three equilibrium points in Stage-I are found to be **stable** while in stage-II only seven are **stable** and remaining six are oscillatory.

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