Solutions of the Homogeneous Cubic Equation with Six Unknowns

$$\alpha xy(x+y)-\beta zw(z+w)=(\alpha-\beta)XY(X+Y)$$

M.A.Gopalan [1], S.Vidhyalakshmi [2] and K.Lakshmi [3]

[1]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, Tamil Nadu, India

[2]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620002, Tamil Nadu, India

[3]Asst.Professor, Department of Mathematics, Shrimati Indira Gandhi College,Trichy-620002, Tamil Nadu, India

***Abstract*: The homogeneous cubic equation with six unknowns represented by the diophantine equation $\alpha xy(x+y)-\beta zw(z+w)=(\alpha-\beta)XY(X+Y)$ is analyzed for its patterns of non-zero distinct integral solutions and different methods of integral solutions are illustrated.**

***Keywords:* Homogeneous cubic equations, integral solutions**.

MSc 2010 mathematics subject classification: 11D25

Notations:

$T_{m,n}$ -Polygonal number of rank n with size m

P_n^m - Pyramidal number of rank n with size m

PR_n - Pronic number of rank n

OH_n - Octahedral number of rank n

SO_n -Stella octangular number of rank n

S_n -Star number of rank n

J_n -Jacobsthal number of rank of n

j_n - Jacobsthal-Lucas number of rank n

KY_n -keynea number of rank n

$CP_{n,3}$ - Centered Triangular pyramidal number of rank n

$CP_{n,6}$ - Centered hexagonal pyramidal number of rank n

$F_{4,n,3}$ - Four Dimensional Figurative number of rank n whose generating polygon is a triangle

$F_{4,n,5}$ - Four Dimensional Figurative number of rank n whose generating polygon is a pentagon.

I. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems [1-3]. Particularly, in [4-9] cubic equations with 4 unknowns and in [10-11] cubic equations with 5 unknowns are studied for their integral solutions. This communication concerns with yet another non-zero cubic equation with six unknowns given by $\alpha xy(x+y)-\beta zw(z+w)=(\alpha-\beta)XY(X+Y)$.Infinitely many non-zero integer triples (x,y,z) satisfying the above equation are obtained. Various interesting properties among the values of x, y and z are presented.

II. METHOD OF ANALYSIS

The diophantine equation representing the cubic equation with six unknowns under consideration is

$$\alpha xy(x+y)-\beta zw(z+w)=(\alpha-\beta)XY(X+Y) \quad (1)$$

Assuming $x=u+p, y=u-p, z=u+q, z=u-q, x=u+v, x=u-v$ (2)

in (1), it reduces to the equation,

$$\alpha p^2 - \beta q^2 = (\alpha - \beta)v^2 \quad (3)$$

Again using the linear transformation

$$p = S + \beta T, q = S + \alpha T \quad (4)$$

iin (3), it reduces to

$$S^2 - \alpha\beta T^2 = v^2 \quad (5)$$

The above equation (5) is solved through different approaches and thus, one obtains different sets of solutions to (1)

A. CaseI: $\alpha\beta$ is not a square.

1)Approach1:

(5) can be written as

$$v^2 + \alpha\beta T^2 = S^2 \quad (6)$$

The solution of (6) can be written as

$$S = \alpha\beta a^2 + b^2, T = 2ab, v = \alpha\beta a^2 - b^2 \quad (7)$$

In view of (7), (4) and (2) the integral solution of (1) are obtained as

$$\left.\begin{aligned} x &= u + \alpha\beta a^2 + b^2 + 2\beta ab \\ y &= u - \alpha\beta a^2 - b^2 - 2\beta ab \\ z &= u + \alpha\beta a^2 + b^2 + 2\alpha ab \\ w &= u - \alpha\beta a^2 - b^2 - 2\alpha ab \\ X &= u + \alpha\beta a^2 - b^2 \\ Y &= u - \alpha\beta a^2 + b^2 \end{aligned}\right\} \quad (8)$$

Properties:

1. The following expressions are nasty numbers

(a) $3k[x(a,a) + z(a,a) - (\alpha\beta + \alpha + \beta + 1)(4T_{3,a} + 2SO_a - 4CP_{a,6})]$

(b) $k[x(a,b) + y(a,b) + z(a,b) + w(a,a) + X(a,b) + Y(a,a)]$

2. $X(a,a) - Y(a,a) + z(a,a) + w(a,a) - (\alpha\beta - 1)(T_{6,a} + 2T_{3,a} - T_{4,a}) \equiv 0 \pmod 2$

3. $X(a(a+1),a) - Y(a(a+1),a) - 2(6\alpha\beta F_{4,a,5} - 2P_a^5 - 2T_{4,a}) = 0$

4. $x(a,a) - y(a,a) - 2(1 + \alpha\beta + 2\beta)(2P_a^8 - SO_a) = 0$

5. $X(a(a+1),a) - \alpha\beta(6F_{4,a,5} - CP_{a,6} - T_{4,a}) - 4\beta P_a^5 - T_{4,a} = k$

6. $x(a,b).y(a,b) - z(a,a).w(a,a) - (k-w)^2 + (k-y)^2 = 0$

7. $4[x(a,1) - y(a,1) + z(a,1) - w(a,1) + X(a,1) - Y(a,1) - 6\alpha\beta T_{4,a} + 4(\alpha+\beta)(2T_{3,a} - T_{4,a})]$ is a cubical integer

8. $x(2^{2n}, 2^{2n}) + y(2^{2n}, 2^{2n}) - 2(\alpha\beta + \alpha + \beta + 1)(j_{2n} + 1) = 0$

2) Approach2:

Let $v = a^2 - \alpha\beta b^2 \quad (9)$

Now, rewrite (5) as

$$S^2 - \alpha\beta T^2 = v^2 \times 1 \quad (10)$$

Also 1 can be written as

$$1 = \frac{(\alpha + \beta + 2\sqrt{\alpha\beta})(\alpha + \beta - 2\sqrt{\alpha\beta})}{(\alpha - \beta)^2} \quad (11)$$

Substituting (11) and (9) in (10) and using the method of factorisation, define

$$(S+\sqrt{\alpha\beta}T)=\frac{(\alpha+\beta+2\sqrt{\alpha\beta})(a+\sqrt{\alpha\beta}b)^2}{(\alpha-\beta)} \tag{12}$$

Equating real and imaginary parts in (11) we get

$$\left.\begin{aligned} S&=\frac{1}{(\alpha-\beta)}[(\alpha+\beta)(a^2+\alpha\beta b^2)+4\alpha\beta ab] \\ T&=\frac{1}{(\alpha-\beta)}[(\alpha+\beta)2ab+2(a^2+\alpha\beta b^2)] \end{aligned}\right\} \tag{13}$$

Considering (2), (4), (9) & (13) and performing some algebra, the corresponding solutions of (1) are given by

$$\left.\begin{aligned} x&=u+(\alpha-\beta)(f_1+\beta g_1) \\ y&=u-(\alpha-\beta)(f_1+\beta g_1) \\ z&=u+(\alpha-\beta)(f_1+\alpha g_1) \\ w&=u-(\alpha-\beta)(f_1+\alpha g_1) \\ X&=u+(\alpha-\beta)^2(A^2-\alpha\beta B^2) \\ Y&=u-(\alpha-\beta)^2(A^2-\alpha\beta B^2) \end{aligned}\right\} \tag{14}$$

where

$$\left.\begin{aligned} f_1&=(\alpha+\beta)(A^2+\alpha\beta B^2)+4\alpha\beta AB \\ g_1&=2(A^2+\alpha\beta B^2)+2AB(\alpha+\beta) \end{aligned}\right\} \tag{15}$$

Properties:

1. $6k[x(a,a)-(\alpha-\beta)\{(\alpha+3\beta)(1+\alpha\beta)+2(3\alpha\beta+\beta^2)\}(2T_{3,a}+SO_a-2CP_{a,6})]$ is a nasty number
2. $z(a,a)-w(a,a)-2(\alpha-\beta)[(3\alpha+\beta)(1+\alpha\beta)+2(\alpha\beta+\alpha^2)][2P_a^5-CP_{a,6}]=0$
3. $2T_{4,a}-X(a,a)+Y(a,a)\equiv 0 \pmod 2$
4. $x(a,a)-y(a,a)=2(\alpha-\beta)[(\alpha+3\beta)(1+\alpha\beta)+2(3\alpha\beta+\beta^2)][2P_a^8-4CP_{a,6}+3(OH_a)]$

Note:
1 can also be written as

$$1=\frac{(\alpha\beta+k^2+2k\sqrt{\alpha\beta})(\alpha\beta+k^2-2k\sqrt{\alpha\beta})}{(\alpha\beta-k^2)^2} \tag{16}$$

Considering (2), (4), (9) & (16) and performing some algebra, the corresponding solutions of (1) are given by

$$\left.\begin{aligned} x&=u+(\alpha\beta-k^2)(f_2+\beta g) \\ y&=u-(\alpha\beta-k^2)(f_2+\beta g) \\ z&=u+(\alpha\beta-k^2)(f_2+\alpha g) \\ w&=u-(\alpha\beta-k^2)(f_2+\alpha g) \\ X&=u+(\alpha\beta-k^2)^2(A^2-\alpha\beta B^2) \\ Y&=u-(\alpha\beta-k^2)^2(A^2-\alpha\beta B^2) \end{aligned}\right\} \tag{17}$$

where

$$\left.\begin{aligned} f_2 &= (\alpha\beta + k^2)(A^2 + \alpha\beta B^2) + 4\alpha\beta kAB \\ g_2 &= (\alpha\beta + k^2)2AB + 2k(A^2 + \alpha\beta B^2) \end{aligned}\right\} \quad (18)$$

3) Approach3:

(5) can be written as $S^2 - v^2 = \alpha\beta T^2$ (19)

Factorisation of the equation (19) gives

$$(S + v)(S - v) = (\alpha T)(\beta T) \quad (20)$$

Considering (20) as a system of double equations and using the method of cross multiplication, the non-zero integral solution of (1) are obtained as

$$x = u + m^2\alpha + n^2\beta + 2\beta mn$$

$$y = u - m^2\alpha - n^2\beta - 2\beta mn$$

$$z = u + m^2\alpha + n^2\beta + 2\alpha mn$$

$$w = u - m^2\alpha - n^2\beta - 2\alpha mn$$

$$X = u + m^2\alpha - n^2\beta$$

$$Y = u - m^2\alpha + n^2\beta$$

Properties:

1. $3k[x(a,a) + z(a,a) - (\alpha + \beta)(T_{10,a} + 6CP_{a,3} - 3CP_{a,6})]$ is a nasty number
2. $4k^2[X(a,a) + Y(a,a) + z(a,a) - w(a,a) - 2(3\alpha + \beta)(T_{6,a} + 3T_{4,a} - 2T_{5,a})]$ is a cubical integer
3. $2(x(a,a) + w(a,a)) - (\beta - \alpha)(T_{10,a} + 9T_{4,a} - 6T_{5,a}) = 4k$
4. $x(a,a) + y(a,a) + w(a,a) - (\beta - \alpha)(T_{8,a} + 4CP_{a,3} - 2CP_{a,6}) \equiv 0 \pmod 3$

4) Approach4:

(20) can be written as a set of double equations in four different ways as below:

Set1: $S - v = \alpha T, S + v = \beta T$

Set2: $S - v = T^2, S + v = \alpha\beta$

Set3: $S - v = \beta T^2, S + v = \alpha$

Set4: $S - v = \alpha\beta, S + v = T^2$

Solving each of the above sets, the corresponding values of v, S *and* T are given by

Set1: $v = (\beta - \alpha)T_1, S = (\alpha + \beta)T_1, T = 2T_1$

Set2: $v = \alpha_1\beta - 2T_1^2, S = \alpha_1\beta + 2T_1^2, T = 2T_1$

Set3: $v = \alpha_1 - \beta_1 T^2, S = \alpha_1 + \beta_1 T^2$

Set4: $v = 2T_1^2 + 2T_1 - 2\alpha_1\beta_1 - \alpha_1 - \beta_1, S = 2\alpha_1\beta_1 + \alpha_1 + \beta_1 + 2T_1^2 + 2T_1 + 1, T = 2T_1 + 1$

In view of (4) and (2), the corresponding solutions to (1) obtained from each of the above sets are as shown below:

Set1**:**

$x = u + (\alpha + 3\beta)T_1$

$y = u - (\alpha + 3\beta)T_1$

$z = u + (3\alpha + \beta)T_1$

$w = u - (3\alpha + \beta)T_1$

$X = u + (\beta - \alpha)T_1$

$Y = u - (\beta - \alpha)T_1$

Properties:

1. $6[X(a).Y(a) - x(a).y(a) + z(a).w(a) + 9\alpha^2 - 7\beta^2 - 2\alpha\beta]$ is a nasty number
2. $x(a) - y(a) + z(a) - w(a) + X(a) - Y(a) - 2(3\alpha + 5\beta)[3(OH_a) - 2CP_{a,6}] = 0$
3. $X(a) + Y(a) - x(a) - y(a) + z(a) - w(a) - 2(3\alpha + \beta)[2T_{4,a} - T_{6,a}] = 0$
4. $X(a) - Y(a) - 4(\beta - \alpha)T_{3,a} = 0$
5. $x(a) - y(a) + z(a) - w(a) - 8(\alpha + \beta)[2CP_{a,3} - CP_{a,6}] = 0$

Set2:

$x = u + \alpha_1\beta + 2T_1^2 + 2\beta T_1$

$y = u - \alpha_1\beta - 2T_1^2 - 2\beta T_1$

$z = u + \alpha_1\beta + 2T_1^2 + 4\alpha_1 T_1$

$w = u - \alpha_1\beta - 2T_1^2 - 4\alpha_1 T_1$

$X = u + \alpha_1\beta - 2T_1^2$

$Y = u - \alpha_1\beta + 2T_1^2$

Properties:

1. $4[3(X(a) - Y(a)) + 2S_a - 6\alpha\beta + 24T_{3,a} - 12T_{4,a}]$ is a nasty number
2. $2\alpha^2\beta^2[x(a) - y(a) + z(a) - w(a) - 2\{2T_{10,a} + (4\alpha + 2\beta + 3)(2T_{3,a} - T_{4,a})\}]$ is a cubical integer
3. $4[2\alpha\beta(T_{6,a} + 2T_{3,a} - T_{4,a}) - X(a).Y(a)]$ is a biquadratic integer
4. $x(a).y(a,a) + z(a).w(a,a) - 4(\beta^2 + 4\alpha^2)T_{4,a} -$
   $4(\beta + 2\alpha)[\alpha\beta(2T_{3,a} - T_{4,a} + 2CP_{a,6}) + 8(6F_{4,a,5} - 3CP_{a,6} - 2T_{4,a})] \equiv 0 \pmod 2$
5. $x(a) - y(a) + z(a) - w(a) - 8T_{4,a} - 4(\beta + 2\alpha)(2T_{3,a} - T_{4,a}) = 0 \pmod 4$

Set3:

$x = u + \beta_1 T^2 + \alpha_1 + 2\beta_1 T$

$y = u - \beta_1 T^2 - \alpha_1 - 2\beta_1 T$

$z = u + \beta_1 T^2 + \alpha_1 + 2\alpha_1 T$

$w = u - \beta_1 T^2 - \alpha_1 - 2\alpha_1 T$

$X = u + \alpha_1 - \beta_1 T^2$

$Y = u - \alpha_1 + \beta_1 T^2$

Properties:

1. $3\alpha[x(a)-y(a)+4\beta(2CP_{a,6}+SO_a)-2\beta T_{4,a}]$ is a nasty number

2. $z(a)-w(a)+4\alpha(2T_{4,a}-T_{6,a})-2\beta(2P_a^5-2CP_{a,6})\equiv 0(\text{mod } 2)$

3. The following expressions are cubical integers

(a) $4\alpha^2[X(a)-Y(a)+\beta(T_{6,a}+3T_{4,a}-2T_{5,a})]$

(b) $\{4X(a)+4Y(a)\}[4x(a).y(a)+4z(a).w(a)+\{x(a)-y(a)\}^2+\{z(a)-w(a)\}^2]$

Set4:

$$x=u+f_3(\alpha_1,\beta_1,T_1)+(2\beta_1+1)(2T_1+1)$$
$$y=u-f_3(\alpha_1,\beta_1,T_1)-(2\beta_1+1)(2T_1+1)$$
$$z=u+f_3(\alpha_1,\beta_1,T_1)+(2\alpha_1+1)(2T_1+1)$$
$$w=u-f_3(\alpha_1,\beta_1,T_1)-(2\alpha_1+1)(2T_1+1)$$
$$X=u+g_3(\alpha_1,\beta_1,T_1)$$
$$Y=u-g_3(\alpha_1,\beta_1,T_1)$$

Where

$$f_3(\alpha_1,\beta_1,k)=2\alpha_1\beta_1+\alpha_1+\beta_1+2T_1^2+2T_1+1$$
$$g_3(\alpha_1,\beta_1,k)=2T_1^2+2T_1-2\alpha_1\beta_1-\alpha_1-\beta_1$$

Properties:

1. $z(a)-Y(a)-4PR_a-(2\alpha+1)(4CP_{a,6}-2SO_a+1)=1$

2. $2[x(a)-y(a)-4T_{4,a}-2(2\alpha\beta+\alpha+3\beta)-(\beta+1)(10T_{4,a}-2T_{12,a})]$ is a cubic integer

3. $X(a)-Y(a)-8T_{3,a}-2(2\alpha\beta+\alpha+\beta)=0$

B. Case II**:** Choose α and β such that $\alpha\beta$ is a perfect square, say, d^2

5) Approach5:

(5) can be written as

$$S^2=v^2+(dT)^2 \qquad (21)$$

The solution of (21) can be written as

$$dT=a^2-b^2, v=2ab, S=a^2+b^2 \qquad (22)$$

Considering (22), (4) & (2) and performing some algebra the integral solution of (1) is obtained as

$$\left.\begin{aligned}
x&=u+d^2(A^2+B^2)+\beta d(A^2-B^2)\\
y&=u-d^2(A^2+B^2)-\beta d(A^2-B^2)\\
z&=u+d^2(A^2+B^2)+\alpha d(A^2-B^2)\\
w&=u-d^2(A^2+B^2)-\alpha d(A^2-B^2)\\
X&=u+2ABd^2\\
Y&=u-2ABd^2
\end{aligned}\right\} \qquad (23)$$

Properties:

1. $3k[x(a,a)+z(a,a)-d^2(2T_{10,a}+6T_{3,a}-3T_{4,a})]$ is a nasty number

2. $X(a,a)-Y(a,a)-4d^2(2P_a^5-CP_{a,6})=0$

3. $x(2a,a)-y(2a,a)-(10d^2+6\beta d)(2P_a^5-CP_{a,6})=0$

4. $6(z(2a,a)-w(2a,a))-(10d^2+6\alpha d)(S_a+6(2CP_{a,6}-SO_a))=0$

Remark:
It is to be noted that (5) can also be written as

$$S^2-(dT)^2=v^2 \tag{24}$$

(24) can be written as a set of double equations as

$$S-dT=v^2, S+dT=1$$

Solving the above set, the corresponding values of $v, S \; and \; T$ are given by

$$v=2v_1d+1, S=2{v_1}^2d^2+2v_1d+1, T=-2({v_1}^2d+v_1)$$

In view of (4) and (2), the corresponding solutions to (1) are obtained as shown below:

$$\left.\begin{array}{l} x=u+2{v_1}^2d^2+2v_1d+1-2\beta({v_1}^2d+v_1) \\ y=u-(2{v_1}^2d^2+2v_1d+1-2\beta({v_1}^2d+v_1)) \\ z=u+2{v_1}^2d^2+2v_1d+1-2\alpha({v_1}^2d+v_1) \\ w=u-(2{v_1}^2d^2+2v_1d+1-2\alpha({v_1}^2d+v_1)) \\ X=u+2v_1d+1 \\ Y=u-2v_1d-1 \end{array}\right\} \tag{25}$$

6) Approach6:
Rewriting (21) as

$$S^2-v^2=(dT)^2 \tag{26}$$

(26) can be written as a set of double equations in different ways as below:

Set1: $S-v=T^2, S+v=d^2$

Set2: $S-v=d, S+v=dT^2$

Set3: $S-v=d^2, S+v=T^2$

Solving each of the above sets, the corresponding values of $v, S \; and \; T$ are given by

Set1: $v=2({d_1}^2-{T_1}^2), S=2({T_1}^2+{d_1}^2), T=2T_1, d=2d_1$

Set2: $v=d_1(T^2-1), S=d_1(1+T^2), d=2d_1$

Set3: $v=2({T_1}^2-{d_1}^2)+2(T_1-d_1), S=2({T_1}^2+{d_1}^2)+2(T_1+d_1)+1, T=2T_1+1, d=2d_1+1$

In view of (4) and (2), the corresponding solutions to (1) are obtained as shown below

Set1:

$x = u + 2(T_1^2 + d_1^2 + \beta T_1)$

$y = u - 2(T_1^2 + d_1^2 + \beta T_1)$

$z = u + 2(T_1^2 + d_1^2 + \alpha T_1)$

$w = u - 2(T_1^2 + d_1^2 + \beta T_1)$

$X = u + 2(d_1^2 - T_1^2)$

$Y = u - 2(d_1^2 - T_1^2)$

Properties:

1The following expressions are nasty numbers

(a) $6[x(a) - y(a) - 4(T_{4,a} + 3\beta T_{4,a} - 2\beta T_{5,a})]$

(b) $3[x(a) - y(a) + X(a) - Y(a) - \beta(3T_{4,a} - 2T_{8,a})]$

2. $2d[x(a) - y(a) + T_{10,a} + 3(2T_{3,a} - T_{4,a})]$ is a cubical integer

3. $4d^2[z(a) - w(a) - 2T_{6,a} - 4T_{3,a} + 2T_{4,a} - 6\alpha T_{4,a} + 2\alpha T_{8,a}]$ is a biquadratic integer

Set2:

$x = u + d_1(1 + T^2) + \beta T$

$y = u - d_1(1 + T^2) - \beta T$

$z = u + d_1(1 + T^2) + \alpha T$

$w = u - d_1(1 + T^2) - \alpha T$

$X = u + d_1(T^2 - 1)$

$Y = u - d_1(T^2 - 1)$

Properties:

1. $2d^2(6P_a^3 - CP_{a,6} - 4T_{3,a} + 2T_{4,a}) - 3d(X(a) - Y(a))$ is a nasty number

2. $z(a) - w(a) - dT_{6,a} - (2\alpha + d)(2CP_{a,3} - CP_{a,6}) \equiv 0 \pmod 2$

3. $x(a) - y(a) = 2d(1 + 2P_a^5 - CP_{a,6}) + 2\beta(3(OH_a) - 2CP_{a,6})$

Set3:

$x = u + 2(T_1^2 + d_1^2) + 2(T_1 + d_1) + 1 + \beta(2T_1 + 1)$

$y = u - 2(T_1^2 + d_1^2) - 2(T_1 + d_1) - 1 - \beta(2T_1 + 1)$

$z = u + 2(T_1^2 + d_1^2) + 2(T_1 + d_1) + 1 + \alpha(2T_1 + 1)$

$w = u - 2(T_1^2 + d_1^2) - 2(T_1 + d_1) - 1 - \alpha(2T_1 + 1)$

$X = u + 2(T_1^2 - d_1^2) + 2(T_1 - d_1)$

$Y = u - 2(T_1^2 - d_1^2) - 2(T_1 - d_1)$

Properties:

1. $2d[8T_{3,a} - X(a) + y(a) - 4d]$ is a cubical integer

2 $x(a) - y(a) - 2(\beta - \alpha)(3T_{4,a} - T_{8,a}) \equiv 0 \pmod 2$

3. $z(a) - w(a) - (4d^2 + 4d + 2\alpha + 2) - 4\alpha(2CP_{a,6} - SO_a) = 4PR_a$

C) Remarkable observation:

If $(x_0, y_0, z_0, w_0, X_0, Y_0)$ be any given integral solution of (1), then the general solution pattern is presented in the matrix form as follows:

Odd ordered solutions:

$$\begin{pmatrix} x_{2n-1} \\ y_{2n-1} \\ z_{2n-1} \\ w_{2n-1} \\ X_{2n-1} \\ Y_{2n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -(\alpha-\beta)^{4n-3}(\beta+\alpha) & 2\beta(\alpha-\beta)^{4n-3} \\ 1 & 0 & (\alpha-\beta)^{4n-3}(\beta+\alpha) & -2\beta(\alpha-\beta)^{4n-3} \\ 1 & 0 & -2\alpha\,(\alpha-\beta)^{4n-3} & (\alpha+\beta)(\alpha-\beta)^{4n-3} \\ 1 & 0 & 2\alpha(\alpha-\beta)^{4n-3} & -(\alpha+\beta)(\alpha-\beta)^{4n-3} \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ p_0 \\ q_0 \end{pmatrix}$$

Even ordered solutions:

$$\begin{pmatrix} x_{2n} \\ y_{2n} \\ z_{2n} \\ w_{2n} \\ X_{2n} \\ Y_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & (\alpha-\beta)^{4n} & 0 \\ 1 & 0 & -(\alpha-\beta)^{4n} & 0 \\ 1 & 0 & 0 & (\alpha-\beta)^{4n} \\ 1 & 0 & 0 & -(\alpha-\beta)^{4n} \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ p_0 \\ q_0 \end{pmatrix}$$

The values of x, y, z, w, X and Y in all the above approaches satisfy the following properties:

1. $3(x+y)(z+w+X+Y)$ is a nasty number

2. $4(X+Y)[2(x^2+y^2+z^2+w^2)-(x-y)^2-(z-w)^2]$ is a cubical integer.

3. $4(xy+zw)+(x-y)^2+(z-w)^2-2(X+Y)^2=0$

4. $2x+2X-z-3w \equiv 0 (mod\, 2)$

5. $x^2-y^2+z^2-w^2+X^2-Y^2=0 \pmod 4$

III CONCLUSION

Instead of (4), the substitution, $p = S - \beta T, q = S - \alpha T$ in (3), reduces it to the same equation (5). Then different solutions can be obtained, using the same patterns.

REFERENCES

[1]. L.E. Dickson, History of Theory of numbers, vol.2, Chelsea Publishing company, New York,1952.

[2]. L.J. Mordell, Diophantine Equations, Academic press, London, 1969.

[3]. Carmichael.R.D, The Theory of numbers and Diophantine Analysis, New York, Dover,1959.

[4]. M.A.Gopalan and S.Premalatha , Integral solutions of $(x+y)(xy+w^2) = 2(k^2+1)z^3$, Bulletin of pure and applied sciences, Vol.28E (No.2), 197-202, 2009.

[5] M.A.Gopalan and V.pandiChelvi, Remarkable solutions on the cubic equation with four unknowns $x^3+y^3+z^3=28(x+y+z)w^2$,

Antarctica J.of maths, Vol.4 , No.4, 393-401, 2010.

[6] M.A.Gopalan and B.Sivagami, Integral solutions of homogeneous cubic equation with four unknowns $x^3 + y^3 + z^3 = 3xyz + 2(x+y)w^3$, Impact.J.Sci.Tech, Vol.4,No.3, 53-60,2010.

[7]. M.A.Gopalan and S.Premalatha , On the cubic Diophantine equation with four unknowns $(x-y)(xy-w^2) = 2(n^2+2n)z^3$, International Journal of mathematical sciences, Vol.9, No.1-2, Jan-June, 171-175, 2010.

[8]. M.A.Gopalan and J.KaligaRani, Integral solutions of $x^3 + y^3 + (x+y)xy = z^3 + w^3 + (z+w)zw$, Bulletin of pure and applied sciences,Vol.29E(No.1),169-173, 2010.

[9]. M.A.Gopalan and S.Premalatha , Integral solutions of $(x+y)(xy+w^2) = 2(k+1)z^3$,The Global Journal of applied mathematics and Mathematical sciences, Vol.3,No.1-2,51-55,2010.

[10].M.A.Gopalan, S.Vidhyalakshmi and T.R.Usha Rani , On the cubic equation with five unknowns $x^3 + y^3 = z^3 + w^3 + t^2(x+y)$, Indian Journal of science, Vol.1, No.1, 17-20, Nov. 2012.

[11].M.A.Gopalan, S.Vidhyalakshmi and T.R.Usha Rani , Integral solutions of the cubic equation with five unknows $x^3 + y^3 + u^3 + v^3 = 3t^3$,IJAMA, Vol.4(2), 147-151, Dec. 2012.