Location of Regions Containing No Zero of a Polynomial

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Abstract: In this paper we locate regions containing no zero of a polynomial whose coefficients are restricted to certain conditions.

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1.Introduction and Statement of Results

The problem of locating the regions containing all, some or no zero of a polynomial is very important in the theory of polynomials. In this connection lots of papers have been published by researchers. Recently M.H.Gulzar [2] proved the following results:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(z)=\beta_j, j=0,1,.....,n$ such that for some $k_1 \geq 1, k_2 \geq 1$, $,0<\tau_1 \leq 1$, $0<\tau_2 \leq 1$, either

$$k_1\alpha_n \geq \alpha_{n-2} \geq .....\geq \alpha_3 \geq \tau_1\alpha_1$$

and $k_2\alpha_{n-1} \geq \alpha_{n-3} \geq .....\geq \alpha_2 \geq \tau_2\alpha_0$, if n is odd

or

$$k_1\alpha_n \geq \alpha_{n-2} \geq .....\geq \alpha_2 \geq \tau_1\alpha_0$$

and $k_2\alpha_{n-1} \geq \alpha_{n-3} \geq .....\geq \alpha_3 \geq \tau_2\alpha_1$, if n is even .

Then, if n is odd , the number of zeros of P(z) in $|z| \leq \frac{R}{c} (R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$, where

$$M_1 = R^{n+2}[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2(|\alpha_1|+|\alpha_0|)-\tau_1(|\alpha_1|+\alpha_1)-\tau_2(|\alpha_0|+\alpha_0)$$
$$+2\sum_{j=0}^{n}|\beta_j|] \qquad \text{for } R\geq 1$$

and

$$M_1 = |a_0|+R[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2|\alpha_1|+|\alpha_0|+|\beta_0|-\tau_1(|\alpha_1|+\alpha_1)-\tau_2(|\alpha_0|+\alpha_0)$$

$$+2\sum_{j=1}^{n}|\beta_j|] \qquad \text{for} \quad R \le 1.$$

If n is even, the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_2}{|a_0|}$, where

$$M_2 = R^{n+2}[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2(|\alpha_1|+|\alpha_0|)-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\alpha_1|+\alpha_1)$$
$$+2\sum_{j=0}^{n}|\beta_j|] \qquad \text{for} \quad R \ge 1$$

and

$$M_2 = |a_0| + R[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2|\alpha_1|+|\alpha_0|+|\beta_0|-\tau_1(|\alpha_0|+\alpha_0)-\tau_2(|\alpha_1|+\alpha_1)$$
$$+2\sum_{j=1}^{n}|\beta_j|] \qquad \text{for} \quad R \le 1.$$

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \ge 1, k_2 \ge 1$, $,0<\tau_1 \le 1, 0<\tau_2 \le 1$, either

$$k_1|a_n| \ge |a_{n-2}| \ge \ldots\ldots \ge |a_3| \ge \tau_1|a_1|$$
and $k_2|a_{n-1}| \ge |a_{n-3}| \ge \ldots\ldots \ge |a_2| \ge \tau_2|a_0|$, if n is odd

or

$$k_1|a_n| \ge |a_{n-2}| \ge \ldots\ldots \ge |a_2| \ge \tau_1|a_0|$$
and $k_2|a_{n-1}| \ge |a_{n-3}| \ge \ldots\ldots \ge |a_3| \ge \tau_2|a_1|$, if n is even .

Then, if n is odd , the number of zeros of P(z) in $|z| \le \frac{R}{c}(R>0, c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$, where

$$M_3 = R^{n+2}[(k_1|a_n|+k_2|a_{n-1}|)(\cos\alpha+\sin\alpha+1)+2(|a_1|+|a_0|)$$
$$-(\tau_1|a_1|+\tau_2|a_0|)(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|]$$
$$\text{for} \quad R \ge 1$$

and

$$M_3 = |a_0| + R[k_1|a_n|+k_2|a_{n-1}|)(\cos\alpha+\sin\alpha+1)+2|a_1|+|a_0|$$
$$-(\tau_1|a_1|+\tau_2|a_0|)(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=2}^{n-2}|a_j|]\text{s}$$
$$\text{for} \quad R \le 1.$$

If n is even, the number of zeros of P(z) in $|z| \le \frac{R}{c} (R > 0, c > 1)$ does not exceed $\frac{1}{\log c} \log \frac{M_4}{|a_0|}$, where

$$M_4 = R^{n+2}[(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) + 2(|a_1| + |a_0|) - (\tau_1|a_0| + \tau_2|a_1|)(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=2}^{n-2} |a_j|]$$

for $R \ge 1$

and

$$M_4 = |a_0| + R[(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) + 2(|a_1| + |a_0|) - (\tau_1|a_0| + \tau_2|a_1|)(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=2}^{n-2} |a_j|]$$

for $R \le 1$.

In this paper, we find regions which contain no zero of the polynomials in Theorems 1 and 2 and prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(z) = \beta_j, j = 0,1,.....,n$ such that for some $k_1 \ge 1, k_2 \ge 1$, $,0 < \tau_1 \le 1$, $0 < \tau_2 \le 1$, either

$$k_1\alpha_n \ge \alpha_{n-2} \ge ..... \ge \alpha_3 \ge \tau_1\alpha_1$$

and $k_2\alpha_{n-1} \ge \alpha_{n-3} \ge ..... \ge \alpha_2 \ge \tau_2\alpha_0$, if n is odd

or

$$k_1\alpha_n \ge \alpha_{n-2} \ge ..... \ge \alpha_2 \ge \tau_1\alpha_0$$

and $k_2\alpha_{n-1} \ge \alpha_{n-3} \ge ..... \ge \alpha_3 \ge \tau_2\alpha_1$, if n is even .

Then, if n is odd , P(z) has no zero in $|z| < \frac{|a_0|}{M_5}$, for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_6}$ for $R \le 1$, where

$$M_5 = R^{n+2}[k_1(|\alpha_n| + \alpha_n) + k_2(|\alpha_{n-1}| + \alpha_{n-1}) + 2|\alpha_1| + |\alpha_0| - \tau_1(|\alpha_1| + \alpha_1) - \tau_2(|\alpha_0| + \alpha_0) + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|]$$

and

$$M_6 = R[k_1(|\alpha_n| + \alpha_n) + k_2(|\alpha_{n-1}| + \alpha_{n-1}) + 2|\alpha_1| + |\alpha_0| - \tau_1(|\alpha_1| + \alpha_1)$$

$$-\tau_2(|\alpha_0|+\alpha_0)+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|].$$

If n is even, then P(z) has no zero in $|z|<\frac{|a_0|}{M_7}$, for $R\geq 1$ and no zero in $|z|<\frac{|a_0|}{M_8}$ for $R\leq 1$, where

$$M_7 = R^{n+2}[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2|\alpha_1|+|\alpha_0|-\tau_1(|\alpha_0|+\alpha_0)$$
$$-\tau_2(|\alpha_1|+\alpha_1)+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|]$$

and

$$M_8 = R[k_1(|\alpha_n|+\alpha_n)+k_2(|\alpha_{n-1}|+\alpha_{n-1})+2|\alpha_1|+|\alpha_0|-\tau_1(|\alpha_0|+\alpha_0)$$
$$-\tau_2(|\alpha_1|+\alpha_1)+|\beta_0|+2\sum_{j=1}^{n}|\beta_j|].$$

Combining Theorem 1 and Theorem A, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(z)=\beta_j, j=0,1,.....,n$ such that for some $k_1\geq 1, k_2\geq 1$, $,0<\tau_1\leq 1$, $0<\tau_2\leq 1$, either

$$k_1\alpha_n \geq \alpha_{n-2}\geq .....\geq \alpha_3 \geq \tau_1\alpha_1$$

and $k_2\alpha_{n-1}\geq \alpha_{n-3}\geq .....\geq \alpha_2\geq \tau_2\alpha_0$, if n is odd

or

$$k_1\alpha_n \geq \alpha_{n-2}\geq .....\geq \alpha_2 \geq \tau_1\alpha_0$$

and $k_2\alpha_{n-1}\geq \alpha_{n-3}\geq .....\geq \alpha_3\geq \tau_2\alpha_1$, if n is even .

Then, if n is odd, the number of zeros of P(z) in $\frac{|a_0|}{M_5}\leq |z|\leq \frac{R}{c}, R\geq 1$ does not exceed $\frac{1}{\log c}\log\frac{M_1}{|a_0|}$ and the number of zeros of P(z) in $\frac{|a_0|}{M_6}\leq |z|\leq \frac{R}{c}, R\leq 1$ does not exceed $\frac{1}{\log c}\log\frac{M_2}{|a_0|}$ where M_1, M_2, M_5, M_6 are as given in Theorem 1 and Theorem A.

If n is even, then the number of zeros of P(z) in $\frac{|a_0|}{M_7} \le |z| \le \frac{R}{c}, R \ge 1$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$ and the number of zeros of P(z) in $\frac{|a_0|}{M_8} \le |z| \le \frac{R}{c}, R \le 1$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$ where M_3, M_4, M_7, M_8 are as given in Theorem 1 and Theorem A.

Theorem 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k_1 \ge 1, k_2 \ge 1,$,$0 < \tau_1 \le 1, 0 < \tau_2 \le 1$, either

$$k_1|a_n| \ge |a_{n-2}| \ge \ldots\ldots \ge |a_3| \ge \tau_1|a_1|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge \ldots\ldots \ge |a_2| \ge \tau_2|a_0|$, if n is odd

or

$$k_1|a_n| \ge |a_{n-2}| \ge \ldots\ldots \ge |a_2| \ge \tau_1|a_0|$$

and $k_2|a_{n-1}| \ge |a_{n-3}| \ge \ldots\ldots \ge |a_3| \ge \tau_2|a_1|$, if n is even .

Then, if n is odd, P(z) has no zero in $|z| < \frac{|a_0|}{M_9}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_{10}}$ for $R \le 1$, where

$$\begin{aligned} M_9 = R^{n+2}[&(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) \\ &- (\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2|a_1| + |a_0| \\ &+ 2\sin\alpha \sum_{j=2}^{n-2}|a_j|] \end{aligned}$$

and

$$\begin{aligned} M_{10} = |G(z)|R[&(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1) \\ &- (\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2|a_1| + |a_0| \\ &+ 2\sin\alpha \sum_{j=2}^{n-2}|a_j|]. \end{aligned}$$

If n is even, then P(z) has no zero in $|z| < \frac{|a_0|}{M_{11}}$ for $R \ge 1$ and no zero in $|z| < \frac{|a_0|}{M_{12}}$ for $R \le 1$, where

$$M_{11} = R^{n+2}[(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1)$$

$$-(\tau_1|a_0|+\tau_2|a_1|)(\cos\alpha-\sin\alpha+1)+2|a_1|+|a_0|$$
$$+2\sin\alpha\sum_{j=2}^{n-2}|a_j|]$$

and

$$M_{12}=R[(k_1|a_n|+k_2|a_{n-1}|)(\cos\alpha+\sin\alpha+1)$$
$$-(\tau_1|a_0|+\tau_2|a_1|)(\cos\alpha-\sin\alpha+1)+2|a_1|+|a_0|$$
$$+2\sin\alpha\sum_{j=2}^{n-2}|a_j|].$$

Combining Theorem 2 and Theorem B, we get the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n such that for some $k_1\geq 1, k_2\geq 1$, $,0<\tau_1\leq 1, 0<\tau_2\leq 1$, either

$$k_1|a_n|\geq|a_{n-2}|\geq.....\geq|a_3|\geq\tau_1|a_1|$$

and $k_2|a_{n-1}|\geq|a_{n-3}|\geq.....\geq|a_2|\geq\tau_2|a_0|$, if n is odd

or

$$k_1|a_n|\geq|a_{n-2}|\geq.....\geq|a_2|\geq\tau_1|a_0|$$

and $k_2|a_{n-1}|\geq|a_{n-3}|\geq.....\geq|a_3|\geq\tau_2|a_1|$, if n is even .

Then, if n is odd , the number of zeros of P(z) in $\frac{|a_0|}{M_9}\leq|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$ for $R\geq 1$ and the the number of zeros of P(z) in $\frac{|a_0|}{M_{10}}\leq|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_3}{|a_0|}$ for $R\leq 1$ where M_3, M_9, M_{10} are as given in Theorem 2 and Theorem B.

If n is even, the number of zeros of P(z) in $\frac{|a_0|}{M_{11}}\leq|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$ for $R\geq 1$ and the the number of zeros of P(z) in $\frac{|a_0|}{M_{12}}\leq|z|\leq\frac{R}{c}(R>0,c>1)$ does not exceed $\frac{1}{\log c}\log\frac{M_4}{|a_0|}$ for $R\leq 1$where M_4, M_{11}, M_{12} are as given in Theorem 2 and Theorem B.

For different values of the parameters, we get many interesting results from the above results.

2. Lemma

For the proofs of the above results, we make use of the following lemma:

Lemma : Let $P(z)=\sum_{j=0}^{\infty} a_j z^j$ be a polynomial o f degree n with complex coefficients such that for some real α,β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}, 0\le j\le n,$ and

$|a_j|\ge|a_{j-1}|, 0\le j\le n,$ then

$$|a_j-a_{j-1}|\le(|a_j|-|a_{j-1}|)\cos\alpha+(|a_j|+|a_{j-1}|)\sin\alpha .$$

The above lemma 3 is due to Govil and Rahman [1].

3. Proofs of Theorems

Proof of Theorem 1: Let n be odd. Consider the polynomial

$$\begin{aligned}
F(z)&=(1-z^2)P(z)\\
&=(1-z^2)(a_nz^n+a_{n-1}z^{n-1}+.....+a_1z+a_0)\\
&=-\alpha_nz^{n+2}-\alpha_{n-1}z^{n+1}+(1-k_1)\alpha_nz^n+(1-k_2)\alpha_{n-1}z^{n-1}\\
&\quad+(k_1\alpha_n-\alpha_{n-2})z^n+(k_2\alpha_{n-1}-\alpha_{n-3})z^{n-1}+(\alpha_{n-2}-\alpha_{n-4})z^{n-2}\\
&\quad+(\alpha_{n-3}-\alpha_{n-5})z^{n-3}+.....+(\alpha_4-\alpha_2)z^4+(\alpha_3-\tau_1\alpha_1)z^3\\
&\quad+(\tau_1\alpha_1-\alpha_1)z^3+(\alpha_2-\tau_2\alpha_0)z^2+\alpha_1z+i\{-\beta_nz^{n+2}-\beta_{n-1}z^{n+1}\\
&\quad+(\beta_n-\beta_{n-2})z^n+.....+(\beta_2-\beta_0)z^2+\beta_1z\}+a_0\\
&=a_0+G(z),\text{ where}
\end{aligned}$$

$$\begin{aligned}
G(z)&=-\alpha_nz^{n+2}-\alpha_{n-1}z^{n+1}+(1-k_1)\alpha_nz^n+(1-k_2)\alpha_{n-1}z^{n-1}\\
&\quad+(k_1\alpha_n-\alpha_{n-2})z^n+(k_2\alpha_{n-1}-\alpha_{n-3})z^{n-1}+(\alpha_{n-2}-\alpha_{n-4})z^{n-2}\\
&\quad+(\alpha_{n-3}-\alpha_{n-5})z^{n-3}+.....+(\alpha_4-\alpha_2)z^4+(\alpha_3-\tau_1\alpha_1)z^3\\
&\quad+(\tau_1\alpha_1-\alpha_1)z^3+(\alpha_2-\tau_2\alpha_0)z^2+(\tau_2\alpha_0-\alpha_0)z^2+\alpha_1z+i\{-\beta_nz^{n+2}\\
&\quad-\beta_{n-1}z^{n+1}+(\beta_n-\beta_{n-2})z^n+.....+(\beta_2-\beta_0)z^2+\beta_1z\}
\end{aligned}$$

For $|z|\le R,$ we have by using the hypothesis

$$
\begin{aligned}
|G(z)| \le |\alpha_n| R^{n+2} &+ |\alpha_{n-1}| R^{n+1} + (k_1 - 1)|\alpha_n| R^n + (k_2 - 1)|\alpha_{n-1}| R^{n-1} + (k_1\alpha_n - \alpha_{n-2}) R^{n-2} \\
&+ (k_2\alpha_{n-1} - \alpha_{n-3}) R^{n-1} + (\alpha_{n-2} - \alpha_{n-4}) R^{n-2} + \ldots\ldots + (\alpha_4 - \alpha_2) R^4 \\
&+ (\alpha_3 - \tau_1\alpha_1) R^3 + (1 - \tau_1)|\alpha_1| R^3 + (\alpha_2 - \tau_2\alpha_0) R^2 + (1 - \tau_2)|\alpha_0| R^2 \\
&+ |\alpha_1| R + |\beta_n| R^{n+2} + |\beta_{n-1}| R^{n+1} + (|\beta_n| + |\beta_{n-2}|) R^n + \ldots\ldots \\
&+ (|\beta_2| + |\beta_0|) R^2 + |\beta_1| R \\
\le R^{n+2} [&|\alpha_n| + |\alpha_{n-1}| + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\alpha_{n-1}| + k_1\alpha_n - \alpha_{n-2} + k_2\alpha_{n-1} \\
&- \alpha_{n-3} + \alpha_{n-2} - \alpha_{n-4} + \ldots\ldots + \alpha_4 - \alpha_2 + \alpha_3 - \tau_1\alpha_1 \\
&+ (1 - \tau_1)|\alpha_1| + \alpha_2 - \tau_2\alpha_0 + (1 - \tau_2)|\alpha_0| + |\alpha_1| + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|] \\
= R^{n+2} [&k_1(|\alpha_n| + \alpha_n) + k_2(|\alpha_{n-1}| + \alpha_{n-1}) + 2|\alpha_1| + |\alpha_0| - \tau_1(|\alpha_1| + \alpha_1) \\
&- \tau_2(|\alpha_0| + \alpha_0) + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|] \\
= M_5 \quad &\text{for } R \ge 1
\end{aligned}
$$

and for $R \le 1$

$$
\begin{aligned}
|G(z)| \le R[&k_1(|\alpha_n| + \alpha_n) + k_2(|\alpha_{n-1}| + \alpha_{n-1}) + 2|\alpha_1| + |\alpha_0| - \tau_1(|\alpha_1| + \alpha_1) \\
&- \tau_2(|\alpha_0| + \alpha_0) + |\beta_0| + 2\sum_{j=1}^{n} |\beta_j|]. \\
= M_6
\end{aligned}
$$

Since G(z) is analytic for $|z| \le R$, G(0)=0, it follows by Schwarz Lemma that for $|z| \le R$, $|G(z)| \le M_5|z|$ for $R \ge 1$ and $|G(z)| \le M_6|z|$ for $R \le 1$.

Hence , for $R \ge 1$,

$$
\begin{aligned}
|F(z)| &= |a_0 + G(z)| \\
&\ge |a_0| - |G(z)| \\
&\ge |a_0| - M_5|z| \\
&> 0
\end{aligned}
$$

if $|z| < \dfrac{|a_0|}{M_5}$.

And for $R \le 1$,

$$
\begin{aligned}
|F(z)| &= |a_0 + G(z)| \\
&\ge |a_0| - |G(z)| \\
&\ge |a_0| - M_6|z| \\
&> 0
\end{aligned}
$$

if $|z| < \frac{|a_0|}{M_6}$.

This shows that F(z) has no zero in $|z| < \frac{|a_0|}{M_5}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_6}$ for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that P(z) has no zero in $|z| < \frac{|a_0|}{M_5}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_6}$ for $R \leq 1$, thereby proving Theorem 1 when n is odd.

The proof for even n is similar and is omitted.

Proof of Theorem 2: : Let n be odd. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z^2)P(z) \\
&= (1-z^2)(a_n z^n + a_{n-1}z^{n-1} + \ldots\ldots + a_1 z + a_0) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + (1-k_1)a_n z^n + (1-k_2)a_{n-1}z^{n-1} \\
&\quad + (k_1 a_n - a_{n-2})z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} \\
&\quad + (a_{n-3} - a_{n-5})z^{n-3} + \ldots\ldots + (a_4 - a_2)z^4 + (a_3 - \tau_1 a_1)z^3 \\
&\quad + (\tau_1 a_1 - a_1)z^3 + (a_2 - \tau_2 a_0)z^2 + (\tau_2 a_0 - a_0)z^2 + a_1 z + a_0 \\
&= a_0 + G(z), \text{ where}
\end{aligned}$$

$$\begin{aligned}
G(z) = &- a_n z^{n+2} - a_{n-1}z^{n+1} + (1-k_1)a_n z^n + (1-k_2)a_{n-1}z^{n-1} \\
&+ (k_1 a_n - a_{n-2})z^n + (k_2 a_{n-1} - a_{n-3})z^{n-1} + (a_{n-2} - a_{n-4})z^{n-2} \\
&+ (a_{n-3} - a_{n-5})z^{n-3} + \ldots\ldots + (a_4 - a_2)z^4 + (a_3 - \tau_1 a_1)z^3 \\
&+ (\tau_1 a_1 - a_1)z^3 + (a_2 - \tau_2 a_0)z^2 + (\tau_2 a_0 - a_0)z^2 + a_1 z
\end{aligned}$$

For $|z| \leq R$, we have by using the hypothesis and Lemma

$$\begin{aligned}
|G(z)| \leq &|a_n|R^{n+2} + |a_{n-1}|R^{n+1} + (k_1 - 1)|a_n|R^n + (k_2 - 1)|a_{n-1}|R^{n-1} + |a_1|R \\
&+ \{(k_1|a_n| - |a_{n-2}|)\cos\alpha + (k_1|a_n| + |a_{n-2}|)\sin\alpha\}R^n \\
&+ \{(k_2|a_{n-1}| - |a_{n-3}|)\cos\alpha + (k_2|a_{n-1}| + |a_{n-3}|)\sin\alpha\}R^{n-1} \\
&+ \{(|a_{n-2}| - |a_{n-4}|)\cos\alpha + (|a_{n-2}| + |a_{n-4}|)\sin\alpha\}R^{n-2} \\
&+ \ldots\ldots + \{(|a_4| - |a_2|)\cos\alpha + (|a_4| + |a_2|)\sin\alpha\}R^4 \\
&+ \{(|a_3| - \tau_1|a_1|)\cos\alpha + (|a_3| + \tau_1|a_1|)\sin\alpha\}R^3 \\
&+ \{(|a_2| - \tau_2|a_0|)\cos\alpha + (|a_2| + \tau_2|a_0|)\sin\alpha\}R^2 \\
&+ (1-\tau_1)|a_1|R^3 + (1-\tau_2)|a_0|R^2
\end{aligned}$$

$$\leq R^{n+2}[(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1)$$
$$-(\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2|a_1| + |a_0|$$
$$+ 2\sin\alpha \sum_{j=2}^{n-2} |a_j|]$$
$$= M_9 \quad \text{for} \quad R \geq 1$$

and for $R \leq 1$

$$|G(z)| \leq R[(k_1|a_n| + k_2|a_{n-1}|)(\cos\alpha + \sin\alpha + 1)$$
$$-(\tau_1|a_1| + \tau_2|a_0|)(\cos\alpha - \sin\alpha + 1) + 2|a_1| + |a_0|$$
$$+ 2\sin\alpha \sum_{j=2}^{n-2} |a_j|]$$
$$= M_{10}.$$

Since G(z) is analytic for $|z| \leq R$, G(0)=0, it follows by Schwarz Lemma that for $|z| \leq R$, $|G(z)| \leq M_9|z|$ for $R \geq 1$ and $|G(z)| \leq M_{10}|z|$ for $R \leq 1$.

Hence , for $R \geq 1$,

$$|F(z)| = |a_0 + G(z)|$$
$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - M_9|z|$$
$$> 0$$

if $|z| < \dfrac{|a_0|}{M_9}$.

And for $R \leq 1$,

$$|F(z)| = |a_0 + G(z)|$$
$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - M_{10}|z|$$
$$> 0$$

if $|z| < \dfrac{|a_0|}{M_{10}}$.

This shows that F(z) has no zero in $|z| < \dfrac{|a_0|}{M_9}$, for $R \geq 1$ and no zero in $|z| < \dfrac{|a_0|}{M_{10}}$ for $R \leq 1$.

Since the zeros of P(z) are also the zeros of F(z) , it follows that P(z) has no zero in $|z| < \frac{|a_0|}{M_9}$, for $R \geq 1$ and no zero in $|z| < \frac{|a_0|}{M_{10}}$ for $R \leq 1$, thereby proving Theorem 2 when n is odd.
The proof for even n is similar and is omitted.

References

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